New Sharp Inequalities in Analysis and Geometry

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Based on a joint paper with
Amir Moradifam, UC Riverside,

and a recent work with
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Outline

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$$\log(\frac{1}{2\pi}\int_{S^1} |e^v|^2 d\theta) \leq \frac{1}{2\pi}\int_{S^1} \bar{v} v_z(z) z d\theta$$

and equality holds if and only if $a_k = \gamma^k/k$ for some $\gamma \in \mathbb{C}$ with $|\gamma| < 1$.

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This is well known in the community of univalent functions, in particular in connection with Bieberbach conjecture.

Real Valued Function: Another Form

Denote D the unit disc on $\mathbb{R}^2$. For any real function $u \in H^{\frac{1}{2}}(S^1)$, the norm of u is identified as the the $H^1(D)$ norm of the harmonic extension of u, which we denote again by u, on the disc D.

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$$\log(\frac{1}{2\pi}\int_{S^1} e^u d\theta) - \frac{1}{2\pi}\int_{S^1} u d\theta \leq \frac{1}{4\pi}||\nabla u||^2_{L^2(D)} \qquad (1)$$

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Note:

$$||\nabla u||^2_{L^2(D)} = \int_{S^1} u\frac{\partial u}{\partial n} d\theta$$

is the $H^{1/2}(S^1)$ norm.

Toeplitz Determinants and the Szego Limit Theorem

Given $f(\theta) \in L^1(S^1)$. Let

$$c_k = \frac{1}{2\pi} \int_{S^1} e^{ik\theta} f(\theta) d\theta, k = 0, \pm 1, \pm 2, \cdots ,$$

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and $T(p,q) = c_{p-q}, p, q \in \mathbb{Z}$ be the Toeplitz matrix, and $T_n(p,q) = c_{p-q}, 0 \le p, q \le n$ be the n-th Toeplitz matrix.

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Then $\ln D_n(e^u) - (n+1)\frac{1}{2\pi}\int_{S^1} u d\theta$ is nondecreasing and

$$\lim_{n\to\infty}\{\ln D_n(e^u) - (n+1)\frac{1}{2\pi}\int_{S^1} u d\theta\} = \frac{1}{4\pi}||\nabla u||^2_{L^2(D)}.$$

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In particular,

$$\ln D_n(e^u) - (n+1)\frac{1}{2\pi}\int_{S^1} ud\theta \leq \frac{1}{4\pi}||\nabla u||^2_{L^2(D)}, \quad n \geq 0.$$

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The first two inequalities in the Szego limit theorem

We have

$$D_1(e^u) = (\frac{1}{2\pi}\int_{S^1} e^u d\theta)^2 - (\frac{1}{2\pi}\int_{S^1} e^u e^{i\theta} d\theta)^2.$$

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When $n = 1$, the second inequality in the Szego limit theorem is

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One notes that in the special case when $\int_{S^1} e^u e^{i\theta} d\theta = 0$, as a direct consequence of above inequality we have

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Question: Any similar inequalities in higher dimensions?

Outline

Trudinger-Moser Inequality (1967, 1971)

Let S^2 be the unit sphere and for $u \in H^1(S^2)$.

$$J_\alpha(u) = \frac{\alpha}{4}\int_{S^2} |\nabla u|^2 d\omega + \int_{S^2} u d\omega - \log \int_{S^2} e^u d\omega \geq C > -\infty,$$

if and only if $\alpha \geq 1$, where the volume form $d\omega$ is normalized so that $\int_{S^2} d\omega = 1$.

Aubin's Result (1979) and Onofri Inequality (1982)

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Aubin observed that for $\alpha \geq \frac{1}{2}$,

$$J_\alpha(u) \geq C > -\infty$$

for

$$u \in \mathcal{M} := \{u \in H^1(S^2) : \int_{S^2} e^u x_i = 0, \quad i = 1, 2, 3\},$$

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$$\frac{\alpha}{2}\Delta u + \frac{e^u}{\int_{S^2} e^u d\omega} - 1 = \sum_{i=1}^{i=3} \mu_i x_i e^u, \quad \text{on} \quad S^2. \tag{4}$$

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Axially symmetric functions

For every function g on $(-1, 1)$ satisfying $\|g\|^2 = \int_{-1}^{1}(1 - x^2)|g'(x)|^2 dx < \infty$ and

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G. and Wei, and independently Lin (2000)

Earlier Result for general functions:

Ghoussoub and Lin (2010):

Conjecture A holds for

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For general functions, to show solutions to (3) are axially symmetric.

Sterographic Projection

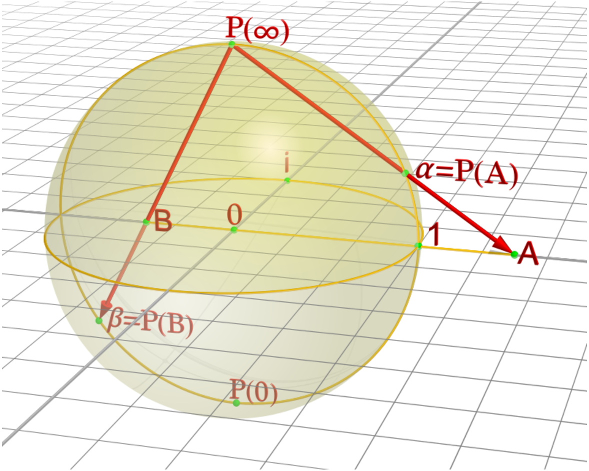

Figure: Sterographic Projection

Equations on $\mathbb{R}^2$

Let Π be the stereographic projection $S^2 \to \mathbb{R}^2$ with respect to the north pole $N = (1, 0, 0)$:

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then v satisfies

$$\Delta v + (1 + |y|^2)^{2(\frac{1}{\alpha} - 1)} e^v = 0 \text{ in } \mathbb{R}^2, \tag{7}$$

and

$$\int_{R^2} (1 + |y|^2)^{2(\frac{1}{\alpha} - 1)} e^v dy = \frac{8\pi}{\alpha}. \tag{8}$$

General Equations

Consider in general the equation

$$\Delta v + (1 + |y|^2)^l e^v = 0 \;\; \text{in} \;\; \mathbb{R}^2, \tag{9}$$

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Conjecture B.
For $0 < l \leq 2$, solutions to (9) and (10) must be radially symmetric.
Note

$$l = 2(\frac{1}{\alpha} - 1) = 2(\frac{\rho}{8\pi} - 1)$$

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A general equation on R^2

Assume $u \in C^2(\mathbb{R}^2)$ satisfies

$$\Delta u + k(|y|)e^{2u} = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{11}$$

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$$\frac{1}{2\pi} \int_{\mathbb{R}^2} k(|y|)e^{2u} dy = \beta < \infty, \tag{12}$$

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$$(K1) \qquad \Delta \ln(k(|y|)) \geq 0, \quad y \in \mathbb{R}^2$$

$$(K2) \qquad \lim_{|y|\to\infty} \frac{|y|k'(|y|)}{k(|y|)} = 2l > 0, \quad y \in \mathbb{R}^2.$$

A general symmetry result

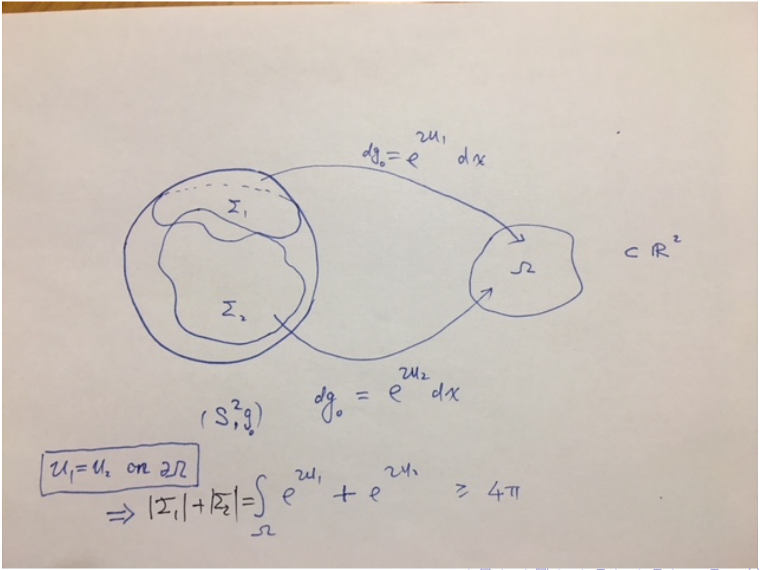
The following general symmetry result is proven.

Proposition

Assume that $K(y) = k(|y|) > 0$ satisfies $(K1) - (K2)$, and u is a solution to (11)-(12) with $l + 1 < \beta \leq 4$. Then u must be radially symmetric.

Outline

The Sphere Covering Inequality: Geometric Description



The Sphere Covering Inequality: Analytic Statement

Theorem (G. and Moradifam, Inventiones, 2018)

Let Ω be a simply connected subset of R^2 and assume $w_i \in C^2(\overline{\Omega})$, $i = 1, 2$ satisfy

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Suppose $w_2 > w_1$ in Ω and $w_2 = w_1$ on $\partial\Omega$, then

The Sphere Covering Inequality: Analytic Statement

Theorem (G. and Moradifam, Inventiones, 2018)

Let Ω be a simply connected subset of R^2 and assume $w_i \in C^2(\overline{\Omega})$, $i = 1, 2$ satisfy

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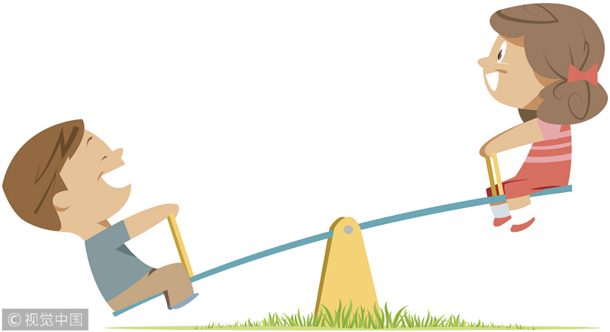
Furthermore if $f_1 \not\equiv 0$ or $f_2 \not\equiv f_1$ in Ω, then $\int_\Omega e^{w_1} + e^{w_2} dy > 8\pi$.

Rigidity of Two Objects: Seesaw Effect

$$\int_{\Omega} e^{w_1} dy \quad vs \quad \int_{\Omega} e^{w_2} dy. \tag{15}$$



Isoperimetric Inequalities

Suppose $\Omega \subset \mathbb{R}^2$, then

$$L^2(\partial\Omega) \geq 4\pi A(\Omega)$$

Equality holds if and only if Ω is a disk.

Levy's Isoperimetric inequalities on spheres (1919)

On the standard unit sphere with the metric induced from the flat metric of $\mathbb{R}^3$,

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If the sphere has radius R, then

$$L^2(\partial\Omega) \geq A(\Omega)\big(4\pi R^2 - A(\Omega)\big)/R^2$$

i.e.,

$$L^2(\partial\Omega) \geq A(\Omega)\big(4\pi - A(\Omega)/R^2\big)$$

Alexandrov-Bol's inequality (1941)

In general, we can identify a sphere with $\mathbb{R}^2$ by a stereographic projection, and equip it with a metric conformal to the flat metric of $\mathbb{R}^2$, i.e., $ds^2 = e^{2v}(dx_1^2 + dx_2^2)$.

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with the gaussian curvature $K \leq 1$.
Then

$$\Big(\int_{\partial\Omega} e^v ds\Big)^2 \geq \Big(\int_\Omega e^{2v}\Big)\Big(4\pi - \int_\Omega e^{2v}\Big)$$

Outline

Logrithemic Determinants and Conformal Geometry

Given a Riemanian surface (M, σ_0) with Gaussian curvature K_0 and normalized area $|M| = 1$. Consider a conformal metric on $\sigma = e^{2u}$ on M.

If $\partial M = \emptyset$, define

$$F(u) = \frac{1}{2}\int_M |\nabla_0 u|^2 dA_0 + \int_M K_0 u dA_0 - \pi\chi(M)\ln(\int_M e^{2u} dA_0).$$

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If ∂M consists of nice boundary with geodesic curvature k_0, assume that (M, σ_0) and (M, σ) are flat. Define

$$F(u) = \frac{1}{2}\int_{\partial M} u\frac{\partial u}{\partial n} ds_0 + \int_{\partial M} k_0 u ds_0 - 2\pi\chi(M)\ln(\int_{\partial M} e^u ds_0).$$

Extremals

B. Osgood, R. Phillips and P. Sarnak. (1988):

$$\log \frac{Det(\Delta_\sigma)}{Det(\Delta_{\sigma_0})} = -\frac{1}{6\pi} F(u) + \frac{1}{4\pi} \int_{\partial M} \frac{\partial u}{\partial n} ds_0$$

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Uniformization, Isospectral Properties, etc.

Widom's observation (1988), Chang-Hang (2019)

If $\int_{S^1} e^{ik\theta} e^u d\theta = 0, -n \leq k \leq n$, then

$$\log\left(\frac{1}{2\pi}\int_{S^1} e^u d\theta\right) - \frac{1}{2\pi}\int_{S^1} u d\theta \leq \frac{1}{4\pi(n+1)}||\nabla u||^2_{L^2(D)} \quad (16)$$

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$$J_{\frac{1}{N(n)}+\epsilon}(u) \geq C_n(\epsilon) > -\infty, \quad \forall u \in H^1(S^2).$$

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A Variant of Aubin-Onofri Inequality, Alice Chang and G., 2019

Let us consider the following functionals in $H^1(S^2)$:

$$I_\alpha(u) = \alpha \int_{S^2} |\nabla u|^2 d\omega + 2 \int_{S^2} u d\omega$$

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But I_α is NOT bounded below in $H^1(S^2)$ for $\alpha < 2/3$.

Euler-Lagrange Equation

Let

$$a_i = \int_{S^2} e^{2u} x_i d\omega, \quad i = 1, 2, 3. \tag{18}$$

Define

$$\mathcal{H} = \{u \in H^1(S^2) : \int_{S^2} e^{2u} d\omega = 1\}. \tag{19}$$

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Proposition

The Euler Lagrange equation for the functional I_α in $\mathcal{H}$ is

$$\alpha \Delta u + \frac{1 - \sum_{i=1}^{3} a_i x_i}{1 - \sum_{i=1}^{3} a_i^2} e^{2u} - 1 = 0 \quad on \quad S^2. \tag{20}$$

Existence and Nonexistence of Solutions

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i) When $\alpha \in [\frac{1}{2}, 1)$ *and* $\alpha \neq \frac{2}{3}$, *equation* (20) *has only constant solutions;*

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In particular, u is axially symmetric about $\vec{a}$ *if* $\vec{a} \neq (0, 0, 0)$.
After a proper rotation, the solution u is explicitly given by the formula in (26) *below.*

Kazdan-Warner condition

For the Gaussian curvature equation:

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$$\int_{S^2} (\nabla K(x) \cdot \nabla x_j) e^{2u} d\omega = 0 \quad \text{for each j} = 1,2,\ 3. \tag{22}$$

Stereographic Project

For $\alpha = \frac{2}{3}$, we assume that $(a_1, a_2, a_3) = (0, 0, a)$ with $a \in (0, 1)$ and consider

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Use the stereographic projection to transform the equation to be on $\mathbb{R}^2$. Let

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Then w satisfies

$$\Delta w + \frac{6}{1 + a}(b^2 + |y|^2)e^{2w} = 0 \quad \text{in} \quad \mathbb{R}^2 \tag{24}$$

where $b^2 = \frac{1+a}{1-a} > 1$, $b > 0$ and

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$$\int_{\mathbb{R}^2} (b^2 + |y|^2)e^{2w} dy = (1 + a)\pi. \tag{25}$$

Exact Solution

Now it is easy to verify directly that

$$w(y) = -\frac{3}{2}\ln(b^2 + |y|^2) + 2\ln b + \frac{1}{2}\ln\frac{2}{1+b^2}$$

is a solution to (24) and (25), and hence $u(x)$ defined by

$$u(x) = u(\Pi^{-1}(y)) := \frac{3}{2}\ln\frac{1+|y|^2}{b^2+|y|^2} + 2\ln b + \frac{1}{2}\ln\frac{2}{1+b^2} \quad (26)$$

is a solution to (23).

Symmetry and Uniqueness of Solutions

Use symmetry result of G.-Moradifam (2018) and uniqueness result of C.S. Lin (2000) on axially symmetric solutions, we know that the solution above is a unique solution.
Define

$$u_{\alpha,b}(x) = u_{\alpha,b}(\Pi^{-1}(y)) := \frac{1}{\alpha} \ln \frac{1+|y|^2}{b^2+|y|^2} + 2\ln b + \frac{1}{2} \ln \frac{2}{1+b^2} \tag{27}$$

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Direct computations show that

$$\lim_{b\to\infty} I_\alpha(u_{\alpha,b}) = -\infty, \quad \text{if } \alpha < \tfrac{2}{3}$$
$$\lim_{b\to\infty} I_\alpha(u_{\alpha,b}) = \infty, \quad \text{if } \alpha > \tfrac{2}{3}$$
$$I_{\frac{2}{3}}(u_{\frac{2}{3},b}) = 0, \quad \forall b > 0$$

Indeed,

$$u_{\alpha,\vec{a}}(x) = -\frac{1}{\alpha} \ln(1 - \vec{a}\cdot x) + \ln(1-|\vec{a}|^2), \quad x \in S^2.$$

Challenge: Compactness?

It is NOT clear if the minimum is attained and a minimizer exists!

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The compactness of the minimizing sequence is NOT known.

A Constrained Minimization Problem and Compactness

For any $\vec{a} = (a_1, a_2, a_3) \in B_1 := \{|a| < 1\} \subset \mathbb{R}^2$, let us define

$$\mathcal{M}_{\vec{a}} := \{u \in H^1(S^2) : \int_{S^2} e^{2u} x_i = a_i, \quad i = 1, 2, 3\} \cap \mathcal{H}. \quad (28)$$

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and recall the following compactness result:

Proposition

For any $\alpha > \frac{1}{2}, \vec{a} = (a_1, a_2, a_3) \in B_1$, there exists $C_{\alpha, \vec{a}} \in \mathbb{R}$ such that

$$I_\alpha(u) \geq C_{\alpha, \vec{a}}, \quad \forall u \in \mathcal{M}_{\vec{a}} \quad (29)$$

E-L Equation of the constrained problem

It is standard to show that there exists a minimizer $u_{\alpha,\vec{a}} \in \mathcal{M}_{\vec{a}}$ of (28) satisfying

$$\alpha \Delta u + e^{2u}(\rho - \sum_{i=1}^{3} \beta_i x_i) = 1, \quad x \in S^2 \tag{30}$$

for some $\rho \in \mathbb{R}$ and $\vec{\beta} = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ with

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$$\min_{u \in \mathcal{M}_{\vec{a}}} I_{\frac{2}{3}}(u) = I_{\frac{2}{3}}(u_{\frac{2}{3},|\vec{a}|}) = 0.$$

When $\alpha \neq \frac{2}{3}$

Rotate the coordinates properly so that $\beta_1 = \beta_2 = 0$. Without loss of generality, we assume that $\beta_3 > 0$.
We can obtain

$$\frac{a_3}{1-a_3^2} \leq \beta_3 \leq \frac{2(\frac{1}{\alpha}-1)a_3}{1-a_3^2}, \quad \text{if } \alpha \in (\frac{1}{2}, \frac{2}{3}] \tag{31}$$

and

$$\frac{a_3}{1-a_3^2} \geq \beta_3 \geq \frac{2(\frac{1}{\alpha}-1)a_3}{1-a_3^2}, \quad \text{if } \alpha \in [\frac{2}{3}, 1]. \tag{32}$$

Difference: Equation on $\mathbb{R}^2$

Let b be a positive constant with $b^2 = \frac{\rho+\beta_3}{\rho-\beta_3} > 1$. Set

$$w_{\alpha,\vec{a}}(y) := u_{\alpha,\vec{a}}(\Pi^{-1}(y)) - \frac{1}{\alpha}\ln(1+|y|^2) + \frac{1}{2}\ln(\frac{4(\rho-\beta_3)}{\alpha}).$$

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Then $w_{\alpha,\vec{a}}$ satisfies

$$\Delta w + k(|y|)e^{2w} = 0 \quad \text{in} \quad \mathbb{R}^2 \tag{33}$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} k(|y|)e^{2w}dy = \frac{2}{\alpha} \tag{34}$$

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When $\frac{1}{2} < \alpha < \frac{2}{3}$, t $k(|y|)$ satisfies $(K1)-(K2)$ with $l = \frac{2}{\alpha}-2$. By G.-Moradifam (2018), $w_{\alpha,\vec{a}}(y)$ must be radially symmetric and hence $u_{\alpha,\vec{a}}(y)$ must be axially symmetric and $a_1 = a_2 = 0$.

Estimate of the minimum $m(\alpha, a)$ of J_α on $\mathcal{M}_a$.

Theorem

There hold pointwise in $a \in [0, 1)$

$$m(\alpha, a) \geq \begin{cases} (\frac{2}{\alpha} - 3)\ln(1 - a^2), & \alpha \in (1/2, 2/3), \\ \alpha(\frac{1}{\alpha} - \frac{3}{2})\ln(1 - a^2), & \alpha \in (2/3, 1). \end{cases} \tag{35}$$

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$$\leq \begin{cases} (\frac{2}{\alpha} - 3)\ln(1 - a^2), & \alpha \in (2/3, 1), \\ \frac{3\alpha}{2a}(\frac{1}{\alpha} - \frac{3}{2})\big(\ln(1 - a^2) - 2(\ln(1 + a) - a)\big), & \forall \alpha \in (1/2, 1). \end{cases} \tag{36}$$

Some Technique Questions

1). Should $u_{\alpha,\vec{a}}(y)$ always be axially symmetric for all $\alpha \in (\frac{1}{2}, 1)$ and $\vec{a} \in B_1$?

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3) Fixed $\alpha \in (\frac{1}{2}, 1), \vec{a} \in B_1$, for any given $\vec{\beta} = \beta_3\vec{a}/|\vec{a}|, 0 < \beta_3 < \frac{1}{1-|\vec{a}|}, \rho = 1 + \beta_3|\vec{a}|$, there is a unique axially symmetric solution u to (30) with the corresponding w solving (33) and (34).

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4) Can we compute or estimate more accurately

$$m(\alpha, \vec{a}) := I_\alpha(u_{\alpha,\vec{a}})?$$

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Higher Dimensions?

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Thank You!