Hidden Convexity in Nonlinear Elasticity

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- Measure-valued convex relaxation of nonlinear elasticity.

Optimal Transport

Move mass from μ to ν, optimally!
(Monge 1781) Find a map

$$x \mapsto T(x)$$

to minimize the total mass distance traveled: cost $c(x, y) = |y - x|$.

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(Kantorovich 1942) Use a joint probability distribution

$$(X, Y) \sim \pi$$

with fixed marginals.

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Duality

Theorem (Kantorovich 1942)

The minimal value of the optimal transport problem with measures $\mu(dx)$, $\nu(dy)$, and cost $c(x,y)$ equals the maximal value of a dual problem for potential functions $\phi(x)$, $\psi(y)$:

$$\min_{\pi \dots} \int\int c(x,y)\pi(dx,dy) = \sup_{\phi,\psi} \int \psi(y)\nu(dy) - \int \phi(x)\mu(dx)$$

with constraint $\psi(y) - \phi(x) \leqslant c(x,y)$ (equality where $\pi(dx,dy) > 0$).

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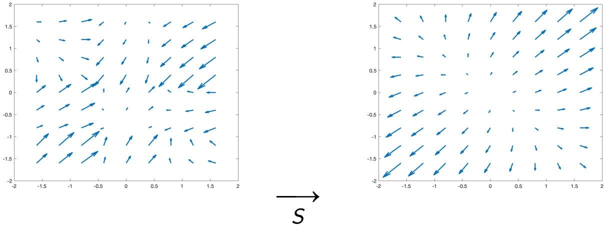
- Linear programming took off with the help of the simplex algorithm (Dantzig 1947)

Contributions of Brenier

- Polar factorization (Brenier 1991): Every (nondegenerate) vector field $f \in L^2(\Omega, \mathbb{R}^d)$ decomposes uniquely as $f = \nabla\phi \circ S$ where ϕ is convex and $S : \Omega \to \Omega$ is volume preserving.



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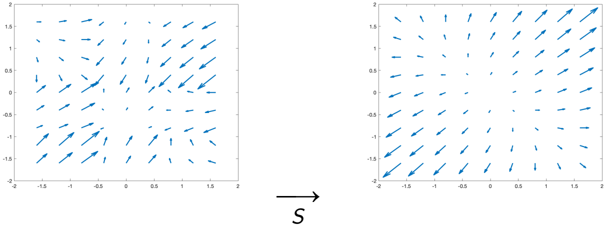

- (Benamou-Brenier 2000) Minimize Lagrangian over velocity field v_t

$$\int_0^1 \int_{\mathbb{R}^d} \frac{1}{2} |v_t|^2 \pi_t(dx) : \quad \partial_t \pi_t + \nabla \cdot v_t \, \pi_t = 0, \quad \pi_0 = \mu, \quad \pi_1 = \nu.$$

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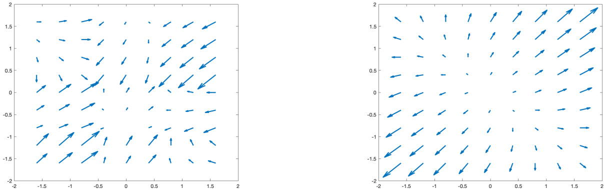

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- [Brenier The initial value problem for the Euler equations of incompressible fluids viewed as a concave maximization problem. CMP 2018]

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- *Dirichlet Boundary conditions*: $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$ (where $\mathbf{g}(\partial\Omega) = \partial D$)
- *Incompressibility*: $\det(\nabla \mathbf{u}) = 1$ or, if $\mathbf{u}$ is injective,

$$\mathbf{u}_\# \mathcal{L}_\Omega = \mathcal{L}_D.$$

Incompressible Elasticity Known / Unkown

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- Global Minimizers (Ball 1976)
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Unknown:

- Existence of pressure for global minimizers.
- Uniqueness of minimizers. (Some examples of nonuniqueness known.)
- Higher regularity.
- A priori bounds.

Elasticity equilibrium as a polar factorization

- Euler-Lagrange equations: pressure, $p : \Omega \to \mathbb{R}$,

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- If ω is *convex* then ω and $\mathbf{u}$ give a polar factorization of the body forces $\nabla \cdot DW(\nabla \mathbf{u})$.

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- The relaxation $\mathbf{u}_{\#}\mathcal{L}_\Omega = \mathcal{L}_D$ is still non-convex.
- If ω and W are convex then $\mathbf{u}$ is the unique minimizer of the convex functional

$$\int_\Omega \Big[W(\nabla \mathbf{u}) + \omega(\mathbf{u}) \Big] d\mathcal{L}_\Omega, \tag{1}$$

with Dirichlet boundary conditions but without incompressibility.

Energy minimization

Theorem

Suppose $\mathbf{u}$ *is an elastic equilibrium with deformed pressure* $\omega = p \circ \mathbf{u}^{-1}$. *If* ω *and* W *are convex, then* $\mathbf{u}$ *is a global energy minimizer and minimizes the convex functional (1).*

Proof.

- $\mathbf{u}$ is a critical point of (1) so is a minimizer of (1) by convexity.
- Let $\mathbf{v}$ be another admissible incompressible deformation. Then

$$\int_\Omega \omega(\mathbf{v})\, d\mathcal{L}_\Omega = \int_D \omega\, d\mathcal{L}_D = \int_\Omega \omega(\mathbf{u})\, d\mathcal{L}_\Omega$$

- It follows

$$\int_\Omega W(\nabla \mathbf{v})\, d\mathcal{L}_\Omega = \int_\Omega \Big[W(\nabla \mathbf{v}) + \omega(\mathbf{v}) - \omega(\mathbf{u}) \Big] d\mathcal{L}_\Omega \geqslant \int_\Omega W(\nabla \mathbf{u})\, d\mathcal{L}_\Omega.$$

□

- Ex: Affine boundary conditions, $\mathbf{u} = \mathbf{A}\mathbf{x} + \mathbf{b}$ on $\partial\Omega$. Then $p = \omega = 0$. The affine map is the energy minimizer.

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- Ex: If pressure, ω_0, is λ-semiconvex at an equilibrium $\mathbf{u}_0$, then modify the energy by $W(\nabla \mathbf{u}) - 2\lambda\, \mathbf{u} \cdot \mathbf{u}_0$, and the pressure becomes $\omega(\mathbf{y}) = \omega_0(\mathbf{y}) + \lambda|y|^2$.

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- Maximum principle (Pogorelov) type arguments to control semiconvexity of ω?
- Handling other boundary conditions? The deformed domain D is no longer fixed.

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$$\Phi(\mu) = \int_D \phi\Big(\frac{d\mu}{d\mathcal{L}_D}\Big) d\mathcal{L}_D$$

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- Same energy minimization result when ω is convex.

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- When ω is convex then it is a solution to the dual problem; this measure-valued relaxed problem coincides with the original.