Asia-Pacific Analysis and PDE Seminar

A variational approach to the regularity theory for optimal transportation

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joint work with Michael Goldman (arXiv '17, Ann. ENS),
with MG & Martin Huesmann (arXiv '18, '19, CPAM)
with Tatsuya Miura (arXiv '20)
with Maxime Prod'homme & Tobias Ried (arXiv '20)

Follow de Giorgi's approach for Minimal Surfaces

Harmonic approximation $\overset{\text{elliptic regularity, change of variables}}{\Longrightarrow}$ One-Step improvement $\overset{\text{Campanato iteration}}{\Longrightarrow}$ ϵ-regularity $\overset{\text{measure theory}}{\Longrightarrow}$ partial regularity.

Here $C^{2,\alpha}$-regularity for Kantorowicz potential ψ

i. e. $C^{1,\alpha}$-regularity for Brenier map $T = \nabla\psi$.

Purely variational/ no maximum principle/ avoids Caffarelli's theory

Applications (old and new):

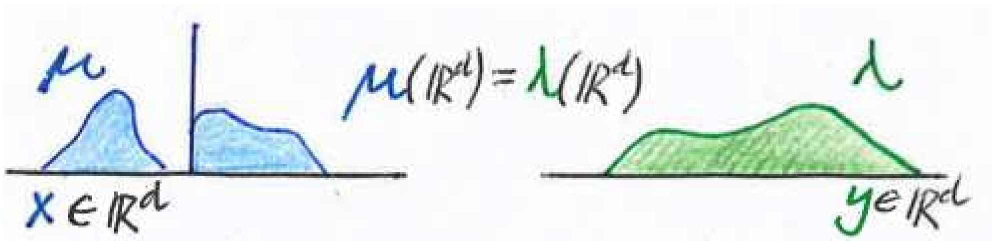
Partial regularity [Figalli-Kim, DePhilippis-Figalli],

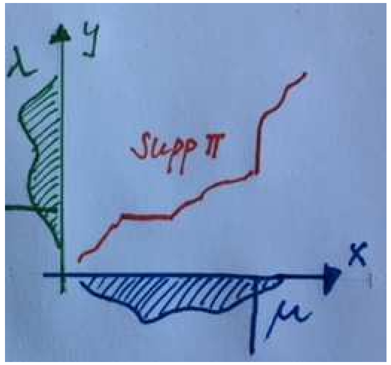
ϵ-regularity at boundary for $\partial\Omega \in C^{1,\alpha}$ [Chen-Figalli],

"Matching" Poisson to Lebesgue [Parisi et. al, Ambrosio et. al]

Recall Optimal Transportation

Given two measures μ and λ





seek transfer plan π, i. e. $\pi(U \times \mathbb{R}^d) = \mu(U)$, $\pi(\mathbb{R}^d \times V) = \lambda(V)$
that minimizes Euclidean transport cost $\int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^2 \pi(dxdy)$.

Minimum $=: W^2(\mu, \lambda)$ (squared) Wasserstein distance.

From optimal transportation to Monge-Ampère

Minimize $\int_{\mathbb{R}^d \times \mathbb{R}^d} |y-x|^2 \pi(dxdy)$
among all $\pi(dxdy)$ with marginals $\mu(dx)$ and dy.

Support of optimal π is cyclically monotone; hence
$\exists$ convex ψ supp$\pi \subset \{\,(x,y)\,|\,y \in$ subgradient $\partial\psi(x)\,\}$.

$\forall$ test functions ζ $\int \zeta(\nabla\psi(x))\mu(dx) = \int \zeta(y)dy$.

In smooth case, this amounts to $\det D^2\psi = \mu$,
an instance of the Monge-Ampère equation.

Nature of the Monge-Ampère equation

Recall Monge-Ampère: $\det D^2\psi = 1$.
Fully non-linear with $F(A) := \det A - 1$.

However elliptic: $F(A) > F(A')$ for $A > A' \geq 0$;
satisfies comparison principle.

However degenerate: $\leftrightarrow$ affine invariant (non-compact SL(n)).
Cf. Laplacian $F(A) = \mathrm{tr} A$: rotation invariant (compact SO(n-1)).

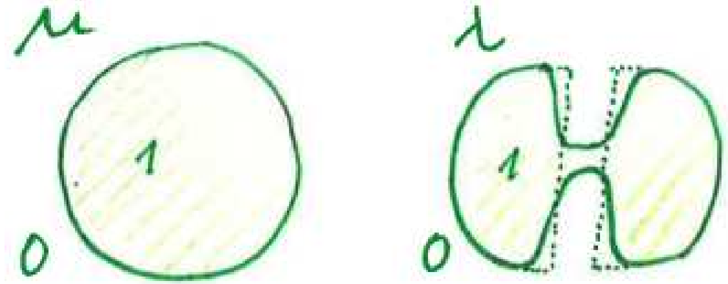
Caffarelli's '90 breakthrough:
comparison principle, affine invariance, compactness.

Monge-Ampère equation at crossroads
of fully nonlinear and variational.

Singularities in optimal transportation are generic

Caffarelli's example:

smooth data μ, λ
do not yield smooth $T = \nabla\psi$.



Riemannian setup: Loeper's example, Ma-Trudinger-Wang theory.

Thus ϵ-regularity is of interest:

$\int |y\text{-}x|^2 \pi(dxdy) \leq \epsilon$ locally, μ, λ smooth locally

$\Longrightarrow$ Kantorowicz potential ψ smooth locally.

Relies on harmonic approximation in our approach:

$\int |y\text{-}x|^2 \pi(dxdy) \leq \epsilon$ locally, $\mu, \lambda \approx 1$ locally

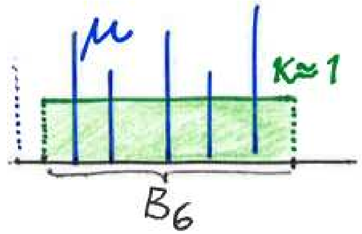
$\Longrightarrow$ displacement $(y\text{-}x)\pi(dxdy) \approx \nabla$harmonic locally.

At the core: harmonic approximation

Local energy $\quad E := \int_{(B_6\times\mathbb{R}^d)\cup(\mathbb{R}^d\times B_6)} |y-x|^2\pi(dxdy),$

Local data size^2 $\quad D := W_{B_6}^2(\mu,\kappa) + (\kappa-1)^2$

$+$ same for λ



Proposition 1 (Goldman&Huesmann&O.)

$\forall\ \tau>0 \quad \exists\ \epsilon(\tau,d)>0,\ C(\tau,d)<\infty \quad$ s. t. $\quad E+D\le\epsilon \Longrightarrow$

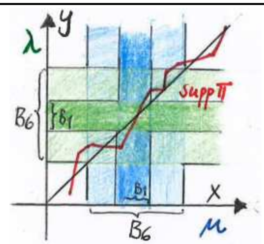
$\exists\ \nabla\phi$ harmonic, $\quad \int_{B_3}|\nabla\phi|^2 \le C(E+D),$

$$\int_{(B_1\times\mathbb{R}^d)\cup(\mathbb{R}^d\times B_1)} \left|y-x-\nabla\phi(\tfrac{1}{2}x+\tfrac{1}{2}y)\right|^2\pi(dxdy) \le \tau E + CD.$$

Amounts to:

Displacement $\ y-x$

$\approx$ harmonic gradient $\ \nabla\phi$



Harmonic approximation: correct homogeneities ...

$E := \int_{(B_6\times\mathbb{R}^d)\cup(\mathbb{R}^d\times B_6)} |y-x|^2\pi(dxdy)$, quadratic in solution,

$D := W^2_{B_6}(\mu,\kappa) + (\kappa-1)^2 +$ same for λ, quadratic in data.

Proposition 1

$\forall\ \tau>0 \quad \exists\ \epsilon(\tau,d)>0,\ C(\tau,d)<\infty \quad$ s. t. $\quad E+D\le\epsilon \implies$

$\exists\ \nabla\phi$ harmonic, $\quad \int_{B_2}|\nabla\phi|^2 \le C(E+D)$,

$$\int_{(B_1\times\mathbb{R}^d)\cup(\mathbb{R}^d\times B_1)} \left|y-x-\nabla\phi(\tfrac{1}{2}x+\tfrac{1}{2}y)\right|^2\pi(dxdy) \le \tau E + CD.$$

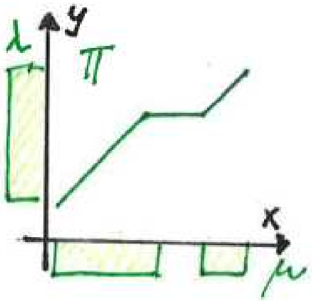
Compare to $\int_{B_1} L(\nabla u-\nabla\phi) \le \tau\int_{B_6} L(\nabla u) + C\int_{B_6}|f|^2$

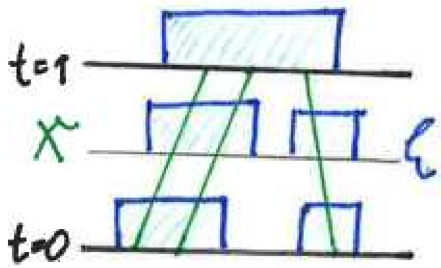
for $\ -\nabla\cdot DL(\nabla u) = \nabla\cdot f\ $ with uniformly convex L .

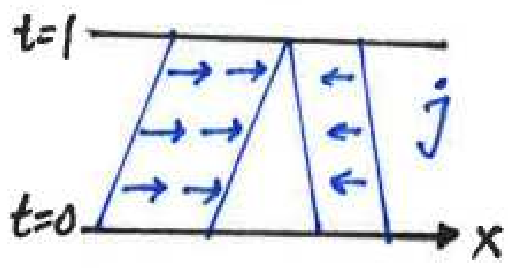
... and correct metric

From Lagrangian to Eulerian (Benamou-Brenier)

Transport plan π, trajectories $X(t)$, density/flux (ρ, j)







Continuity eqn. $\partial_t \rho + \nabla \cdot j = 0$, kinetic energy $\int_{\mathbb{R}^d \times (0,1)} \frac{1}{\rho} |j|^2$.

$$W^2(\mu, \lambda) = \inf\{ \int \frac{1}{\rho} |j|^2 \mid \partial_t \rho + \nabla \cdot j = 0,\ \rho(t{=}0){=}\mu,\ \rho(t{=}1){=}\lambda \}$$

Kinetic energy density $(\rho, j) \mapsto \frac{1}{\rho} |j|^2$ is strictly convex.

Linearization $\frac{1}{\rho} |j|^2 \rightsquigarrow |j|^2$ amounts to $W^2(\mu, \lambda) \rightsquigarrow \|\mu - \lambda\|^2_{\dot{H}^{-1}}$.

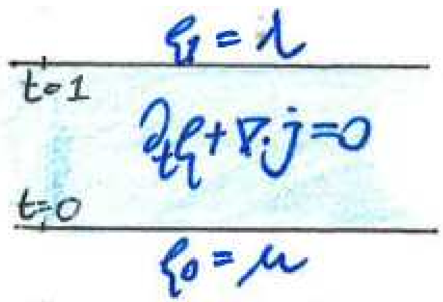
Analogy for minimal surfaces: varifolds vs. currents.

Eulerian version of harmonic approximation

Density/flux $(\rho, j) = (\rho_t dt, j_t dt)$ where

$\int \zeta d\rho_t = \int \zeta(ty+(1\text{-}t)x)\pi(dxdy), \ \int \xi \cdot dj_t = \int \xi(ty+(1\text{-}t)x)\cdot(y\text{-}x)\pi(dxdy)$

continuity equation



$E \rightsquigarrow \int_{B_5\times(0,1)} \frac{1}{\rho}|j|^2$

reveals strict convexity of variational problem

Proposition 1'

$\forall \ \tau > 0 \quad \exists \ \epsilon(\tau, d) > 0, \ C(\tau, d) < \infty \quad$ s. t. $\quad E + D \leq \epsilon \Longrightarrow$

$\exists \ \nabla\phi$ harmonic, $\quad \int_{B_2} |\nabla\phi|^2 \leq C(E + D),$

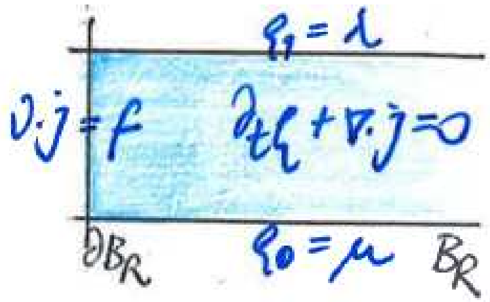
$\int_{B_2\times(0,1)} \frac{1}{\rho}\left|j - \rho\nabla\phi\right|^2 \leq \tau E + CD.$

Amounts to: Eulerian velocity $\frac{j}{\rho} \approx \nabla\phi$ harmonic gradient

Construct $\nabla\phi$ via flux (Neumann) data

Normal flux $f := \nu \cdot j$ across bdry ∂B_R,

its time integral $\bar{f} := \int_0^1 f\,dt$.



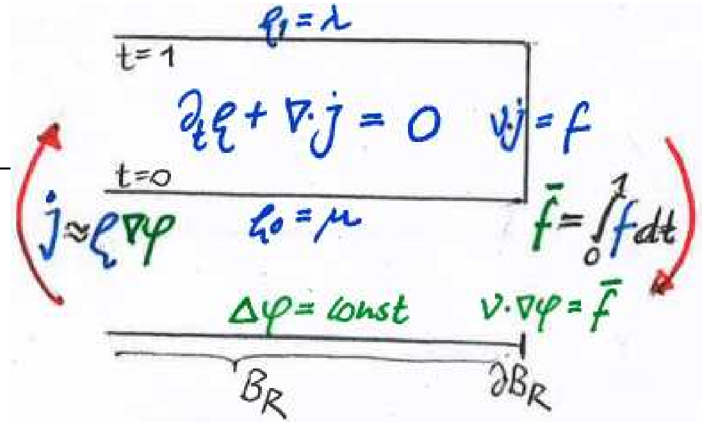
Proposition 1” $\forall\, \tau > 0 \;\exists\, \epsilon > 0, C < \infty :\; E + D \leq \epsilon \implies$
$\exists\, R \in (3,4)$ s. t. $\triangle\phi = \text{const}$ in B_R, $\nu \cdot \nabla\phi = \bar{f}$ on ∂B_R

satisfies $\displaystyle\int_{B_2} |\nabla\phi|^2 \leq C(E+D)$,

$$\int_{B_2\times(0,1)} \frac{1}{\rho}\left|j - \rho\nabla\phi\right|^2 \leq \tau E + CD.$$

cf. Dacorogna-Moser.

Choice of “good” radius R.



Old application: ϵ-regularity as Schauder theory

Recall Hölder semi-norms $[f]_{\alpha,B} := \sup_{x\neq x'\in B}\frac{|f(x)-f(x')|}{|x-x'|^\alpha}$, $\alpha\in(0,1)$.

Suppose $\mu = f\,dx,\ \lambda = g\,dy$

with Hölder continuous f, g and $f(0) = g(0) = 1$.

Monitor (the dimensionless) $E := \frac{1}{R^2|B_R|}\int_{\Omega\cap B_{2R}}|T-x|^2dx$

with $T = \nabla\psi$ Brenier map.

Monitor (the dimensionless) $D := R^{2\alpha}[f]^2_{\alpha,B_{2R}} + R^{2\alpha}[g]^2_{\alpha,B_{2R}}$.

Theorem 1 (Goldman&O., à la DePhilippis&Figalli)
If $E + D \ll 1$ then $R^{2\alpha}[\nabla T]^2_{\alpha,B_R} \lesssim E + D$.

Amounts to $C^{2,\alpha}$-regularity for Monge-Ampère $\det D^2\psi = f$ ($g\equiv 1$).

Comparison: DePhilippis&Figalli vs. Goldman&O.

Theorem 1 (Goldman&O., à la DePhilippis&Figalli)
$E+[f]^2_{\alpha,B_2}+[g]^2_{\alpha,B_2}\ll 1 \implies [D^2\psi]^2_{\alpha,B_1}\lesssim E+[f]^2_{\alpha,B_2}+[g]^2_{\alpha,B_2}.$

Perturbation around linear $\triangle\phi = const$,
as opposed to nonlinear $\det D^2\psi = 1$
(following Caffarelli '92 approach to $W^{2,p}/C^{2,\alpha}$, but using ϵ-regularity Figalli&Kim).

Get (immediately) linear homogeneities $[D^2\psi]_{\alpha,B_1}\lesssim [f]_{\alpha,B_2}+[g]_{\alpha,B_2}$.

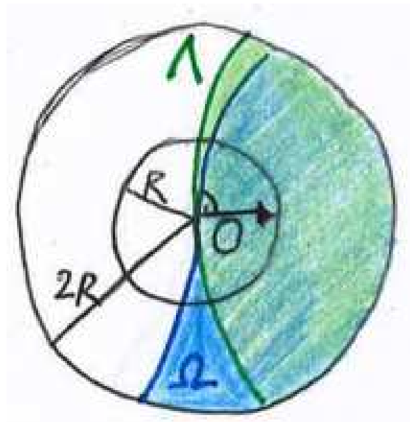
Get $\psi\in C^{2,\alpha}$ in one step (also for general cost functions),
as opposed to three-step bootstrap $C^{1,\alpha}$, $C^{1,1}$, $C^{2,\alpha}$.

Competitors in strictly convex variational problem,
instead of comparison principle.

Refined application: Boundary ϵ-regularity

Suppose $\mu = dx_{|\Omega}$, $\lambda = dy_{|\Lambda}$ for two open sets $\Omega, \Lambda \subset \mathbb{R}^d$.

Consider $E := \frac{1}{R^2|B_R|}\int_{\Omega\cap B_{2R}} |T-x|^2 dx$ where T Brenier map



Topological condition:

$T(\Omega \cap B_R) \subset \Lambda \cap B_{2R}, \quad T^{-1}(\Lambda \cap B_R) \subset \Omega \cap B_{2R}.$

Geometric condition:

$\Omega \cap B_{2R} = \{x_1 > g(x')\} \cap B_{2R}$, $\Lambda \cap B_{2R} = \{x_1 > h(x')\} \cap B_{2R}$

with $g(0) = h(0) = 0$, $\nabla' g(0) = \nabla' h(0) = 0$.

Consider $D := R^{2\alpha}[\nabla' g]^2_{\alpha, B_{2R}} + R^{2\alpha}[\nabla' h]^2_{\alpha, B_{2R}}$.

Theorem 2 (Miura&O., à la Chen-Figalli)

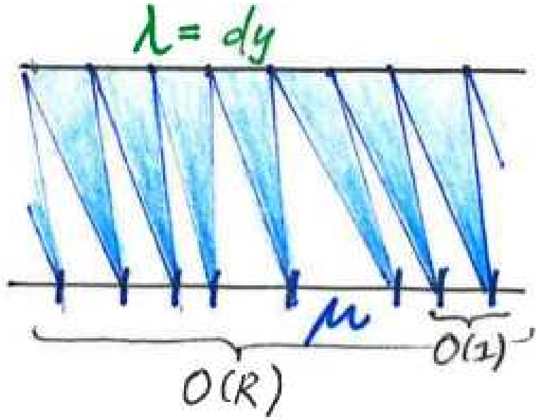
If $E + D \ll 1$ then $R^{2\alpha}[\nabla T]^2_{\alpha, B_R} \lesssim E + D$

Optimal by Jhaveri '19

New application: Quantitative linearization ...

Now general μ but $\lambda = dy$ Lebesgue. Expect:

$\mu \approx dx \implies y - x \approx \nabla u$ under π, where $\triangle u = \mu - 1$.



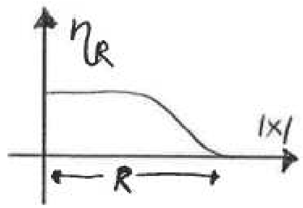
With an eye on
matching Poisson (scale 1)
to Lebesgue (on scale $R \gg 1$),

want: $\mu \approx dx$ *on large-scale average*,

eg. control of $\frac{1}{|B_R|} W^2_{B_R}(\mu, \kappa) + (\kappa - 1)^2$

$\implies$ $y - x \approx \nabla u$ *on large-scale average*,

eg. control of $\int \eta_R (y - x)\pi(dxdy) - \int \eta_R \nabla u$.

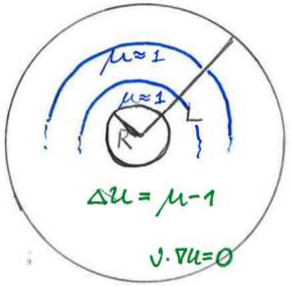


... on large scales for matching problems

Quantitative linearization on large scales

Admissible rate function:

$$\frac{1}{C}\sum_{k=0}^{\infty}\frac{\beta(2^kR)}{2^kR}\le\frac{\beta(R)}{R}\le C\quad\text{for}\quad R\ge 1.$$



Poisson: $\beta(R)\sim\ln R$ for $d=2$,

$\beta(R)\sim 1$ for $d>2$.

Theorem 3 (Goldman&Huesmann&O.)

Suppose $\operatorname{supp}\mu\subset B_L$, $\mu(\mathbb{R}^d)=|B_L|$

and $\frac{1}{|B_R|}W^2_{B_R}(\mu,\kappa)+(\kappa\text{-}1)^2\le\beta(R)$ for all $1\le R\le L$.

Then $$\left|\frac{\int\eta_{R(x)}(y-x)\pi(dxdy)}{\int\eta_{R(x)}\pi(dxdy)}-\frac{\int\eta_R\nabla u}{\int\eta_R}\right|\lesssim\frac{\beta(R)}{R},\ 1\ll R\le L,$$

where $\triangle u=\mu-1$ in B_L, $\nu\cdot\nabla u=0$ on ∂B_L.

Harmonic approximation even interesting for μ, λ (locally) constant

Suppose $\mu = dx$ and $\lambda = dy$ on B_6 i.e. $D = 0$

Recall: $f := \nu \cdot j$, $\bar{f} := \int_0^1 f\,dt$, $E := \int_{B_5 \times (0,1)} \frac{1}{\rho}|j|^2$.

Lemma 1 (Goldman&O.)

$\exists\ R \in (3,4)$ s. t. $\left\{ \begin{array}{rcl} \triangle\phi & = & 0 \text{ in } B_R, \\ \nu \cdot \nabla\phi & = & \bar{f} \text{ on } \partial B_R \end{array} \right\}$ satisfies

$$\int_{B_R} |\nabla\phi|^2 \le C(d)E,$$

$$\int_{B_R \times (0,1)} \frac{1}{\rho}\left|j - \rho\nabla\phi\right|^2 \le C(d)E^{\frac{d+2}{d+1}}.$$

transport velocity $\frac{j}{\rho} \approx \nabla\phi$ gradient of harmonic function

Reflects that Wasserstein $W_2 \approx \dot{H}^{-1}$ for densities ≈ 1.

Ingredients for Lemma 1

Lemma 1: $\exists\,\phi$ harmonic such that

$$\int_0^1\int_{B_1}\frac{1}{\rho}|j\text{-}\rho\nabla\phi|^2 \lesssim \Big(\int_0^1\int_{B_2}\frac{1}{\rho}|j|^2\Big)^{\frac{d+2}{d+1}}.$$

1) Construction of ϕ: $\triangle\phi = 0$ in B_R, $\nu\cdot\nabla\phi = \bar f$ on ∂B_R,

where $\bar f := \int_0^1 f$, $f := \nu\cdot j$

and $R\in[1,2]$ such that $\int_0^1\int_{\partial B_R}\frac{1}{\rho}|j|^2 \lesssim \int_0^1\int_{B_2}\frac{1}{\rho}|j|^2$ (good radius)

2) Orthogonality:

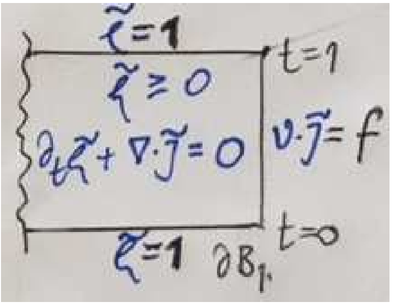
$$\int_0^1\int_{B_R}\frac{1}{\rho}|j\text{-}\rho\nabla\phi|^2 \le \int_0^1\int_{B_R}\frac{1}{\rho}|j|^2 - \int_{B_R}|\nabla\phi|^2.$$

Follows from McCann's *displacement convexity* in form of $\rho\le 1$.

3) Construction of competitor: $(\tilde\rho,\tilde j)$ admissible with bc of (ρ,j) and

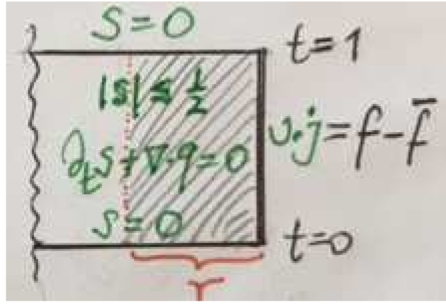
$$\int_0^1\int_{B_R}\frac{1}{\tilde\rho}|\tilde j|^2 - \int_{B_R}|\nabla\phi|^2 \lesssim \Big(\int_0^1\int_{\partial B_R} f^2\Big)^{\frac{d+2}{d+1}}.$$

Ingredients for Lemma 1: boundary layer construction



Ansatz $(\tilde{\rho}, \tilde{j}) = (1, \nabla\phi) + (s, q)$

$\mathrm{supp}(s, q) \subset [0,1] \times (\bar{B}_R - B_{R-r})$



Enough to construct (s, q) such that $|s| \le \frac{1}{2}$ and

$$\int_0^1 \int_{B_R} |q|^2 \lesssim r \int_0^1 \int_{\partial B_R} (f\text{-}\bar{f})^2 \quad \text{for} \quad r \gg \Big(\int_0^1 \int_{\partial B_R} (f\text{-}\bar{f})^2\Big)^{\frac{1}{d+1}}.$$

Reduce to *isoperimetric estimate* $\Big(\int_0^1 \int_{\partial B_R} (\zeta - \bar{\zeta})^2\Big)^{\frac{1}{2}}$

$$\lesssim r^{\frac{1}{2}} \Big(\int_0^1 \int_{B_R - B_{R-r}} |\nabla\zeta|^2\Big)^{\frac{1}{2}} + r^{-\frac{d+1}{2}} \int_0^1 \int_{B_R - B_{R-r}} |\partial_t \zeta|.$$

Reminiscent of Alberti&Choksi&O. '09.

Analogies to minimal surfaces (Schoen&Simon '82)

Approximate minimal surface by harmonic graph /
approximate displacement by harmonic gradient.

Use: Object is minimizing under compact perturbations.
Don't use: Euler-Lagrange equation (= first variation).

Mismatch of type of boundary condition
for construction of harmonic competitor:
graph vs. non-graph / time-averaged vs. time-resolved;
Lower-dimensional isoperimetric estimate:
error is of higher-order (choice of good radius).

Use of strict convexity to convert energy gap
into distance ("approximate orthogonality");
need to smooth out boundary data.

From Lemma 1 to Theorem 1

Inner regularity for ϕ. Eulerian $(\rho, j) \rightsquigarrow$ Lagrangian T.

For $\quad Q \in \mathbb{R}^{d\times d}$, $b \in \mathbb{R}^d \quad$ consider

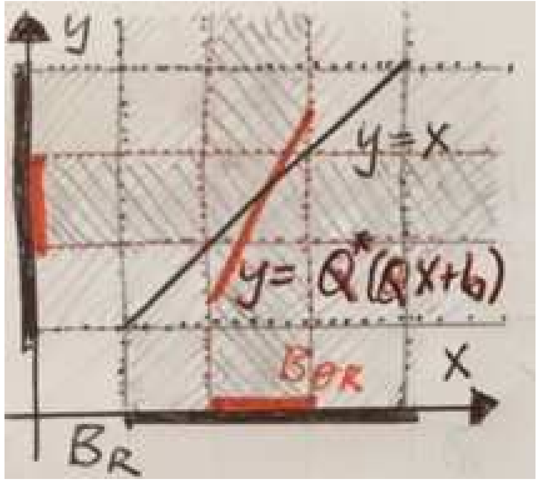
$$E(R,Q,b) := \frac{1}{R^{d+2}}\int_{B_R}|Q^{-*}T-(Qx+b)|^2dx$$

$E(R) := E(R,\mathrm{id},0) \ll 1 \Longrightarrow$

$\exists\, Q, b \quad \forall\, 0<\theta\ll 1$

$$E(\theta R, Q, b) \lesssim \theta^2 E(R) + \theta^{-(1+\frac{d}{2})}E^{\frac{d+2}{d+1}}(R)$$

$$|Q-\mathrm{id}|^2 + \frac{1}{R^2}|b|^2 \lesssim E(R)$$



Affine change of variables $\hat{x} = Qx+b$, $\hat{y} = Q^{-*}y$

preserves optimality of transport. Campanato iteration.

Changes for rough μ, λ, flux boundary data

Recall: $D := W^2_{B_6}(\mu,\kappa)+(\kappa\text{-}1)^2, \quad E := \int_{(B_6\times\mathbb{R}^d)\cup(\mathbb{R}^d\times B_6)} |y-x|^2\pi(dxdy).$

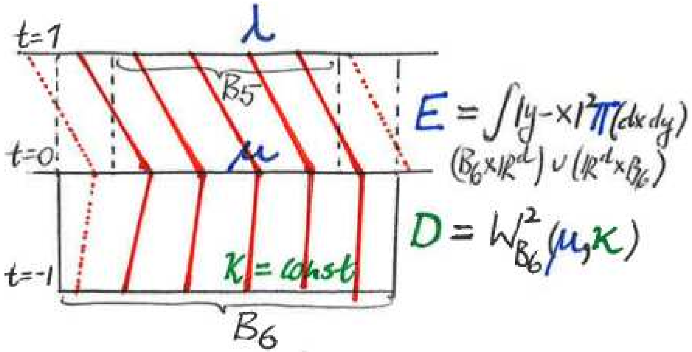
Issue: $\bar{f} = \int_0^1 f\,dt, \quad f = \nu\cdot j$ measure $\notin L^2$.

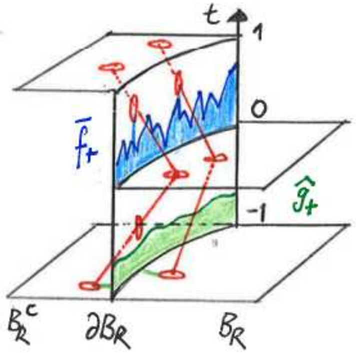
Replace $\bar{f}$ by $\hat{g}\in L^2$ in

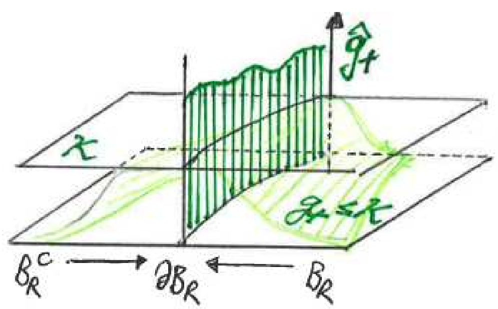
$$\triangle\phi = c \text{ in } B_R, \qquad \nu\cdot\nabla\phi = \hat{g} = \hat{g}_+ - \hat{g}_- \text{ on } \partial B_R$$

Lemma (nonlin. L^2-approx.) $\exists\, R\in(3,4)$, $\hat{g}_\pm$ on ∂B_R s. t.

$$\int_{\partial B_R} \hat{g}_\pm^{\,2} \lesssim E+D, \qquad W^2_{\partial B_R}(\hat{g}_\pm,\bar{f}_\pm) \lesssim (E+D)^{1+\frac{1}{d+2}}$$







Changes for rough μ, λ, L^∞-bound

Recall: $D := W^2_{B_6}(\mu,\kappa)+(\kappa\text{-}1)^2, \quad E := \int_{(B_6\times\mathbb{R}^d)\cup(\mathbb{R}^d\times B_6)} |y-x|^2\pi(dxdy).$

Lemma (nonlinear L^2-approximation)

$\exists R \in (3,4)$, $\hat{g}_\pm$ on ∂B_R s. t.

$$\int_{\partial B_R} \hat{g}_\pm^2 \lesssim E+D, \quad W^2_{\partial B_R}(\hat{g}_\pm, \bar{f}_\pm) \lesssim (E+D)^{1+\frac{1}{d+2}}$$

Superlinear exponent comes from

Lemma (L^∞-estimate)

For $(x_0, x_1) \in \mathrm{supp}\pi \cap (B_5 \times \mathbb{R}^d) \cup (\mathbb{R}^d \times B_5)$

$$|x_0 - x_1| \lesssim (E+D)^{\frac{1}{d+2}}$$

Follows from monotonicity of suppπ

Summary and outlook

Harmonic approximation $\Longrightarrow$ One-Step improvement
$\overset{\text{Campanato iteration}}{\Longrightarrow}$ small or large-scale-regularity

Applications: ϵ-regularity at boundary for $\partial\Omega \in C^{1,\alpha}$

[Savin-Yu, Chen-Figalli, Chen-Liu-Wang],

Matching Poisson to Lebesgue

[Sturm-Huesmann, Parisi et.al, Ambrosio et.al, Ledoux, Talagrand]

general cost functions $c \in C^{2,\alpha}$ with Prod'homme & Ried,

Next step:

thermodynamic limit for Matching in $d = 2$.