Ancient mean curvature flow and singularity analysis

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Mean curvature flow

(**Mean curvature flow**) Suppose that $X_0 : M^n \to \mathbb{R}^{n+1}$ is a smooth immersion and consider a solution $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ to the partial differential equation

$$X_t = \Delta_g X,$$

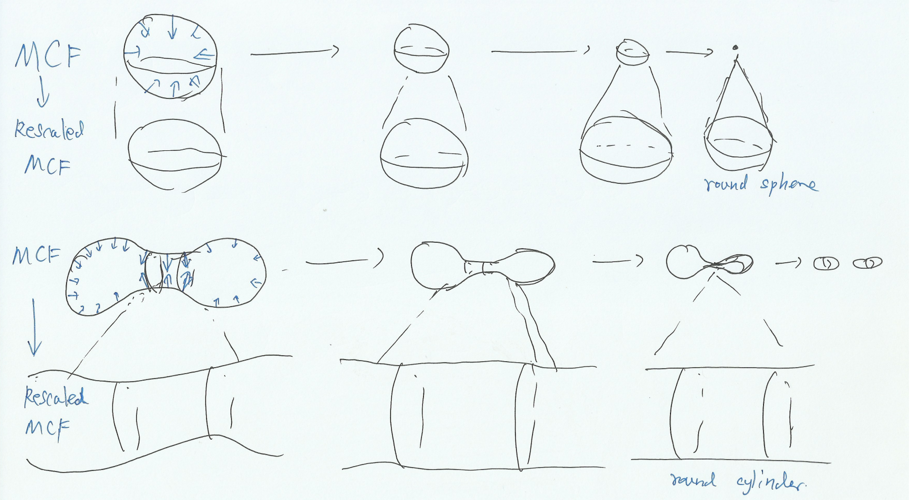
satisfying the initial condition $X(\cdot, 0) = X_0$. Then, the one-parameter family of hypersurfaces $X(M, t)$ is the mean curvature flow.

Notice that $\Delta_g X = -H\nu$ where H the mean curvature and ν is the outward pointing unit normal vector.

In particular, if $X(\cdot, t)$ is the graph of a function $u : \mathbb{R}^n \to \mathbb{R}$ then u satisfies

$$u_t = (1 + |Du|^2)^{\frac{1}{2}} \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right).$$

Singularity example



Round spheres and round cylinders are self-similarly shrinking solutions

Rescaled flow

Given a singularity $(x_0, t_0) \in \mathbb{R}^{n+1} \times [0, T]$, we define the rescaled mean curvature flow by

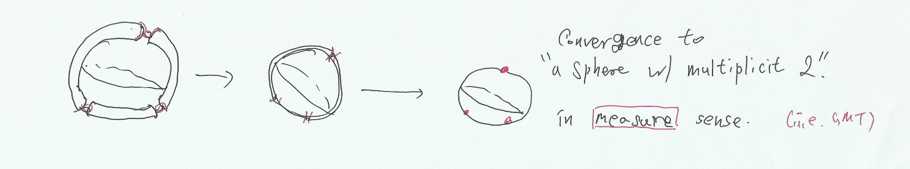
$$\hat{X}_\tau = \Delta_{\hat{g}} \hat{X} - \tfrac{1}{2}\hat{X},$$

where $\tau = -\log(t_0 - t)$ and $\hat{X} = (t_0 - t)^{\frac{1}{2}}(X - x_0)$.

The Huisken's monotonicity formula

$$\frac{d}{d\tau}\int e^{-\frac{|\hat{x}|^2}{4}} d\hat{g} = -\int \left|\hat{H} - \tfrac{1}{2} < \hat{X}, \hat{\nu} >\right|^2 e^{-\frac{|\hat{x}|^2}{4}} d\hat{g} \leq 0,$$

implies that $\hat{X}(M, \tau)$ converges to a shrinker with multiplicity as $\tau \to +\infty$, where a hypersurface Σ is a shrinker if $\sqrt{-t}\,\Sigma$ is a solution to the MCF.



Limit flow, Tangent flow, and Ancient flow

Consider $(x_i, t_i, \lambda_i) \to (x_0, t_0, +\infty)$ and

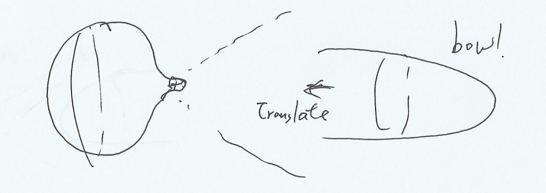
$$X^i(M, \lambda_i^2(t - t_i)) = \lambda_i(X^i(M, t) - x_i).$$

The limit $\bar{X}(M, t) = \lim X^i(M, t)$ exists, then we call it a limit flow at (x_0, t_0).

In particular, it $(x_i, t_i) = (x_0, t_0)$ then $\bar{X}(M, t)$ is a tangent flow.

Notice that limit flows exist from $-\infty$, namely it is ancient. Thus, we can expect classification results by parabolic Liouville theory.

Translaters (Travelling waves) can be limit flows at singularities. c.f. The bowl soltion at a degenerated neck in Angenent-Velazquez 97.



Applications to topology

(**Schoenflies theorem**) A closed curved embedded in the place bounds a disk, namely the curve is the boundary of a disk.

(**3D Schoenflies problem**) A closed surface embedded in $\mathbb{R}^3$ is homeomorphic to the sphere. Does the surface bound a ball?
(**Answer**) No. There exists a counter-example; Alexander's horned sphere.

(**3D smooth Schoenflies theorem**) A closed surface embedded in $\mathbb{R}^3$ is diffeomorphic to the sphere. Then, it bounds a ball.

(**4D smooth Schoenflies problem**) A closed hypersurface embedded in $\mathbb{R}^4$ is diffeomorphic to S^3. Does the hypersurface bound B^4?

The mean curvature flow changes the topology of the hypersurface at singularities, and the blow-up at each singularity subsequentially converges to a self-similarly shrinking solution (shrinker) with multiplicity.

However, there exist infinitely many shrinkers, and thus we can not see how the topology changes.

Stability

Consider a shrinker Σ and its Jacobi operator (linearied operator of nonlinear PDE)

$$L = \Delta_\Sigma - \frac{1}{2} X \cdot \nabla_\Sigma + \frac{1}{2} + |A_\Sigma|^2.$$

Then, L has unstable eigenfunctions, the mean curvature H and coordinate vectors $\hat{X}^i = \langle \hat{X}, E_i \rangle$. Indeed, we have $LH = H$ and $LX^i = \frac{1}{2} X^i$.

Colding-Minicozzi 12 introduced an entropy of the MCF

$$\mathrm{Ent}(t) = \sup_{Y \in \mathbb{R}^{n+1}, \lambda > 0} \int_{X(M,t)} \frac{1}{(4\pi)^{\frac{n}{2}} \lambda} e^{-\frac{|X-Y|^2}{4\lambda}} dg$$

and they showed that if only unstable eigenfunctions of the Jacobi operator are H and X^i, then the shrinker is the hyperplanes, round spheres, and the round cylinders. c.f. Huisken-Sinestrari 99, a shrinker with positive mean curvature must be a round sphere or a round cylinder.

(**Conjecture**) Given a closed smooth embedded hypersurface $X_0(M)$ and arbitrarily small $\epsilon > 0$, there exists an ϵ-graph $\hat{X}_0(M)$ over $X_0(M)$ such that the mean curvature flow $\hat{X}(M,t)$ (from $\hat{X}_0(M)$) develops finitely many singularities which are the round sphere or round cylinders.

(**Generic mean curvature flow**) $\hat{M}_t$ is called a generic mean curvature flow.

Expected steps

(**Multiplicity one conjecture**) A closed (generic) mean curvature flow must develop a multiplicity one singularity at the first singular time.

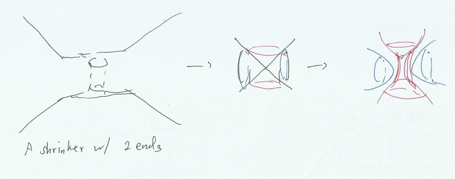
(**Well-posedness around stable singularities**) The mean curvature flow is well-posed in a neighborhood of multiplicity one spherical and cylindrical singularities.

(**Avoidance principle**) There exists a generic mean curvature flow which does not develops multiplicity one unstable singularities.

(**Generic isolation of stable singularities**) There exists a generic mean curvature flow which isolates stable singularities.

Well-posedness

(**Il-posed example**)



By Huisken-Sinestrari 99, the MCF with $H > 0$ has multiplicity one spherical or cylindrical singularities.
(**Mean convex neighborhood conjecture**) Suppose that a MCF has a multiplicity one spherical or cylindrical singularity at (x_0, t_0). Then, there exists a space-time neighborhood $B_r(x_0) \times (t_0 - r, t_0 + r)$ where the MCF satisfies $H > 0$.

(**Non-fattening**) Hershkovits-White 20' (arXiv 17') showed that if mean convex neighborhood conjecture is true then the Brakke flow (weak MCF) is well-posed in a space-time neighborhood of multiplicity one spherical or cylindrical singularities. In particular, if all limit flows at multiplicity one spherical or cylindrical singularities are convex, then the conjecture must be true.

Ancient flows asymptotic to round cylinders

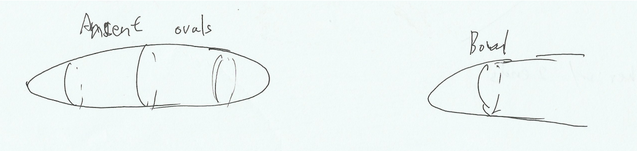
Colding-Minicozzi 15' showed the uniqueness of tangent flow at multiplicity one cylindrical singularities.

Angenent-Dskalopoulos-Sesum 19' and (arXiv) 18' showed that a closed two-convex non-collapsed ancient mean curvature flow must be a shrinking round sphere or an ancient oval.

Brendle-C 19' and (arXiv) 18' showed that a complete non-compact two-convex non-collapsed ancient mean curvature flow must be a bowl soliton.

See White 00' and Andrews 12' for the definition of non-collapsedness.

C-Haslhofer-Hershkovits (arXiv) 18' proved that (rescaled) ancient flows asymptotic to a round sphere or a round cylinder must be a shrinking sphere, a shrinking cylinder, an ancient oval, or a bowl soliton, which are all convex. Jointly with White (arXiv) 19', they extend the result for $S^{n-1} \times \mathbb{R}$.



Avoidance

L. Wang (arXiv 16') showed that each end of a non-compact complete shrinker in $\mathbb{R}^3$ must be asymptotically conical or cylindrical.

(**No cylinder conjecture**) A shrinker with a cylindrical end in $\mathbb{R}^3$ must be a round cylinder.

See L. Wang 14' for the uniqueness result by a conical asymptotic behavior.

(**Wang-Bernstein**) A low entropy MCF is unknotted. See 16, 17a-b, 18a-d, 19a-b, 20.

C-Chodosh-Mantoulidis-Schulze 20' provide an alternative proof by using one-side ancient flows.

Idea: Rescaled ancient flows located in one-side of an asymptotically conical or compact shrinkers is unique and its speed is positive. Hence, by HW20 the one-sided ancient flow only develops multiplicity one spherical or cylindrical singularities.

On the other hand, generic MCF not touching the given data initially converge to the one-sided ancient flow by blow-ups due to the maximum principle. Hence, we can choose a generic mean curvature flow avoiding multiplicity one conical and compact singularities.

Merle-Zaag's ODE dynamic

A rescaled ancient MCF asymptotic to a shrinker Σ can be (locally) considered as the graph of a function $u : \Sigma \times (-\infty, T] \to \mathbb{R}$ such that

$$u_\tau = Lu + E,$$

where L is the Jacobi operator of Σ and E is a quadratic error term. Moreover, u locally converges 0 in C^k-sense.

Hence, u behaves like a solution to the linear equation $u_\tau = Lu$, namely

$$u \approx \sum_i c_i e^{-\lambda_i \tau} \varphi_i$$

where (φ_i, λ_i) are eigenpairs. To converges to zero, $c_i = 0$ for stable eigenfunctions φ_i. By using MZ 98' (cf ADS 19'), we can show that there exists a certain eigenpair such that

$$u \approx c_i e^{-\lambda_i \tau} \varphi_i + Error.$$

In particular, if $\lambda_i = 0$ then we replace $e^{0\tau} = 1$ by $1/|\tau|^{p_i}$. In addition, if $i = 1$ then $\varphi_1 > 0$ implies $u_\tau > 0$ which is a key idea of CCMS20.

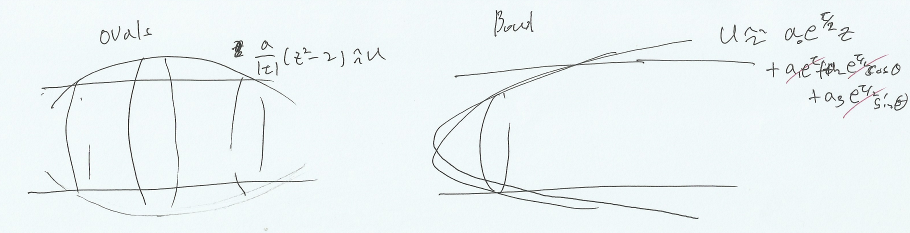
Eigenfunctions on round cylinder

Let $\Sigma = \sqrt{-2}\, S^1 \times \mathbb{R} = \{x_1^2 + x_2^2 = 2\} \subset \mathbb{R}^3$, the round cylindrical shrinker. Then,

$$L = \frac{\partial^2}{\partial z^2} + \frac{1}{2}\frac{\partial^2}{\partial \theta^2} - \frac{z}{2}\frac{\partial}{\partial z} + 1.$$

In addition, $\varphi_1 = 1/\sqrt{2} = H$, $\varphi_2 = \sqrt{2}\cos\theta = x_1$, $\varphi_3 = \sqrt{2}\sin\theta = x_2$, $\varphi_4 = z = x_3$ are the only unstable eigenfunctions and $\varphi_5 = z^2 - 2$, $\varphi_6 = z\cos\theta$, $\varphi_7 = z\sin\theta$ are the only Jacobi fields.

If $u \approx \frac{a}{|\tau|}(2 - z^2)$ then the ancient flow must be an ancient oval. Otherwise, u must be a bowl or a round cylinder.



Thank you