Maximal estimates for the Schrödinger equation with orthonormal initial data

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Asia-Pacific Analysis and PDE Seminar

Plan

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2. Pointwise convergence for infinitely many particles (Main results).
3. Endpoint 1d-Strichartz estimate for the orthonormal initial data (Almost sharp answer to works by Frank-Lewin-Lieb-Seiringer and Frank-Sabin).

Classical pointwise convergence convegence

Consider the free Schrödinger equation:

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\begin{aligned}
&i\partial_t u(t,x) + \partial_x^2 u(t,x) = 0, \quad (t,x) \in \mathbb{R}^{1+1}, \\
&u(0,x) = f(x),
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whose solution is explicitly given by

$$e^{it\partial_x^2} f(x) = C \int_{\mathbb{R}} e^{i(x\cdot\xi + t|\xi|^2)} \hat{f}(\xi)\, d\xi \quad (x \in \mathbb{R}).$$

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holds true for all f? Or equivalently,

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$\rightsquigarrow$ Mathematically, this is a problem if we can exchange the order:

$$\lim_{t\to 0} \int_{\mathbb{R}} \, d\xi \stackrel{??}{=} \int_{\mathbb{R}} \lim_{t\to 0} \, d\xi.$$

Few observations

Question: pointwise convergence

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$\rightsquigarrow$ Identify the smallest $s \geq 0$ for which (1) holds for all $f \in H^s(\mathbb{R})$. (Small $s \geq 0$ means large class of initial data).

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Answer $\rightsquigarrow$ Carleson and Dahlberg-Kenig: the pointwise convergence (1):

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holds for all $f \in H^s(\mathbb{R})$ if and only if $s \geq \frac{1}{4}$.

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- Pointwise convergence $\lim_{t\to 0} u(t,x) \to f(x)$, u: solution to the NLS due to Compaan-Lucá-Staffilani.

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$\rightsquigarrow$ Analyze the system $(e^{it\partial_x^2}f_j(x))_j$ as $t \to 0$.

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Dynamics of the system of N-many Fermions interacting with each other by a potential $w : \mathbb{R} \to \mathbb{C}$:

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As $N \to \infty$, operator valued equation:

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$\rightsquigarrow$ Pointwise convergence of $\gamma(t) \to \gamma_0 \quad (t \to 0)$ in appropriate sense??

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$\rightsquigarrow$ Formulate the problem by using the density function:

$$\lim_{t\to 0} \rho_\gamma(t,x) = \rho_{\gamma_0}(x) \quad \text{a.e. } x \in \mathbb{R}.$$

Compare to the original Carleson's problem

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which is (up to signature) the original Carleson's problem.

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$$\lim_{t\to 0} |e^{it\partial_x^2} f(x)| = |f(x)| \quad \text{a.e. } x \in \mathbb{R}$$

which is (up to signature) the original Carleson's problem.

$\rightsquigarrow$ If $\gamma_0 = |f\rangle\langle f|$, $f \in H^{\frac{1}{4}}(\mathbb{R})$, then (3) follows from the classical result.

Precise formulation

Problem: Identify the largest class of γ_0 for which

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- If one can prove (4) for all $\gamma_0 = \sum_j \nu_j |f_j\rangle\langle f_j|$ with $\nu \in \ell^\beta$ and for some $\beta > 1$, then this is an improvement of the classical result:

$$\forall f \in H^{\frac{1}{4}}(\mathbb{R}), \quad \lim_{t\to 0} |e^{it\partial_x^2} f(x)| = |f(x)| \quad \text{a.e. } x \in \mathbb{R}$$

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Problem 1

Identify the largest $\beta \geq 1$ for which

$$\lim_{t \to 0} \rho_\gamma(t,x) = \rho_{\gamma_0}(x) \quad \text{a.e. } x \in \mathbb{R} \tag{5}$$

holds for all $\gamma_0 \in \mathcal{C}^\beta(H^{\frac{1}{4}})$.

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- Improvement is up to near $\mathcal{C}^2(H^{\frac{1}{4}})$.
- No idea if $\beta < 2$ is sharp for the pointwise convergence (5) or not.

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For instance, Kenig-Ponce-Vega proved a bit stronger estimate:

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$\rightsquigarrow \beta > 1$ means non-trivial (and improved) estimate.

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Theorem 3 (Bez-Lee-N)

For all families of the orthonormal functions $(f_j)_j$ in $H^{\frac{1}{4}}(\mathbb{R})$ and coefficients $\nu = (\nu_j)_j$,

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- This gives the pointwise convergence result for infinitely many particles.

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Note: Applying their idea to our setting is not straightforward and need an extra twist.

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for all families of orthonormal functions $(f_j)_j$ in $L^2(\mathbb{R})$ and coefficient $(\nu_j)_j$ and for some $\beta \geq 1$.

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- The case $\beta = 1$ is equivalent to the classical Strichartz estimate and (9) with $\beta > 1$ means an improvement.

General result

General form: Let (q, r) be the admissible pair i.e. $\frac{2}{q} = \frac{1}{2} - \frac{1}{r}$, $q, r \geq 2$.

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holds true?

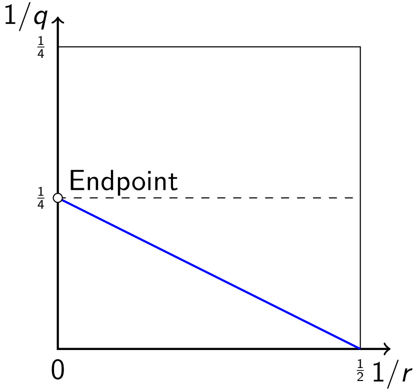
Theorem 4 (Frank-Lewin-Lieb-Seiringer (2014), Frank-Sabin(2015))

Suppose

$$\frac{2}{q}=\frac{1}{2}-\frac{1}{r},\quad 2\leq r<\infty.$$

Then (10) *holds with* $\beta=\frac{2r}{r+2}$ *and this is sharp.*

Picture...



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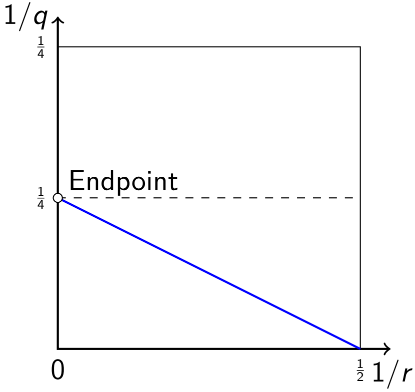

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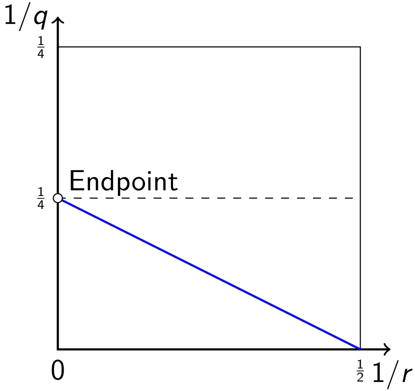

$\rightsquigarrow$ On the endpoint $(q, r) = (4, \infty)$, the problem to find the sharp β is open. Frank-Sabin's argument does NOT work at $(q, r) = (4, \infty)$.

Third result (Positive answer up to Lorentz exponent)

Problem: Find the sharp $\beta \geq 1$ for

$$\Big\| \Big(\sum_j \nu_j |e^{it\partial_x^2} f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L_t^4 L_x^\infty(\mathbb{R}^{1+1})} \lesssim \|\nu\|_{\ell^\beta}^{\frac{1}{2}} \tag{11}$$

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Theorem 5 (Bez-Lee-N)

For all families of o/n functions $(f_j)_j$ in $L^2(\mathbb{R})$ and coefficients $(\nu_j)_j$,

$$\Big\| \Big(\sum_j \nu_j |e^{it\partial_x^2} f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L_t^{4,\infty} L_x^\infty(\mathbb{R}^{1+1})} \lesssim \|\nu\|_{\ell^\beta}^{\frac{1}{2}} \tag{12}$$

holds as long as $\beta < 2$ and this is sharp.

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$\rightsquigarrow$ Using Kenig-Ponce-Vega's idea + α, we obtain Theorem 1 (maximal in time estimate).

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- Schatten-2 class is the Hilbert-Schmit class. In particular, if the integral kernel of A is $K(x,y)$, then

$$\|A\|_{\mathcal{C}^2}^2 = \int_{\mathbb{R}\times\mathbb{R}} |K(x,y)|^2 \, dxdy.$$

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For L.H.S,

$$\big\| WUU^* \overline{W} \big\|^4_{\mathcal{C}^4} = \big\| WUU^* |W|^2 UU^* \overline{W} \big\|^2_{\mathcal{C}^2}.$$

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where K_0 is the integral kernel of UU^*:

$$K_0(t-t', x-x') = \int_{\mathbb{R}} e^{i\left[(x-x')\xi+(t-t')\xi^2\right]}\, d\xi = O(|t-t'|^{-\frac{1}{2}}).$$

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the estimate are reduced to the multilinear fractional integral:

$$\int_{\mathbb{R}^4} |t_1 - t_2|^{-\frac{1}{2}} |t_2 - t_4|^{-\frac{1}{2}} |t_1 - t_3|^{-\frac{1}{2}} |t_3 - t_4|^{-\frac{1}{2}} \prod_{i=1}^{4} w(t_i) \, d\vec{t},$$

where $w(t_i) = \|W(t_i, \cdot)\|_{L^2_x}^2$.

Further details (Multilinear fractional integral)

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Then (14) becomes

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$$\begin{aligned} UU^*[F](t,x) &= \int_{\mathbb{R}^{1+1}} K_0(t-t',x-x')F(t',x')\,dt'dx' \\ &= \sum_{j\in\mathbb{N}_0} \int_{|t-t'|\sim 2^j} K_0(t-t',x-x')F(t',x')\,dt'dx' \\ &=: \sum_{j\in\mathbb{N}_0} T_j[F](t,x) \end{aligned}$$

and employ the bilinear real interpolation argument.

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In the proof, We simply use the triangle inequality for Schatten norm:

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- Higher dimension problem would be challenging.

Thank you for your attention.