An extension problem for the logarithmic Laplacian

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Asia-Pacific Analysis and PDE Seminar

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Joint work with

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- Tobias Weth (Goethe-Universität Frankfurt, Germany)

THE FRACTIONAL LAPLACIAN.

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For $0 < s < 1$, and $u \in C_c^2(\mathbb{R}^N)$, the *fractional Laplacian* can be defined as the singular integral operator

$$(-\Delta)^s u(x) = c_{N,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,$$

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for every $x \in \mathbb{R}^N$ where $c_{N,s}$ is a normalized constant, see for example [Ann. Inst. H. Poincaré-AN (2014)] by Cabré & Sire,

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which makes

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \hat{u}(\xi) \qquad \text{for every } \xi \in \mathbb{R}^N.$$

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ASYMPTOTIC PROPERTIES.

Let $u \in C_c^2(\mathbb{R}^N)$. Then, one has that

$$\lim_{s\to 0^+} (-\Delta)^s u(x) = u(x) \quad \text{and} \quad \lim_{s\to 1^-} (-\Delta)^s u(x) = -\Delta u(x)$$

for every $x \in \mathbb{R}^N$.

The logarithmic Laplacian.

THE LOGARITHMIC LAPLACIAN.

Chen & Weth [Comm. Part. Diff. Eq. (2019)] introduced the

logarithmic Laplacian L_Δ

such that

$$(-\Delta)^s u(x) = u(x) + s\, L_\Delta u(x) + o(s) \qquad \text{as } s \to 0^+$$

for $u \in C_c^2(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$.

THE LOGARITHMIC LAPLACIAN.

A.1 THEOREM [CHEN & WETH'2019].

Let $u \in C_c^{\alpha}(\mathbb{R}^N)$, $\alpha > 0$. Then, one has that

1. $\frac{\mathrm{d}}{\mathrm{d}s}(-\Delta)^s u(x)_{|_{s=0}} = L_{\Delta} u(x)$,

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1. $\frac{\mathrm{d}}{\mathrm{d}s}(-\Delta)^s u(x)_{|s=0} = L_{\Delta} u(x)$,
2. $L_{\Delta} u(x) = c_N \,\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)\mathbb{1}_{B_1(x)} - u(y)}{|x-y|^N}\mathrm{d}y + \rho_N\, u(x)$,

   where

$$c_N = \frac{\Gamma(\frac{N}{2})}{\pi^{N/2}} = \frac{2}{\omega_N}, \qquad \rho_N := 2\ln(2) + \psi(\frac{N}{2}) - \gamma,$$

   $\gamma = -\Gamma'(1)$ is the Euler Mascheroni constant, and $\psi = \Gamma'/\Gamma$ is the Digamma function.

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3. $\mathcal{F}(L_{\Delta} u)(\xi) = 2\ln|\xi|\hat{u}(\xi)$ for every $\xi \in \mathbb{R}^N$.

THE LOGARITHMIC LAPLACIAN.

APPLICATIONS.

1. Determing the asymptotics as $s \to 0^+$ of the Dirichlet eigenvalues and eigenfunctions of $(-\Delta)^s$ [J. Fourier Anal. Appl.(2022)] by Feulefack, Jarohs, and Weth;

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2. In the geometric context of the 0-fractional perimeter, see [Ann. Scuola Norm-SCI (2021)] by De Luca, Novaga, and Ponsiglione.

INDEPENDENT DISCOVERY.

In the study of classifying all finite energy solutions of an equation arising from the Euler-Lagrange equation of a conformally invariant logarithmic Sobolev inequality, Rupert, T. König & Tang [Adv. in Math., 2020] also arrived to the logarithmic Laplacian.

The fractional Laplacian.

THE FRACTIONAL LAPLACIAN.

A.2 THEOREM [CAFFARELLI & SILVESTRE, COMM. PDE (2007)].

Let $0 < s < 1$ and $u \in C_c^\infty(\mathbb{R}^N)$. Then, there is an s-*harmonic extension* $w_s : \mathbb{R}_+^{N+1} \to \mathbb{R}$ of u;

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(1) $$\begin{cases} -\operatorname{div}\left(t^{1-2s}\nabla w_s\right) = 0 & \text{in} \quad \mathbb{R}_+^{N+1}, \\ w_s = u & \text{on} \quad \mathbb{R}^N = \partial\mathbb{R}_+^{N+1}, \end{cases}$$

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and

(2)
$$(-\Delta)^s u = -d_s \lim_{t\to 0^+} t^{1-2s}\partial_t w_s$$

with constant $d_s = 2^{2s-1}\dfrac{\Gamma(s)}{\Gamma(1-s)}$.

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$$C_c^\infty(\mathbb{R}^N) \ni u \mapsto \Lambda_s u := -\lim_{t\to 0^+} t^{1-2s}\partial_t w_s.$$

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Poisson kernel representation (Caffarelli & Silvestre [Comm. PDE'07], Cabré & Sire [Ann. Inst. H. Poin.'14].

Let $0 < s < 1$ and $u \in C_c(\mathbb{R}^N)$. Then, the weak solution w_s of the exentions problem (1) admits the representation

$$w_s(x,t) = p_{N,s}\, t^{2s} \int_{\mathbb{R}^N} (|x-\tilde{x}|^2 + t^2)^{-\frac{N+2s}{2}} u(\tilde{x})\, \mathrm{d}\tilde{x}$$

for every $(x,t) \in \mathbb{R}^{N+1}_+$, where the constant $p_{N,s}$ is given by

$$p_{N,s} = \pi^{-\frac{N}{s}}\, s\, \frac{\Gamma(\frac{N}{2}+s)}{\Gamma(1+s)}.$$

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for every $(x,t) \in \mathbb{R}^{N+1}$ with $t \neq 0$, one obtains that w_s is a weak solution of

$$-\operatorname{div}\left(|t|^{1-2s}\nabla w_s\right) = 0 \qquad \text{on } \mathbb{R}^N \times \{t\,:\,|t|>0\}.$$

Let $u \in C_c^2(\mathbb{R}^N)$. Since

$$v_s(x,t) := 2\,\frac{w_s(x,t) - (1 - |t|^{2s})\,u(x)}{s\,|t|^{2s}} = \mathcal{O}(1) \qquad \text{as } s \to 0^+$$

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we make the following

Asymptotic Ansatz.

$$w_s(x,t) = (1-|t|^{2s})\,u(x) + \frac{s\,|t|^{2s}}{2}\,v_s(x,t)$$

for ever $(x,t) \in \mathbb{R}^{N+1}$ with $t \neq 0$ and every small enough $s > 0$.

Recall that $(-\Delta)^s = d_s\, \Lambda_s$,

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Recall that $(-\Delta)^s = d_s\,\Lambda_s$, i.e.,

$$(-\Delta)^s u(x) = -d_s \lim_{t\to 0^+} t^{1-2s}\partial_t w_s(x,t).$$

By the asymptotic Ansatz $(0 < s << 1)$, for $t > 0$,

$$w_s(x,t) = (1-t^{2s})\,u(x) + \frac{s\,t^{2s}}{2}\,v_s(x,t),$$

one has that

$$\partial_t w_s(x,t) = -2\,s\,t^{2s-1}\,u(x) + s^2\,t^{2s-1}\,v_s(x,t) + \frac{s\,t^{2s}}{2}\,\partial_t v_s(x,t)$$

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and so,

$$t^{1-2s}\partial_t w_s(x,t) = -2\,s\,u(x) + s^2\,v_s(x,t) + \frac{s\,t}{2}\,\partial_t v_s(x,t).$$

Therefore for $0 < s << 1$,

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we get for for $0 < s << 1$,

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From this,

$$\frac{(-\Delta)^s u(x) - u(x)}{s} = \frac{2\,s\,d_s - 1}{s}\,u(x) - d_s\,s\,v_s(x,0)$$

for every $x \in \mathbb{R}^N$ and every $0 < s << 1$.

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where $\gamma = -\Gamma'(1)$ is the Euler Mascheroni constant.

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Further, we make the following assumption.

Assumption.

There exist $v_0 \in C(\mathbb{R}^{N+1})$ such that

$$\lim_{s\to 0^+} \sup_{(x,t)\in B} |v_s(x,t) - v_0(x,t)| = 0 \qquad \text{for every } B \Subset \mathbb{R}^{N+1}.$$

Then, sending $s \to 0^+$ in

$$\frac{(-\Delta)^s u(x) - u(x)}{s} = \frac{2\, s d_s - 1}{s}\, u(x) - d_s\, s\, v_s(x,0),$$

Then, sending $s \to 0^+$ in

$$\frac{(-\Delta)^s u(x) - u(x)}{s} = \frac{2\,sd_s - 1}{s}\,u(x) - d_s\,s\,v_s(x,0),$$

one obtains

(3) $$L_{\Delta}\,u(x) = \lim_{s\to 0^+} \frac{(-\Delta)^s u(x) - u(x)}{s} = 2\,(\ln 2 - \gamma)\,u(x) - \frac{1}{2}\,v_0(x,0).$$

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Aim.

We need to identify v_0!

LEMMA 1.

Let $0 < s < \frac{1}{2}$ and $u \in C^2_c(\mathbb{R}^N)$. Then the weak solution $w_s : \mathbb{R}^{N+1} \to \mathbb{R}$ of the extension problem (1) satisfies

$$(4) \qquad -\operatorname{div}\left(|t|^{1-2s}\nabla w_s\right) = \frac{2}{d_s}\left[(-\Delta)^s u\right] \mathcal{L}^N \times \delta_{\{0\}} \quad \text{in } \mathbb{R}^{N+1}$$

in the distributional sense,

LEMMA 1.

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in the distributional sense, i.e.,

$$\int_{\mathbb{R}^{N+1}} w_s\Big(-\operatorname{div}\left(|t|^{1-2s}\nabla\varphi\right)\Big)\,\mathrm{d}(x,t) = \frac{2}{d_s}\int_{\mathbb{R}^N}[(-\Delta)^s u](x)\varphi(x,0)\,\mathrm{d}x$$

for every $\varphi \in C_c^\infty(\mathbb{R}^{N+1})$.

LEMMA 2.

Let $0 < s < \frac{1}{2}$ and $u \in C_c^2(\mathbb{R}^N)$. Then the function $v_s : \mathbb{R}^{N+1} \to \mathbb{R}$ given by (for $0 < s << 1$)

$$v_s(x,t) := 2\,\frac{w_s(x,t) - (1 - |t|^{2s})\,u(x)}{s\,|t|^{2s}}$$

is a distributional solution in $\mathbb{R}^{N+1}$ of

$$-\operatorname{div}\left(|t|\nabla v_s\right) = 2\,s\frac{t}{|t|}\partial_t v_s + 2(sv_s - 2u)\,\mathcal{L}^N \times \delta_{\{0\}} + \frac{4}{s\,d_s}(-\Delta)^s u\,\mathcal{L}^N \times \delta_{\{0\}} + 2\,t|\frac{|t|^{-2s} - 1}{s}\Delta_x u.$$

This means that

$$\begin{aligned}
&-\frac{4}{s\,d_s}\int_{\mathbb{R}^N}(-\Delta)^s u(x)\varphi(x,0)\,\mathrm{d}x\\
&\quad= 2\int_{\mathbb{R}^{N+1}}|t|\frac{|t|^{-2s}-1}{s}\big(\Delta_x u(x)\big)\varphi(x,t)\,\mathrm{d}(x,t) - 8\int_{\mathbb{R}^N}u(x)\varphi(x,0)\,\mathrm{d}x\\
&\qquad+\int_{\mathbb{R}^{N+1}}v_s\,\mathrm{div}\,(|t|\nabla\varphi)\,\mathrm{d}(x,t) + 4s\int_{\mathbb{R}^N}v_s(x,0)\varphi(x,0)\,\mathrm{d}x\\
&\qquad\quad+2s\int_{\mathbb{R}^{N+1}}\partial_t v_s\frac{t}{|t|}\varphi\,\mathrm{d}(x,t)
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&\qquad+\int_{\mathbb{R}^{N+1}}v_s\,\mathrm{div}\,(|t|\nabla\varphi)\,\mathrm{d}(x,t)+4s\int_{\mathbb{R}^N}v_s(x,0)\varphi(x,0)\,\mathrm{d}x\\
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for every $\varphi\in C_c^\infty(\mathbb{R}^{N+1})$.

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\end{aligned}
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for every $\varphi \in C_c^\infty(\mathbb{R}^{N+1})$. Since

$$
\lim_{s\to 0^+}(-\Delta)^s u(x) = u(x) \qquad \text{on } \mathbb{R}^d
$$

This means that

$$-\frac{4}{s\,d_s}\int_{\mathbb{R}^N}(-\Delta)^s u(x)\varphi(x,0)\,\mathrm{d}x$$

$$= 2\int_{\mathbb{R}^{N+1}}|t|\frac{|t|^{-2s}-1}{s}\big(\Delta_x u(x)\big)\varphi(x,t)\,\mathrm{d}(x,t) - 8\int_{\mathbb{R}^N}u(x)\varphi(x,0)\,\mathrm{d}x$$

$$+\int_{\mathbb{R}^{N+1}}v_s\,\mathrm{div}\,(|t|\nabla\varphi)\,\mathrm{d}(x,t) + 4s\int_{\mathbb{R}^N}v_s(x,0)\varphi(x,0)\,\mathrm{d}x$$

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for every $\varphi\in C_c^\infty(\mathbb{R}^{N+1})$. Since

$$\lim_{s\to 0^+}(-\Delta)^s u(x) = u(x) \qquad \text{on } \mathbb{R}^d$$

and by the assumption

$$\lim_{s\to 0^+} v_s = v_0 \qquad \text{in } L^\infty_{loc}(\mathbb{R}^{N+1}),$$

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\begin{aligned}
-8\int_{\mathbb{R}^N} u(x)\varphi(x,0)\,\mathrm{d}x = 2\int_{\mathbb{R}^{N+1}} |t|(-2\ln|t|)\Delta_x u(x)\,\varphi(x,t)\,\mathrm{d}(x,t)\\
-8\int_{\mathbb{R}^N} u(x)\varphi(x,0)\,\mathrm{d}x\\
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for every $\varphi \in C_c^\infty(\mathbb{R}^{N+1})$.

LEMMA 3.

Let $u \in C_c^2(\mathbb{R}^N)$ and suppose that there is a $v_0 \in C(\mathbb{R}^{N+1})$ such that

$$\lim_{s \to 0^+} v_s = v_0 \qquad \text{in } L^\infty_{loc}(\mathbb{R}^{N+1}).$$

Then, v_0 is a distributional solution of

$$-\operatorname{div}\left(|t| \nabla v_0\right) = -4\,|t|\,\ln|t|\,\Delta_x u \qquad \text{in } \mathbb{R}^{N+1}.$$

LEMMA 4.

Let $u \in C_c^2(\mathbb{R}^N)$ and suppose that there is a $v_0 \in C(\mathbb{R}^{N+1})$ such that

$$\lim_{s\to 0^+} v_s = v_0 \qquad \text{in } L^\infty_{loc}(\mathbb{R}^{N+1}).$$

Then, w_u given by

$$w_u(x,t) := \frac{1}{4} v_0(x,t) - u(x) \ln|t|$$

for every $(x,t) \in \mathbb{R}^{N+1}$ with $t \neq 0$, is a distributional solution of

(5) $$-\operatorname{div}\left(|t|\nabla w_u\right) = 2\, u\, \mathcal{L}^N \times \delta_{\{0\}} \qquad \text{in } \mathbb{R}^{N+1}.$$

Proof of Lemma 4.

Note,

$$-\operatorname{div}(|t|\nabla w_u) = -\frac{1}{4}\operatorname{div}(|t|\nabla v_0) + \operatorname{div}(|t|\nabla(u\ln|t|))$$

Proof of Lemma 4.

Note,

$$-\operatorname{div}(|t|\nabla w_u) = -\frac{1}{4}\operatorname{div}(|t|\nabla v_0) + \operatorname{div}(|t|\nabla(u\ln|t|))$$

and $v = \ln|t|$ is a distributional solution of

$$\partial_t(|t|\partial_t \ln|t|) = 2\,\delta_{\{0\}} \qquad \text{in } \mathbb{R}.$$

Thus and by Lemma 3.,

$$\begin{aligned} -\operatorname{div}(|t|\nabla w_u) &= -|t|\big(\ln|t|\big)\Delta_x u + |t|\big(\ln|t|\big)\Delta_x u + u\,\partial_t(|t|\partial_t \ln|t|) \\ &= u\,2\,\mathcal{L}^N \times \delta_{\{0\}} \end{aligned}$$

in the distributional sense in $\mathbb{R}^{N+1}$.

Due to Lemma 4, we have that

$$w_u(x,t) := \frac{1}{4}v_0(x,t) - u(x)\ \ln|t|$$

is a distributional solution of

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Due to Lemma 4, we have that

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(5) $$-\operatorname{div}\left(|t|\nabla w_u\right) = 2\,u\,\mathcal{L}^N \times \delta_{\{0\}} \qquad \text{in } \mathbb{R}^{N+1}.$$

Note, (5) means that

w_u is a distributional solution of the *inhomogeneous Neumann problem*

$$\begin{cases} -\operatorname{div}\left(t\nabla w_u\right) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0^+} t\partial_t w_u = u & \text{on } \mathbb{R}^N. \end{cases}$$

Further,

$$\frac{1}{2}v_0(x,0) = 2\lim_{t\to 0}\Big(w_u(x,t) + u(x)\,\ln|t|\Big)$$

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So, we get for $u \in C_c^2(\mathbb{R}^N)$ that

$$L_{\Delta}u(x) = 2\,(\ln 2 - \gamma)\,u(x) - 2\lim_{t\to 0}\Big(w_u(x,t) + u(x)\,\ln|t|\Big).$$

Therefore, we have formally justified that the logarithmic Laplacian L_Δ admits the following extension property.

Therefore, we have formally justified that the logarithmic Laplacian L_{Δ} admits the following extension property.

One has that

$$L_{\Delta} = 2\,(\ln 2 - \gamma)\,\mathrm{id} - 2\,\Lambda_0^{\mathrm{Ex},-1}$$

where $\Lambda_0^{\mathrm{Ex},-1}$ is the *Neumann-to-Dirichlet map with an excess term* associated with $-\operatorname{div}(t\nabla\cdot)$ given by

$$u \mapsto \Lambda_0^{\mathrm{Ex},-1}u := \lim_{t\to 0}\Big(w_u(x,t) + u(x)\,\ln|t|\Big).$$

If one inserts the Poisson kernel representation

$$w_s(x,t) = p_{N,s}\, t^{2s} \int_{\mathbb{R}^N} (|x-\tilde{x}|^2 + t^2)^{-\frac{N+2s}{2}} u(\tilde{x})\, \mathrm{d}\tilde{x}$$

of the weak solution w_s of the exentions problem (1) for $(-\Delta)^s$ into

$$\tfrac{1}{2} v_s(x,t) = \frac{w_s(x,t) - (1 - |t|^{2s})\, u(x)}{s\, |t|^{2s}}$$

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and subsequently, sends $s \to 0^+$, then one finds

$$\tfrac{1}{2} v_0(x,t) = c_N \int_{\mathbb{R}^N} (|x-\tilde{x}|^2 + |t|^2)^{-\frac{N}{2}} u(\tilde{x})\, d\tilde{x} + 2\,\ln|t|\, u(x)$$

for every $(x,t) \in \mathbb{R}^{N+1}$ with $t \neq 0$, and $c_N = 2/\omega_N$.

If one inserts the Poisson kernel representation

$$w_s(x,t) = p_{N,s}\, t^{2s} \int_{\mathbb{R}^d} (|x-\tilde{x}|^2 + t^2)^{-\frac{N+2s}{2}} u(\tilde{x})\,\mathrm{d}\tilde{x}$$

of the weak solution w_s of the exentions problem (1) into

$$\tfrac{1}{2} v_s(x,t) = \frac{w_s(x,t) - (1-|t|^{2s})\,u(x)}{s\,|t|^{2s}}$$

and subsequently, sends $s \to 0^+$, then one finds

$$\tfrac{1}{2} v_0(x,t) = \underbrace{c_N \int_{\mathbb{R}^d} (|x-\tilde{x}|^2 + |t|^2)^{-\frac{N}{2}} u(\tilde{x})\, d\tilde{x}}_{=2\,w_u(x,t)} + 2\,\ln|t|\,u(x)$$

for every $(x,t) \in \mathbb{R}^{N+1}$ with $t \neq 0$, and $c_N = 2/\omega_N$.

Consider the space

$$L^1_0(\mathbb{R}^N) := L^1(\mathbb{R}^N, \mathrm{d}x/(1+|x|)^N) := \Big\{ u \in L^1_{loc}(\mathbb{R}^N) \,\Big|\, \int_{\mathbb{R}^N} \frac{|u(x)|\mathrm{d}x}{(1+|x|)^N} < \infty \Big\}.$$

Theorem 1 [Chen, H., Weth'23].

For every $u \in L^1_0(\mathbb{R}^N)$, there is a unique distributional solution

$$w_u \in L^1_{loc}(\mathbb{R}^{N+1}) \cap C^\infty(\mathbb{R}^N \times (\mathbb{R} \setminus \{0\}))$$

of the Poisson problem

$$-\operatorname{div}(|t|\,\nabla w_u) = 2\,u\,\mathcal{L}^N \otimes \delta_0 \quad \text{in } \mathbb{R}^{N+1} \tag{6}$$

satisfying

$$\lim_{|t|\to\infty} w_u(x,t) = 0 \qquad \text{for every } x \in \mathbb{R}^N,$$

where $\mathcal{L}^N$ denotes the Lebesgue-measure on $\mathbb{R}^N$ and δ_0 the Dirac-measure on $\mathbb{R}$ at $t = 0$.

THEOREM 1 [CHEN, H., WETH'23] (CONT.).

In particular, for $u \in L^1_0(\mathbb{R}^N)$, the following statements hold.

1. w_u can be represented via the Poisson formula

$$w_u(x,t) = \frac{c_N}{2} \int_{\mathbb{R}^N} \frac{u(\tilde{x})}{(|x-\tilde{x}|^2 + |t|^2)^{N/2}}\, d\tilde{x}$$

for every $(x,t) \in \mathbb{R}^N \times (\mathbb{R} \setminus \{0\})$, and w_u satisfies

$$L_\Delta u = 2(\ln 2 - \gamma)u - 2 \lim_{t\to 0}\Big(w_u + u \log|t|\Big)$$

in the distributional sense in $\mathbb{R}^N$, and

$$\lim_{|t|\to 0^+} \frac{w_u(x,t)}{\ln|t|} = -u(x) \quad \text{in } L^1_{loc}(\mathbb{R}^N).$$

THEOREM 1 [CHEN, H., WETH'23] (CONT.).

Further, the following statement holds.

2. If $u \in L^1_0(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, then w_u satisfies the Neumann boundary condition
$$-\lim_{t\to 0+} t\partial_t w_u(\cdot,t) = u \qquad \text{in } L^1_{loc}(\mathbb{R}^N),$$
and, in particular, w_u is a distributional solution of the Neumann problem on the half-space $\mathbb{R}^{N+1}_+$,
$$\begin{cases} -\operatorname{div}(t\nabla w_u) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0+} t\partial_t w_u(\cdot,t) = u & \text{on } \mathbb{R}^N. \end{cases}$$

Theorem 1 [Chen, H., Weth'23] (cont.).

Further, the following statement holds.

3. If $u \in L^1_0(\mathbb{R}^N)$ and Dini continuous at $x \in \mathbb{R}^N$, then

$$L_\Delta u(x) = 2\,(\ln 2 - \gamma)\,u(x) - 2\left(w_u(x,t) + \ln|t|\,u(x)\right)(1 + o(1))$$

(in the strong sense) in $\mathbb{R}$ as $|t| \to 0^+$, where $o(1) \to 0$ as $|t| \to 0^+$.

Remark.

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- By definition, the distributional limit

$$L_{\Delta} u = 2(\ln 2 - \gamma)u - 2\lim_{t\to 0}\Big(w_u + u\log|t|\Big)$$

means that

$$\int_{\mathbb{R}^N} u L_{\Delta}\phi dx = 2(\ln 2 - \gamma)\int_{\mathbb{R}^N} u\phi\, dx - 2\lim_{t\to 0^+}\int_{\mathbb{R}^N}\Big(w_u(x,t) + u\ln t\Big)\phi(x)\, dx$$

for all $\phi \in C_c^{\infty}(\mathbb{R}^N)$.

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for all $\phi \in C_c^{\infty}(\mathbb{R}^N)$.

- In fact, we shall show that this property already holds if $\phi \in C_c^D(\mathbb{R}^N)$, where $C_c^D(\mathbb{R}^N)$ denotes the space of uniformly Dini continous functions on $\mathbb{R}^N$ with compact support.

Remark.

- A direct consequence of this property is an alternative representation of the energy

$$\phi \mapsto \mathcal{E}_L(\phi,\phi) = \frac{c_N}{2}\int_{|x-\tilde{x}|<1} \frac{(\phi(x)-\phi(\tilde{x}))^2}{|x-\tilde{x}|^N} dx d\tilde{x} - \frac{c_N}{2}\int_{|x-\tilde{x}|\geq 1} \frac{\phi(x)\phi(\tilde{x})}{|x-\tilde{x}|^N}\, dx d\tilde{x} + \frac{\rho_N}{2}\int_{\mathbb{R}^N} \phi(x)^2\, dx.$$

  associated with L_Δ, which has been introduced by Chen & Weth [Comm. Part. Diff. Eq. (2019)].

Corollary 1 [Chen, H., Weth'23].

For every $\phi \in C_c^D(\mathbb{R}^N)$, one has that

$$\begin{aligned}\mathcal{E}_L(\phi,\phi) &= \int_{\mathbb{R}^N} \phi\, L_{\Delta} \phi\, dx \\ &= 2(\ln 2 - \gamma)\|\phi\|^2_{L^2(\mathbb{R}^N)} \\ &\qquad - 2 \lim_{t\to 0^+} \int_{\mathbb{R}^N} \Big(\phi(x) w_\phi(x,t) + \phi^2(x) \ln t \Big)\, dx.\end{aligned}$$

Theorem 2 [Chen, H., Weth'23].

For $u \in L^1_0(\mathbb{R}^N)$, let w_u be a solution of

$$-\operatorname{div}\left(|t|\nabla w_u\right) = 2\,u\,\mathcal{L}^N \otimes \delta_{\{0\}} \qquad \text{in } \mathcal{D}'(\mathbb{R}^{N+1}). \tag{5}$$

Theorem 2 [Chen, H., Weth'23].

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If one considers $w_u(x,|y|)$ as a function on $\mathbb{R}^{N+2}$, then

(7) $$-\Delta w_u = 2\pi\, u\, \mathcal{L}^N \otimes \delta_{\{(0,0)\}} \qquad \text{in } \mathcal{D}'(\mathbb{R}^{N+2}).$$

Proof.

PROOF.

Let $\varphi \in C_c^\infty(\mathbb{R}^{N+2})$,

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Let $\varphi \in C_c^\infty(\mathbb{R}^{N+2})$, and define

$$\tilde{\varphi}(x,t) = \int_0^{2\pi} \varphi(x, |t|\, e^{i\theta})\, \mathrm{d}\theta \qquad \text{for } (x,t) \in \mathbb{R}^{N+1}.$$

Proof.

Let $\varphi \in C_c^\infty(\mathbb{R}^{N+2})$, and define

$$\tilde{\varphi}(x,t) = \int_0^{2\pi} \varphi(x, |t|\, e^{i\theta})\, \mathrm{d}\theta \qquad \text{for } (x,t) \in \mathbb{R}^{N+1}.$$

Then, for $|y| = |(y_1, y_2)|$ and by representing y in polar coordinates,

$$\begin{aligned}
\int_{\mathbb{R}^{N+2}} w_u \Delta\varphi \, \mathrm{d}(x,y) &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^2} w_u(x,|y|)\Delta_x\varphi(x,y) + \Delta_y\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \int_0^\infty w_u(x,r) \int_0^{2\pi} \Big[\Delta_x\varphi(x,re^{i\theta}) + \frac{\partial_r(r\,\partial_r\varphi(x,re^{i\theta}))}{r} \\
&\qquad\qquad + \frac{\partial_{\theta\theta}\varphi(x,re^{i\theta})}{r^2}\Big]\, d\theta\, r\mathrm{d}r\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \int_0^\infty w_u(x,r)\Big[r\Delta_x\tilde{\varphi}(x,r) + \partial_r(r\,\partial_r\tilde{\varphi}(x,r))\Big]\, dr\, dx \\
&= \int_{\mathbb{R}^N} \int_0^\infty w_u(x,r)\,\mathrm{div}\,(r\nabla\tilde{\phi}(x,r))\, dr\, dx
\end{aligned}$$

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Thus,

$$\begin{aligned}
\int_{\mathbb{R}^{N+2}} w_u \Delta\varphi \,\mathrm{d}(x,y) &= \int_{\mathbb{R}^N}\int_0^\infty w_u(x,r)\operatorname{div}(r\nabla\tilde{\varphi}(x,r))\,dr\,dx\\
&= \frac{1}{2}\int_{\mathbb{R}^{N+1}} w_u(x,t)\operatorname{div}(|t|\nabla\tilde{\varphi}(x,t))\,dxdt\\
&= -\int_{\mathbb{R}^N} u(x)\tilde{\phi}(x,0)\,dx\\
&= -2\pi\int_{\mathbb{R}^N} u(x)\phi(x,0)\,dx.
\end{aligned}$$

PROOF.

Recall,

- w_u is radially symmetric in the r-variable,
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Thus,

$$\begin{aligned}
\int_{\mathbb{R}^{N+2}} w_u \Delta\varphi \, \mathrm{d}(x,y) &= \int_{\mathbb{R}^N}\int_0^\infty w_u(x,r)\operatorname{div}\left(r\nabla\tilde{\varphi}(x,r)\right) dr\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^{N+1}} w_u(x,t)\operatorname{div}\left(|t|\nabla\tilde{\varphi}(x,t)\right) dxdt \\
&= -\int_{\mathbb{R}^N} u(x)\tilde{\phi}(x,0)\,dx \\
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\end{aligned}$$

This complete the proof of Theorem 2.

The weak unique continuation property.

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Theorem 3 [Chen, H., Weth'23].

Let $u \in L^1_0(\mathbb{R}^N)$ and suppose there is an open, non-empty subset $\Omega \subseteq \mathbb{R}^N$ such that

$$u = 0 \text{ on } \Omega \text{ and } L_\Delta u = 0 \text{ on } \Omega.$$

THE WEAK UNIQUE CONTINUATION PROPERTY.

THEOREM 3 [CHEN, H., WETH'23].

Let $u \in L^1_0(\mathbb{R}^N)$ and suppose there is an open, non-empty subset $\Omega \subseteq \mathbb{R}^N$ such that

$$u = 0 \text{ on } \Omega \text{ and } L_\Delta u = 0 \text{ on } \Omega.$$

Then, $u = 0$ on $\mathbb{R}^N$.

Thank you for your attention!!!