Critical space illposedness for incompressible Euler

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Organization of the talk

- Pictures
- 2D Euler, heuristics (stability/instability)
- Well-posedness and Critical regularity
- Proof

I. Gallery

2D flows

Large-scale atmosphere and ocean dynamics: essentially 2D

2D flows

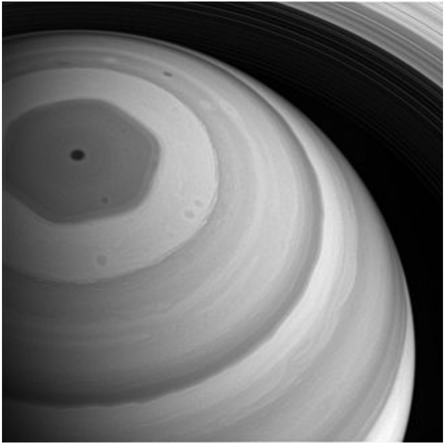


Figure: Saturn's hexagon (2009)

2D flows

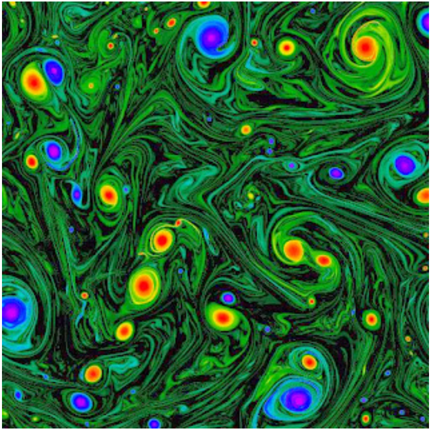


Figure: Turbulence in 2D

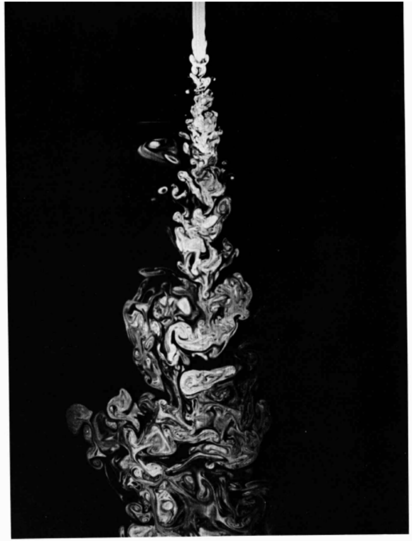


Figure: Axisymmetric water jet at Re $\sim$ 2300

2D flows

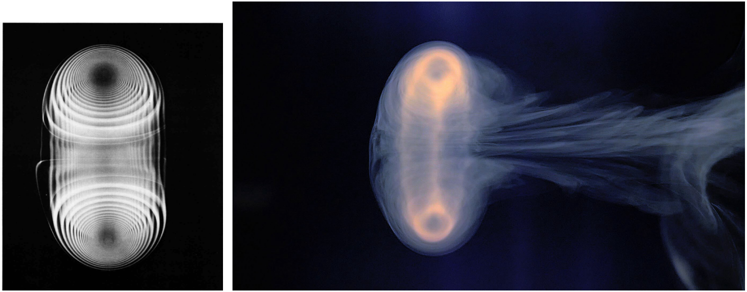


Figure: Hill's smoke ring (1894) at Re $\sim$ 10000

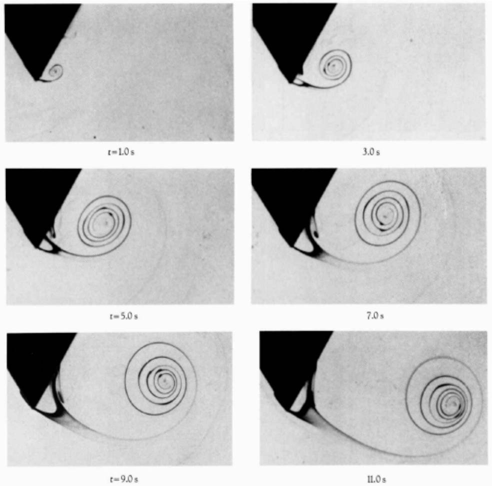


Figure: Birth of a Kaden spiral (1931) at Re $\sim$ 1000

2D flows

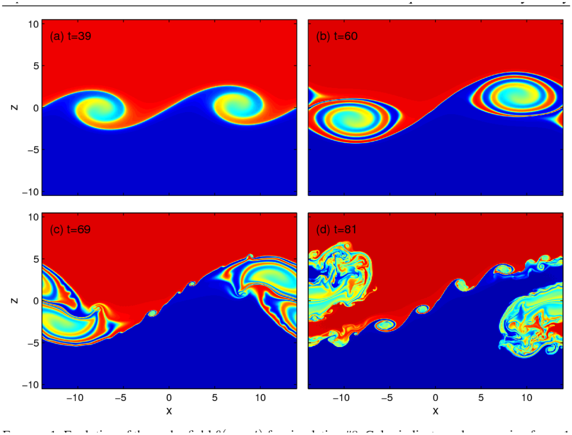


Figure: Kelvin-Helmholtz instability (1871, 1868)

2D flows

Figure: Kelvin-Helmholtz in real life

2D flows

Figure: Kármán vortex street (1963)

2D flows

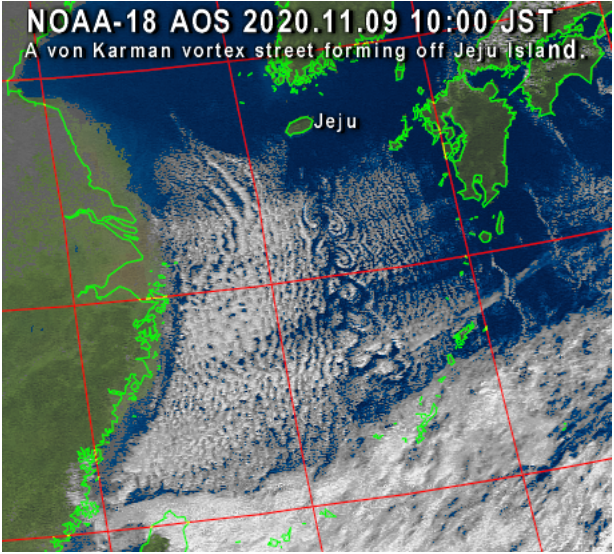
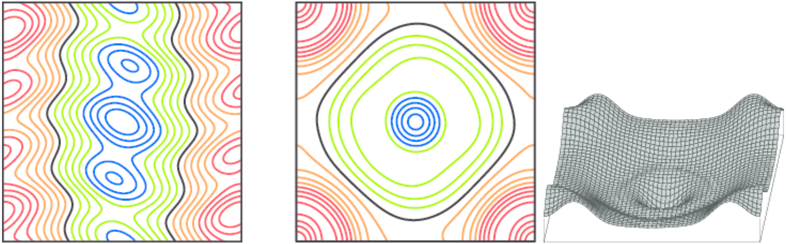


Figure: Karman vortex street behind Jeju island

Absence of Turbulence in 2D

- Kolmogorov conjecture (1940s)
- Yudovich (1956)
- Arnol'd (1960)
- Meshalkin-Sinai (1961)



(a) (b) (c)

Figure: Kim-Okamoto (2010)

Summary: 2D flows

- Large-scale coherent structures at low viscosity
- Stability/Instability coexistence
- Some rigorous results exist

II. Equations and Heuristics

Incompressible Euler equations in 2D

2D Euler

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ u(t=0) = u_0. \end{cases} \tag{Euler}$$

2D Euler in vorticity form: $\omega = \nabla \times u$

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Here $u : [0, \infty) \times \Omega \to \mathbb{R}^2$, $p : [0, \infty) \times \Omega \to \mathbb{R}$, $\omega : [0, \infty) \times \Omega \to \mathbb{R}$ with a two-dimensional domain Ω.

Vorticity formulation for Euler

- Flow:

$$\frac{d}{dt}\Phi(t,x) = u(t,\Phi(t,x)), \quad \Phi(0,x) = x.$$

For fixed t,

$$\Phi(t,\cdot) : \Omega \to \Omega$$

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- Along the flow:

$$\omega(t,\Phi(t,x)) = \omega_0(x).$$

Mechanism of stability and instability

Stability

- Kinetic energy conservation: $\|u\|_{L^2} = \|\omega\|_{H^{-1}}$ (low frequency control)
- Transport, incompressibility: $\|\omega\|_{L^p}$ for any p (*distribution function*).
- e.g. Constant vorticity in a domain, piecewise constant with odd symmetry

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Instability (quantify by growth of $\|\omega\|_{H^1}$, $\|\omega\|_{C^1}$, etc.)

- Incompressibility implies *vortex thinning*

$$\partial_t \nabla\omega + u \cdot \nabla\omega = [\nabla u]^T \nabla\omega.$$

Rigorous instability results

Well-posedness: "reasonable" function space X

- global regularity based on the a priori estimate

$$\frac{d}{dt}\|\omega\|_X \lesssim \|\omega\|_X\|\omega\|_{L^\infty}\log(10+\frac{\|\omega\|_X}{\|\omega\|_{L^\infty}})$$

  together with $\|\omega\|_{L^\infty} = \|\omega_0\|_{L^\infty}$, we obtain

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Recent progress: Denissov, Kiselev-Sverak, Zlatos, ... **Use stability to prove instability**.

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- Infinite time result requires a special argument: *ingenuity*.

III. Mathematical Theory

Notion of wellposedness

Fix some Banach space X. Given initial data $\omega_0 \in X$, we say (Euler) is wellposed in X if:

- (Existence) for some $T > 0$, there is a solution in $L^\infty([0,T];X)$.
- (Uniqueness) the solution is unique in the class $L^\infty([0,T];X)$.

Really the basic requirement! Transport system: expect the regularity of the solution to be preserved in time. (Not even asking for continuity of the solution map.)
In practice, want $\|\omega\|_{L^\infty_t X} \lesssim \|\omega_0\|_X$. Necessary for continuity of the solution operator.

Sobolev wellposedness and critical regularity

Theorem (classical)

2D Euler is well-posed with $X = W^{s,p}$ if $sp > 2$. In higher dimensions, $sp > n$ suffices.

Definition: critical Sobolev spaces

The space $X = W^{s,p}$ is called critical (with respect to Euler) if $sp = n$ in n spatial dimensions.

Theorem (Bourgain-Li '15 '19, Elgindi-J. '17)

Euler is *illposed* in $W^{s,p}$ with $sp = n$ if $0 < s < n$.

Sobolev wellposedness and critical regularity

- Supercritical case $sp < n$.
- Yudovich theory in 2D: Existence and **uniqueness** in L^∞.
- Precise illposedness statement (Bourgain-Li):
  - (norm inflation) for any $\epsilon, \delta > 0$, there exists $\omega_0 \in C^\infty$ s.t.

$$\|\omega_0\|_{W^{s,p}} < \epsilon \qquad \sup_{t\in(0,\delta)} \|\omega(t)\|_{W^{s,p}} > \frac{1}{\epsilon}.$$

  - (nonexistence–2D) there exists $\omega_0 \in W^{s,p} \cap L^\infty$ such that the Yudovich solution escapes $W^{s,p}$ instantaneously. That is,

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- Similar result in C^m spaces: (Misiolek-Yoneda '16, Elgindi-Masmoudi '17, Bourgain-Li '15)
- Open problems.

Understanding criticality

Try a priori estimate: e.g. H^1 for ω,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 = -\int \nabla u \cdot \nabla\omega\nabla\omega.$$

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Recall $\nabla u = \nabla\nabla \times (-\Delta)^{-1}\omega$.
$\omega \in H^1 \iff \nabla u \in H^1 \not\Rightarrow \nabla u \in L^\infty$.

Motivations

- Strongest conservation, uniqueness class, scaling invariance
- Nontrivial $o(1)$ time dynamics (cf. singular vortex patches)
- Slightly subcritical dynamics (cf. Elgindi '21)
- Slightly supercritical dynamics

Failure is $\log^{\frac{1}{2}}$ in H^1.

Wellposedness in slightly regularized systems:

- Critical Besov wellposedness
- Logarithmically regularized Euler
- Logarithmically dissipative Euler
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Enhanced dissipation in the Navier-Stokes case

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Critical Besov wellposedness

n-dim'l Euler is LWP in $B^{n/p}_{p,1}, 1 \le p \le \infty$.
Vishik ('98, '99), Pak-Park ('04, '13), Chae ('04).

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Logarithmically regularized Euler

Replace $u = \nabla \times \Delta^{-1} \ln^{-\gamma}(10 - \Delta)\omega$ (Chae-Constantin-Wu '11).
Critical Sobolev Well-posed for $\gamma > \frac{1}{2}$ (Chae-Wu '12)
Critical Sobolev ill-posed for $\gamma \leq \frac{1}{2}$ (Kwon '20)

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Logarithmical dissipation

Consider $\partial_t\omega + u \cdot \nabla\omega = -\ln^{\gamma}(10 - \Delta)\omega$.
Critical Sobolev Well-posed for $\gamma > \frac{1}{2}$
Critical Sobolev ill-posed for $\gamma \leq \frac{1}{2}$?

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Losing estimate in 2D

(Elgindi-J. '17, Brue-Nguyen '20)
$\omega_0 \in H^1 \cap L^\infty \to \omega(t) \in H^{1-Ct} \cap W^{1,2-Ct}$.

Think about the "correct" proof of the CK theorem.

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Question

2D Navier-Stokes is globally well-posed for $\omega_0 \in H^1$ for any $\nu > 0$:

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Theorem (J.-Yoneda '20)

There exists H^1 convergent sequence ω_0^ν with solutions

$$\|\omega^\nu(t)\|_{H^1} \gtrsim (\ln \frac{1}{\nu})^{c(t)} \|\omega_0\|_{H^1}$$

for any $t > 0$ and some $c(t) > 0$.

SQG case

Modify the velocity as follows: $u = \nabla \times (-\Delta)^{-\frac{1}{2}}\omega$. (Called SQG)

Theorem (J.-Kim '21+)

SQG is illposed in the critical Sobolev space H^2, in the same sense as in Bourgain-Li.

The proof extends to many other related systems.

IV. Proof

The proof

- Key points: choice of initial data, dynamic propagation of

$$\int \nabla u \nabla \omega \nabla \omega \gg 1.$$

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- Key Lemma + Yudovich estimates: EJ '17 proof.

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- Key Lemma + Yudovich estimates: EJ '17 proof.
- Key Lemma + dyadic bootstraping arguments: EY '20 proof. Without contradiction, quantitative local growth rate.
- Extension to the SQG case: Key Lemma, dyadic bootstrap with refined geometric control.

Dynamics of Bourgain-Li bubbles

Define Bourgain-Li bubbles:

- Fix some smooth $\varphi \geq 0$ supported in a neighborhood of $(\frac{1}{2}, \frac{1}{2})$.
- For some bounded non-negative sequence $\{a_j\}_{j\geq 0}$, define

$$\omega_0 = \sum_{j\geq 0} a_j \varphi(2^j x).$$

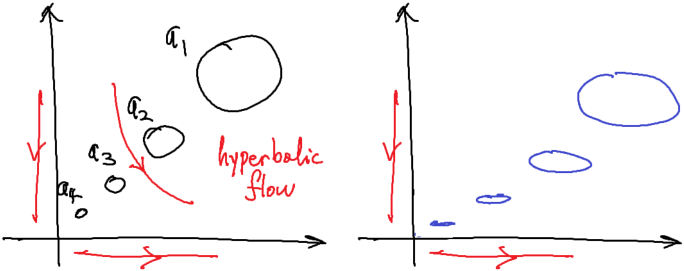
- Extend to $\mathbb{R}^2$ (or $\mathbb{T}^2$) using odd symmetry.

Observations:

- $\omega_0 \in L^\infty \implies$ unique global solution $\omega \in L^\infty([0,\infty); L^\infty)$ (Yudovich theory).
- $\omega_0 \in H^1 \iff \{a_j\} \in \ell_2$.

Dynamics of Bourgain-Li bubbles

A stable vortex configuration for instability. Difficulty: **inviscid damping, inverse energy cascade** (well-known in physics).



Stability of the instability again!

Quantify small-scale creation in Bourgain-Li bubbles

- The j-th bubble is **almost invariant** for the timescale

$$\tau_j \sim \frac{1}{S_j}, \quad S_j = \sum_{k=0}^{j-1} a_k.$$

  Improvement over any existing WP theory, using geometry of the data (stability).

- The j-th bubble is stretched as follows (instability):

$$\|\omega(t)\|_{H^1(\Phi(t,B_j))} \gtrsim a_j (S_j)^{ct}.$$

  Square summation in j gives the H^1 norm.

- Corollary: 2D Euler is **illposed** in H^1, by taking initial data with $\{a_j\} \in \ell^2 \setminus \ell^1$. Application to turbulent flows?

Thank you for listening!