Temperature optimimzation problems governed by semi-discrete phase-field models of grain boundary motions

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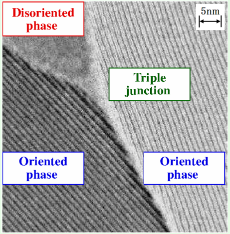
0. Kobayashi–Warren–Carter type system of grain boundary motion

$0 < T < \infty$, $\Omega \subset \mathbb{R}^N$: b.d.d. domain ($N \in \{1,2,3,4\}$), $\Gamma = \partial\Omega$: smooth, n_Γ: unit outer normal, $Q := (0,T) \times \Omega$

KWC system: cf. [Kobayashi–Warren–Carter](2000)

$$\begin{cases} \partial_t \eta - \Delta \eta + g(\eta) + \alpha'(\eta)|\nabla\theta| = u(t,x), \ (t,x) \in Q, \\ \alpha_0(\eta)\partial_t\theta - \mathrm{div}\left(\alpha(\eta)\dfrac{D\theta}{|D\theta|} + \nu^2 \nabla\theta\right) = v(t,x), \ (t,x) \in Q, \\ \nabla\eta \cdot n_\Gamma = 0, \ \ \theta = 0 \text{ on } \Sigma := (0,T) \times \Gamma, \\ \eta(0,x) = \eta_0(x), \ \theta(0,x) = \theta_0(x), \ x \in \Omega. \end{cases}$$



- $\eta = \eta(t,x)$: orientation order ($\eta \geq 1$: oriented, $\eta \leq 0$: disoriented),
- $\theta = \theta(t,x)$: orientation angle of grain
- $u = u(t,x)$: temperature, $v = v(t,x)$: forcing for θ
- $g \in C^2(\mathbb{R})$: Lipschitz, $\exists\, G \in C^3(\mathbb{R})$ s.t. $G' = g$ on $\mathbb{R}$
- $0 < \alpha \in C^2(\mathbb{R})$: convex • $\alpha_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$: fixed function
- $[\eta_0, \theta_0] = [\eta_0(x), \theta_0(x)]$: fixed initial data of $[\eta, \theta]$ • $\nu > 0$: fixed constant

The Gradient system of free-energy:

$$[\eta,\theta] \mapsto \mathscr{F}(\eta,\theta) := \frac{1}{2}\int_\Omega |\nabla\eta|^2\,dx + \int_\Omega G(\eta)\,dx + \int_\Omega \left(\alpha(\eta)|\nabla\theta| + \frac{\nu^2}{2}|\nabla\theta|^2\right)dx.$$

◇ Sketch of history

[1] Existence and large-time behavior

(∗) the case of $\nu > 0$: [Ito–Kenmochi–Yamazaki](2008–2011)
the results in active case of $\nu \Delta\theta$

(∗∗) the case of $\nu = 0$: [Moll–S.–Watanabe](2011–)
the results in very singular case of $-\mathrm{div}\left(\alpha(\eta)\frac{D\theta}{|D\theta|}\right)$

[2] Uniqueness (a few, and only in the case of $\nu > 0$)

(i) 1D-case of Ω: [Ito–Kenmochi–Yamazaki](2008)

(ii) η-independent case of $\alpha_0 \partial_t \theta = \alpha_0(t,x)\partial_t\theta$: [Antil–Kubota–S.–Yamazaki](2020–)
Ω is higher dimensional, but $0 < \alpha_0 \in W^{1,\infty}(Q)$ (positive, b.d.d., Lipschitz)

[3] Optimal control problem ($\nu > 0$)

(iii) continuation work of (ii): [Antil–Kubota–S.–Yamazaki](2020–)
Existence, parameter dependence of optmial controls, necessary condition of optimality

This talk: Optimal control problem under η-dependent case of α_0 $\Longrightarrow$ time-discrete setting

1. Time-discrete Kobayashi–Warren–Carter type system of grain boundary motion

$\Omega \subset \mathbb{R}^N$: b.d.d. domain ($N \in \{1,2,3,4\}$), $\Gamma = \partial\Omega$: smooth, n_Γ: unit outer normal

$n \in \mathbb{N}$, $\tau = T/n$ (time-step-size) $X := L^2(\Omega)$, $\mathbb{X} := [X]^n$

State-system $(\mathrm{S})_0$: cf. [Moll–S.–Watanabe](2014–)

$$\begin{cases} \frac{1}{\tau}(\eta_i - \eta_{i-1}) - \Delta\eta_i + g(\eta_i) + \alpha'(\eta_i)|\nabla\theta_{i-1}| = u_i \text{ in } \Omega, & \text{(1st.eq)} \\ \frac{1}{\tau}\alpha_0(\eta_{i-1})(\theta_i - \theta_{i-1}) - \mathrm{div}\left(\alpha(\eta_i)\frac{D\theta_i}{|D\theta_i|} + \nu^2\nabla\theta_i\right) = v_i \text{ in } \Omega, & \text{(2nd.eq)} \\ \nabla\eta_i \cdot n_\Gamma = 0, \ \theta_i = 0 \text{ on } \Gamma, i = 1,2,3,\dots,n, \\ \eta_0 \in H^1(\Omega), \ \theta_0 \in H^1_0(\Omega). \end{cases}$$

- $\eta = \{\eta_i\}_{i=1}^n \in \mathbb{X}$: orientation order
- $\theta = \{\theta_i\}_{i=1}^n \in \mathbb{X}$: orientation angle of grain
- $u = \{u_i\}_{i=1}^n \in \mathbb{X}$: temperature, $v = \{v_i\}_{i=1}^n \in \mathbb{X}$: forcing for θ

$\dagger_1$. The state system $(\mathrm{S})_0$ is a coupled system of two schemes of minimizing movements: (1st.eq) and (2nd.eq)

$\dagger_2$. The coupled minimizing movements can be solved separately, in the order of (1st.eq) $\to$ (2nd.eq)

2. Temperature constrained optimal control problem

$\Omega \subset \mathbb{R}^N$, $\Gamma := \partial\Omega$: (smooth), $X := L^2(\Omega)$, $\mathbb{X} := [X]^n$

Problem (OP)$_0$: find $[u^*, v^*] = [\{u_i^*\}_{i=1}^n, \{v_i^*\}_{i=1}^n] \in [\mathbb{X}]^2$, called optimal control, s.t.

$$[u^*, v^*] = \text{arg-min}\mathcal{J} = \mathcal{J}(u, v) \text{ on a constrained class } \mathcal{U}_{\rm ad},$$

with a cost functional $\mathcal{J} : [u, v] \in [\mathbb{X}]^2 \mapsto \mathcal{J}(u, v) \in [0, \infty)$, defined as

$$\mathcal{J}(u, v) := \frac{1}{2}\big|[\eta, \theta] - [\eta_{\rm ad}, \theta_{\rm ad}]\big|_{[\mathbb{X}]^2}^2 + \frac{1}{2}\big|[u, v]\big|_{[\mathbb{X}]^2}^2.$$

In the context,

- $u = \{u_i\}_{i=1}^n$: the control for $\eta = \{\eta_i\}_{i=1}^n$ (temperature), $v = \{v_i\}_{i=1}^n$: the control for $\theta = \{\theta_i\}_{i=1}^n$
- $[\eta, \theta] = [\{\eta_i\}_{i=1}^n, \{\theta_i\}_{i=1}^n] \in [\mathbb{X}]^2$: the solution to the state-system (S)$_0$, for any $[u, v] \in [\mathbb{X}]^2$.
- $[\eta_{\rm ad}, \theta_{\rm ad}] = [\{\eta_{{\rm ad},i}\}_{i=1}^n, \{\theta_{{\rm ad},i}\}_{i=1}^n] \in [\mathbb{X}]^2$: the admissible target profile for $[\eta, \theta]$
- $\mathcal{U}_{\rm ad}$: a class of admissible controls $[u, v] \in [\mathbb{X}]^2$, which fulfill:
  - box-constraint: $\sigma_{*,i} \leq u_i \leq \sigma_i^*$ a.e. in Ω, $i = 1, 2, 3, \ldots, n$, with fixed obstacle sequences $\sigma_* := \{\sigma_{*,i}\}_{i=1}^n, \sigma^* = \{\sigma_i^*\}_{i=1}^n \in [L^\infty(\Omega)]^n$

$\dagger_1$. The time-discrete setting can be applied to the numerical scheme, directly

3. Approximating problems

Problem $(\mathrm{OP})_\varepsilon$ with $\varepsilon \geq 0$: find $[u^*, v^*] = [\{u_i^*\}_{i=1}^n, \{v_i^*\}_{i=1}^n] \in [\mathbb{X}]^2$, called optimal control, s.t.

$$[u_\varepsilon^*, v_\varepsilon^*] = \text{arg-min}\mathcal{J}_\varepsilon = \mathcal{J}_\varepsilon(u, v) \text{ on a constrained class } \mathcal{U}_{\mathrm{ad}},$$

with a regularized cost functional $\mathcal{J}_\varepsilon : [u, v] \in [\mathbb{X}]^2 \mapsto \mathcal{J}_\varepsilon(u, v) \in [0, \infty)$, defined as

$$\mathcal{J}_\varepsilon(u, v) := \frac{1}{2}\big|[\eta_\varepsilon, \theta_\varepsilon] - [\eta_{\mathrm{ad}}, \theta_{\mathrm{ad}}]\big|_{[\mathbb{X}]^2}^2 + \frac{1}{2}\big|[u, v]\big|_{[\mathbb{X}]^2}^2.$$

State-system $(\mathrm{S})_\varepsilon$:

$$\begin{cases} \frac{1}{\tau}(\eta_{\varepsilon,i} - \eta_{\varepsilon,i-1}) - \Delta\eta_{\varepsilon,i} + g(\eta_{\varepsilon,i}) + \alpha'(\eta_{\varepsilon_i}) f_\varepsilon(\nabla\theta_{\varepsilon,i-1}) = u \text{ in } \Omega, \\ \frac{1}{\tau}\alpha_0(\eta_{\varepsilon,i-1})(\theta_{\varepsilon,i} - \theta_{\varepsilon,i-1}) - \mathrm{div}\big(\alpha(\eta_{\varepsilon_i})\partial f_\varepsilon(\nabla\theta_{\varepsilon,i}) + \nu^2\nabla\theta_{\varepsilon,i}\big) \ni v \text{ in } \Omega, \\ \nabla\eta_{\varepsilon_i} \cdot n_\Gamma = 0, \;\; \theta_{\varepsilon,i} = 0 \text{ on } \Gamma, \\ \eta_{\varepsilon,0}(x) = \eta_0(x), \; \theta_{\varepsilon,0}(x) = \theta_0(x), \; x \in \Omega. \end{cases}$$

- $f_\varepsilon(\omega) := \sqrt{\varepsilon^2 + |\omega|^2}, \forall \omega \in \mathbb{R}^N, \varepsilon > 0 \;\big(f_\varepsilon \to f_0 := |\cdot| \text{ in } L^\infty(\mathbb{R}^N) \text{ as } \varepsilon \downarrow 0\big)$

  $\partial f_\varepsilon \subset \mathbb{R} \times \mathbb{R}$: subdifferential of f_ε $\big(\partial f_0(\nabla\theta) \sim$ "set-valued sign function" $\sim \frac{\nabla\theta}{|\nabla\theta|}\big)$

4. Main Theorems

Former part: mathematical analysis by means of calculus of variation

Theorem A. Existence and parameter-dependence of optimal controls for $\varepsilon \geq 0$

Theorem B. The first order necessary condition in the case of $\varepsilon > 0$ (regular case of the problem $(\mathrm{OP})_\varepsilon$)

Keypoint: linearlization method for the state-system $\Longleftrightarrow$ Gâteaux differential of the cost

quasilinear diffusion in $(\mathrm{S})_\varepsilon$ $\quad\longrightarrow\quad$ diffusion in the necessary condition

$$-\mathrm{div}\left(\alpha(\eta)\frac{\nabla\theta}{\sqrt{\varepsilon^2+|\nabla\theta|^2}}\right) \longrightarrow -\mathrm{div}\left(\alpha(\eta)\frac{(\varepsilon^2+|\nabla\theta|^2)I-\nabla\theta\otimes\nabla\theta}{\sqrt{\varepsilon^2+|\nabla\theta|^2}^3}\nabla v\right)$$

($v \in \mathbb{X}$: component of optimal control)

Theorem C. The limiting observation of the necessary condition as $\varepsilon \downarrow 0$

Keypoint: limiting approach to the singular case of the problem $(\mathrm{OP})_0$

singular diffusion in $(\mathrm{S})_0$ $\quad\longrightarrow\quad$ limiting expression as $\varepsilon \downarrow 0$

$$-\mathrm{div}\left(\alpha(\eta)\frac{D\theta}{|D\theta|}\right) \longrightarrow \boxed{\zeta^\circ \in [\mathscr{D}'(\Omega)]^n \ (?)}$$

Latter part: precise observation under 1D-setting of $\Omega = (0,1)$

Theorem D. Limiting necessary condition, on some neighborhood of the grain boundary

Keypoint: H^2-regularity of solutions to $(\mathrm{S})_0$

Decomposition property of quasilinear diffusion: $-\partial_x\left(\alpha(\eta)\frac{D\theta}{|D\theta|}+\nu^2\partial_x\theta\right) = -\partial_x\left(\alpha(\eta)\frac{D\theta}{|D\theta|}\right) - \nu^2\partial_x^2\theta$ in $\mathbb{X}$

Assumptions and notations.

(A0) $N \in \{1,2,3,4\}$, $\nu > 0$, $\sigma_* = \{\sigma_{*,i}\}_{i=1}^n$, $\sigma^* = \{\sigma_i^*\}_{i=1}^n \in [L^\infty(\Omega)]^n$; $\sigma_{*,i} \leq \sigma_i^*$ a.e. in Ω, $i = 1, \dots, n$,

$$X := L^2(\Omega),\ \mathbb{X} := [X]^n,\ Y := H^1(\Omega),\ \mathbb{Y} := [Y]^n,\ Y_0 := H_0^1(\Omega),\ \mathbb{Y}_0 := [Y_0]^n$$

(A1) $\alpha_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ and $\alpha_0 > 0$ on $\overline{\Omega}$.

(A2) $\alpha \in C^2(\mathbb{R})$, s.t. $\alpha'(0) = 0,\ \alpha'' \geq 0$, and $\delta_\alpha := \inf \alpha(\mathbb{R}) \cup \alpha_0(\mathbb{R}) > 0$

(A3) $g \in C^2(\mathbb{R})$: Lipschitz, having a non-negative potential $0 \leq G \in C^3(\mathbb{R})$, and there exist constants $-\infty < r_* \leq 0 < r^* < \infty$, which satisfy:

$$\begin{cases} g(\eta_i) \leq -|\sigma_{*,i}|_{L^\infty(\Omega)}, \text{ for } \eta_i \leq r_*, \text{ for any } i = 1,2,3,\dots,n, \\ g(\eta_i) \geq |\sigma_i^*|_{L^\infty(\Omega)}, \text{ for } \eta_i \geq r^*, \text{ for any } i = 1,2,3,\dots,n. \end{cases}$$

(A4) $[\eta_0, \theta_0] \in \big[Y \cap L^\infty(\Omega)\big] \times Y_0$, and $r_* \leq \eta_0 \leq r^*$ a.e. in Ω

(A5) Let $n \in \mathbb{N}$ be a large number, s.t.:

$$0 < \tau = \frac{T}{n} < \frac{\nu^2}{4(\nu^2(1 + |g'|_{L^\infty(\mathbb{R})}) + |\alpha'|_{L^\infty(\mathbb{R})})}$$

†1. The conditions colored blue are for the L^∞-boundedness of the orientation order $\eta = \{\eta_i\}_{i=1}^n \in \mathbb{X}$

†2. The assumption (A5) is for the strict coercivity (existence and uniqueness) of $(\mathrm{S})_\varepsilon$

Proposition 1 (Solvability of the state-system $(\mathrm{S})_\varepsilon$) cf. [Moll–S.–Watanabe](2013–)

Let us assume (A0)–(A5). Then, for $\varepsilon \geq 0$, and $[u, v] \in \mathcal{U}_{\mathrm{ad}}$, the state-system $(\mathrm{S})_\varepsilon$ admits a unique solution $[\eta, \theta]$, defined as follows.

(S0) $[\eta, \theta] = [\{\eta_i\}_{i=1}^n, \{\theta_i\}_{i=1}^n] \in ([H^2(\Omega)]^n \times [L^\infty(\Omega)]^n) \times [Y_0]^n$

(S1) $\dfrac{1}{\tau}(\eta_i - \eta_{i-1}) - \Delta\eta_i + g(\eta_i) + \alpha'(\eta_i) f_\varepsilon(\nabla\theta_{i-1}) = u_i$ in Ω,

subject to $r_* \leq \eta_i \leq r^*$ a.e. in Ω, $\nabla\eta_i \cdot n_\Gamma = 0$ on Γ, for $i = 1, 2, 3, \dots, n$

(S2) $\dfrac{1}{\tau}\alpha_0(\eta_{i-1})(\theta_i - \theta_{i-1}) - \mathrm{div}\big(\alpha(\eta_i)\omega_i^* + \nu^2\nabla\theta_i\big) = v_i$ in Ω,

with $\omega_i^* \in L^\infty(\Omega)$ satisfying $\omega_i^* \in \partial f_\varepsilon(\nabla\theta_i)$ a.e. in Ω,

subject to $\theta_i = 0$ on Γ, for $i = 1, 2, 3, \dots, n$

- $f_\varepsilon(\omega) := \sqrt{\varepsilon^2 + |\omega|^2}$, $\forall \omega \in \mathbb{R}$, $\varepsilon > 0$ $\big(f_\varepsilon \to |\cdot|$ in $L^\infty(\mathbb{R})$ as $\varepsilon \downarrow 0\big)$
- the case when $\varepsilon = 0$, $\omega_i^* \in \mathrm{Sgn}(\nabla\theta_i)$ a.e. in Ω
- $\mathrm{Sgn} : \omega \in \mathbb{R}^N \longrightarrow \mathrm{Sgn}(\omega)\{\omega^* \in \mathbb{R}^N : \omega^* \cdot (z - \omega) \leq |z| - |\omega|, \ \forall z \in \mathbb{R}^N\}$ set-valued sign function

†. cf. [Moll–S.–Watanabe](2015): If $[u, v] = \big[\{u_i\}_{i=1}^n, \{v_i\}_{i=1}^n\big] = 0$ in $[\mathbb{X}]^2$, then we further verify energy-disspation for the following sequence of free-energy:

$$\{\mathscr{F}_i\}_{i=1}^n := \left\{ \frac{1}{2}\int_\Omega |\nabla\eta_i|^2\,dx + \int_\Omega G(\eta_i)\,dx + \int_\Omega \alpha(\eta_i)|\nabla\theta_i|\,dx + \frac{\nu^2}{2}\int_\Omega |\nabla\theta_i|^2\,dx \right\}_{i=1}^n$$

Proposition 2 (Continuous dependence for the state-system $(\mathrm{S})_\varepsilon$) cf. [Kubota–S.](2020–)

Under (A0)–(A5), let us define:

$$\mathcal{S}_\varepsilon : [u, v] \in \mathcal{U}_{\mathrm{ad}} \mapsto [\eta_\varepsilon, \theta_\varepsilon] := \mathcal{S}_\varepsilon[u, v] : \text{ the solution to } (\mathrm{S})_\varepsilon, \text{ for } \varepsilon \geq 0$$

Then,

$$\{\varepsilon_m\}_{m=1}^\infty \subset [0, 1],\ \varepsilon_m \to \varepsilon,\ [u_m, v_m] \to [u, v] \text{ weakly in } [\mathbb{X}]^2, \text{ as } m \to \infty$$
$$\Longrightarrow [\eta_m, \theta_m] := \mathcal{S}_{\varepsilon_m}[u_m, v_m] \to [\eta, \theta] := \mathcal{S}_\varepsilon[u, v] \text{ in } [Y]^n \times [Y_0]^n, \text{ as } m \to \infty$$

Theorem A (Existence and parameter dependence of optimal contorols)

(I) Under (A0)–(A5) $\varepsilon \geq 0$, the following two items holds.

The problem $(\mathrm{OP})_\varepsilon$ admits at least one optimal control $[u_\varepsilon^*, v_\varepsilon^*] \in \mathcal{U}_{\mathrm{ad}}$

(II) Under (A0)–(A5) $\varepsilon_0 \geq 0$, let $\{[u_\varepsilon^*, v_\varepsilon^*]\}_{\varepsilon \geq 0}$ be a sequence optimal controls

$\exists \{\varepsilon_m\}_{m=1}^\infty \subset \{\varepsilon\}_{\varepsilon \geq 0}$, $\exists [u^*, v^*] \in \mathcal{U}_{\mathrm{ad}}$ s.t. :

$$\begin{cases} \varepsilon_m \to \varepsilon_0, [u_{\varepsilon_m}^*, v_{\varepsilon_m}^*] \to [u^*, v^*] \text{ weakly in } [\mathbb{X}]^2 \text{ as } m \to 0, \\ [u^*, v^*]\text{: optimal control of } (\mathrm{OP})_{\varepsilon_0} \end{cases}$$

†. Theorem A will be obtained as a consequence of the argument of minimizing sequence

◇ Adjoint of the linearlized state-system (necessary condition of optimality)

Adjoint system (A)$_\varepsilon$ ($\varepsilon > 0$): to find $[p, z] = \bigl[\{p_i\}_{i=1}^n, \{z_i\}_{i=1}^n\bigr] \in [\mathbb{X}]^2$ s.t.

$$\begin{cases} \dfrac{p_i - p_{i+1}}{\tau} - \Delta p_i + \mu_i p_i + \lambda_i p_i + \omega_i \cdot \nabla z_i + \tilde{\lambda}_i z_i = h_i, \text{ in } \Omega, & \text{(ad.1)} \\ \dfrac{a_i z_i - a_{i+1} z_{i+1}}{\tau} - \mathrm{div}\,(A_i \nabla z_i + \nu \nabla z_i + p_{i+1}\, \tilde{\omega}_i) = k_i, \text{ in } \Omega, & \text{(ad.2)} \\ \nabla p_i \cdot n_\Gamma = z_i = 0, \text{ on } \Gamma, \end{cases}$$

$$\text{for } i = n, \dots, 3, 2, 1, \text{ with } [p_{n+1}, z_{n+1}] = [0, 0] \text{ in } \Omega$$

In this context,

$$[h, k] = \bigl[\{h_i\}_{i=1}^n, \{k_i\}_{i=1}^n\bigr] \in [\mathbb{X}]^2 : \text{forcing}, \quad [\eta_\varepsilon, \theta_\varepsilon] = \bigl[\{\eta_{\varepsilon,i}\}_{i=1}^n, \{\theta_{\varepsilon,i}\}_{i=1}^n\bigr] : \text{sol. to } (\mathrm{S})_\varepsilon$$

$$\begin{cases} a_i = \alpha_0(\eta_{\varepsilon,i-1}),\ \mu_i = \alpha''(\eta_{\varepsilon,i}) f_\varepsilon(\nabla \theta_{\varepsilon,i-1}),\ \lambda_i = g'(\eta_i),\ \omega_i = \alpha'(\eta_{\varepsilon,i}) \nabla f_\varepsilon(\nabla \theta_{\varepsilon,i}), \\ A_i = \alpha(\eta_{\varepsilon,i}) \nabla^2 f_\varepsilon(\nabla \theta_{\varepsilon,i}),\ \tilde{\omega}_i = \alpha'(\eta_{\varepsilon,i+1}) \nabla f_\varepsilon(\nabla \theta_{\varepsilon,i}),\ \tilde{\lambda}_i = \alpha_0'(\eta_{\varepsilon,i}) \frac{\theta_{\varepsilon,i} - \theta_{\varepsilon,i-1}}{\tau}, \end{cases} \quad i = n, \dots, 3, 2, 1$$

Keypoint: • $N \in \{1, 2, 3, 4\} \implies H^1(\Omega) \subset L^4(\Omega) \implies 0 \leq \mu_i \in X$ and $p_i \in Y$ imply $\mu_i p_i \in H^1(\Omega)^*$

- (ad.1)(ad.2) are backward scheme, for the time-step $i = n, \dots, 3, 2, 1$, with the zero-terminal condition
- adjoint sytem (A)$_\varepsilon$ is solved separately, in the order of (ad.2)→(ad.1)
- the time-step-size τ **is a fixed constant**

Theorem B (Necessary condition of optimality in regular problem (OP)$_\varepsilon$ for $\varepsilon > 0$)

Under (A0)–(A5), let $\varepsilon > 0$ and $[u_\varepsilon^*, v_\varepsilon^*] \in \mathcal{U}_{\mathrm{ad}}$ be the optimal control for (OP)$_\varepsilon$. Then, it holds that:

$$\left(u_\varepsilon^* + p_\varepsilon^*, h - u_\varepsilon^*\right)_{\mathbb{X}} \geq 0, \quad \forall h \in \mathcal{U}_{\mathrm{ad}} \ \ (\sigma_{*,i} \leq h_i \leq \sigma_i^*) \ \text{ and } \ v_\varepsilon^* + z_\varepsilon^* = 0 \ \text{ in } \ \mathbb{X}$$

In the context, $[\eta_\varepsilon^*, \theta_\varepsilon^*] := \mathcal{S}_\varepsilon[u_\varepsilon^*, v_\varepsilon^*]$ in $[\mathbb{X}]^2$ and $[p_\varepsilon^*, z_\varepsilon^*] \in [\mathbb{X}]^2$is a unique solution to:

$$\begin{cases} \frac{1}{\tau}(p_{\varepsilon_i}^* - p_{\varepsilon,i+1}^*) - \Delta p_{\varepsilon,i}^* + \left(g'(\eta_{\varepsilon,i}^*) + \alpha''(\eta_{\varepsilon,i}^*) f_\varepsilon(\nabla \theta_{\varepsilon,i-1}^*)\right) p_{\varepsilon,i}^* + \alpha'(\eta_{\varepsilon,i}^*)[\nabla f_\varepsilon](\nabla \theta_{\varepsilon,i}^*) \cdot \nabla z_{\varepsilon_i}^* \\ \qquad + \frac{1}{\tau}\alpha_0'(\eta_{\varepsilon,i}^*)(\theta_{\varepsilon,i+1}^* - \theta_{\varepsilon,i}^*) z_{\varepsilon,i+1}^* = \eta_{\varepsilon,i}^* - \eta_{\mathrm{ad},i} \ \text{ in } \Omega, \\ \frac{1}{\tau}(\alpha_0(\eta_{\varepsilon,i-1}^*) z_{\varepsilon,i}^* - \alpha_0(\eta_{\varepsilon,i}^*) z_{\varepsilon,i+1}^*) - \mathrm{div}\left(\alpha(\eta_{\varepsilon,i}^*)[\nabla^2 f_\varepsilon](\nabla \theta_{\varepsilon,i}^*) \nabla z_{\varepsilon,i}^* \right. \\ \qquad \left. + \nu^2 \nabla z_{\varepsilon,i}^* + \alpha'(\eta_{\varepsilon,i+1}^*) p_{\varepsilon,i+1}^* [\nabla f_\varepsilon](\nabla \theta_{\varepsilon,i}^*)\right) = \theta_{\varepsilon,i}^* - \theta_{\mathrm{ad},i} \ \text{ in } \Omega, \\ \nabla p_{\varepsilon,i}^* \cdot n_\Gamma = 0, \ \ z_{\varepsilon,i}^* = 0 \ \text{ on } \Gamma, \text{ for any } i = n, \ldots, 3, 2, 1, \\ p_{\varepsilon,n+1}^* = z_{\varepsilon,n+1}^* = 0, \text{ in } \Omega. \end{cases}$$

Keypoint:

- the temperature $u = \{u_i\}_{i=1}^n \in [\mathbb{X}]^2$ is constrained on $\mathcal{U}_{\mathrm{ad}}$
  $\Longrightarrow$ the necessary condition for the 1st component u is obtained as a variational inequality
- there is no constraint for the component $v = \{v_i\}_{i=1}^n \in [\mathbb{X}]^2$
  $\Longrightarrow$ the necessary condition for the 2nd component v is obtained as an equality

Theorem B (Necessary condition of optimality in regular problem $(\mathrm{OP})_\varepsilon$ for $\varepsilon > 0$)

Under (A0)–(A5), let $\varepsilon > 0$ and $[u_\varepsilon^*, v_\varepsilon^*] \in \mathcal{U}_{\mathrm{ad}}$ be the optimal control for $(\mathrm{OP})_\varepsilon$. Then, it holds that:

$$\left(u_\varepsilon^* + p_\varepsilon^*, h - u_\varepsilon^*\right)_{\mathbb{X}} \geq 0, \quad \forall h \in \mathcal{U}_{\mathrm{ad}} \ \ (\sigma_{*,i} \leq h_i \leq \sigma_i^*) \ \text{ and } \ v_\varepsilon^* + z_\varepsilon^* = 0 \ \text{ in } \ \mathbb{X}$$

In the context, $[\eta_\varepsilon^*, \theta_\varepsilon^*] := \mathcal{S}_\varepsilon[u_\varepsilon^*, v_\varepsilon^*]$ in $[\mathbb{X}]^2$ and $[p_\varepsilon^*, z_\varepsilon^*] \in [\mathbb{X}]^2$is a unique solution to:

$$u_{\varepsilon,i}^*(x) = \mathrm{proj}_{[\sigma_{*,i}(x), \sigma_i^*(x)]}(-p_{\varepsilon,i}^*(x)) = \begin{cases} -p_{\varepsilon,i}^*(x), \text{ if } \sigma_{*,i}(x) \leq -p_{\varepsilon,i}^*(x) \leq \sigma_i^*(x) \\ \sigma_i^*(x), \text{ if } -p_{\varepsilon,i}^*(x) \geq \sigma_i^*(x) \\ \sigma_{*,i}(x), \text{ if } -p_{\varepsilon,i}^*(x) \leq \sigma_{*,i}(x) \end{cases}$$

$$\text{a.e. } x \in \Omega, \ i = n, \dots, 3, 2, 1$$

- for $-\infty \leq a < b \leq \infty$, $\mathrm{proj}_{[a,b]} : \mathbb{R} \longrightarrow [a,b] \cap \mathbb{R}$ is the projection on to $[a,b] \cap \mathbb{R}$

Keypoint:

- the temperature $u = \{u_i\}_{i=1}^n \in [\mathbb{X}]^2$ is constrained on $\mathcal{U}_{\mathrm{ad}}$
  $\Longrightarrow$ the necessary condition for the 1st component u is obtained as a variational inequality
- there is no constraint for the component $v = \{v_i\}_{i=1}^n \in [\mathbb{X}]^2$
  $\Longrightarrow$ the necessary condition for the 2nd component v is obtained as an equality

Theorem C (Limiting observation of necessary condition for $(\mathrm{OP})_\varepsilon$, as $\varepsilon \downarrow 0$)

Under (A0)–(A5), there exist an optimal control $[u^\circ, v^\circ] \in \mathcal{U}_{\rm ad}$ for $(\mathrm{OP})_0$, $[\eta^\circ, \theta^\circ] = \mathcal{S}_0[u^\circ, v^\circ]$, and $[\xi^\circ, \zeta^\circ, \omega^\circ] = [\{\xi_i^\circ\}_{i=1}^\circ, \{\zeta_i^\circ\}_{i=1}^n, \{\omega_i^\circ\}_{i=1}^n] \in \mathbb{X} \times [H^{-1}(\Omega)]^n \times [L^\infty(\Omega)]^n$, s.t.:

$$\left(u^\circ + p^\circ, h - u^\circ\right)_{\mathbb{X}} \geq 0, \ \forall h \in \mathcal{U}_{\rm ad} \ (\sigma_{*,i} \leq h_i \leq \sigma_i^*), \ \ v^\circ + z^\circ = 0 \ \text{in } \mathbb{X},$$

$$\text{and } \ \omega_i^\circ \in \mathrm{Sgn}(\nabla \theta_i^\circ) \text{ a.e. in } \Omega,$$

$$\begin{cases} \dfrac{1}{\tau}(p_i^\circ - p_{i+1}^\circ) - \Delta p_i^\circ + \left(g'(\eta_i^\circ) + \alpha''(\eta_i^\circ)|\nabla \theta_{i-1}^\circ|\right)p_i^\circ + \alpha'(\eta_i^\circ)\xi_i^\circ \\ \qquad + \dfrac{1}{\tau}\alpha_0'(\eta_i^\circ)(\theta_{i+1}^\circ - \theta_i^\circ)z_{i+1}^\circ = \eta_i^\circ - \eta_{\mathrm{ad},i} \ \text{ in } X, \\ \dfrac{1}{\tau}(\alpha_0(\eta_{i-1}^\circ)z_i^\circ - \alpha_0(\eta_i^\circ)z_{i+1}^\circ) + \zeta_i^\circ - \mathrm{div}\left(\nu^2 \nabla z_i^\circ + \alpha'(\eta_{i+1}^\circ)\omega_i^\circ p_{i+1}^\circ\right) = \theta_i^\circ - \theta_{\mathrm{ad},i} \ \text{ in } H^{-1}(\Omega) = Y_0^*, \\ \nabla p_i^\circ \cdot n_\Gamma = 0, \ \ z_i^\circ = 0 \ \text{ on } \Gamma, \text{ for any } i = n, \dots, 3, 2, 1, \\ p_{n+1}^\circ = z_{n+1}^\circ = 0, \ \text{in } \Omega. \end{cases}$$

Keypoint: • $\xi_i^\circ \sim \frac{D\theta_i^\circ}{|D\theta_i^\circ|} \cdot \nabla z_i^\circ$, $\quad \zeta_i^\circ \sim -\mathrm{div}\left(\alpha(\eta_i^\circ)[\nabla \mathrm{Sgn}]\left(\nabla \theta_i^\circ\right)\nabla z_i^\circ\right)$

- estimate of perturbed Poisson eq. to have strong convergence $p_\varepsilon^* \to p^\circ$ in $\mathbb{X}$ (H^1-boundedness)
  $\Longrightarrow$ we obtain the limiting necessary condition of variational inequality
- the necessary condition of equality, and the linearity of adjoint system
  $\Longrightarrow$ we need only weak-convergences $p_\varepsilon^* \to p^\circ$ weakly in Y, $z_\varepsilon^* \to z^\circ$ weakly in Y_0 (H^1-boundedness)

5. Precise observation in 1D case

$\Omega := (0,1) \subset \mathbb{R}$ (1D-domain), $\Gamma = \partial\Omega = \{0,1\}$, $\quad X := L^2(\Omega)$, $\quad \mathbb{X} := [X]^n$

Problem (OP)$_\varepsilon$ ($\varepsilon \geq 0$): to find $[u^*, v^*] = [\{u_i^*\}_{i=1}^n, \{v_i^*\}_{i=1}^n] \in [\mathbb{X}]^2$, called optimal control, s.t.

$$[u^*, v^*] = \text{arg-min}\mathcal{J}_\varepsilon = \mathcal{J}_\varepsilon(u,v) \text{ on } [\mathbb{X}]^2 \text{ (constraint-free setting)},$$

with a cost functional $\mathcal{J} : [u,v] \in [\mathbb{X}]^2 \mapsto \mathcal{J}(u,v) \in [0,\infty)$, defined as

$$\mathcal{J}(u,v) := \frac{1}{2}\left|[\eta,\theta] - [\eta_{\rm ad}, \theta_{\rm ad}]\right|_{[\mathbb{X}]^2}^2 + \frac{1}{2}\left|[u,v]\right|_{[\mathbb{X}]^2}^2.$$

State-system (S)$_\varepsilon$:

$$\begin{cases} \frac{1}{\tau}(\eta_i - \eta_{i-1}) - \partial_x^2 \eta_i + g(\eta_i) + \alpha'(\eta_i)|\partial_x \theta_{i-1}| = u_i \text{ in } \Omega, \\ \frac{1}{\tau}\alpha_0(\eta_{i-1})(\theta_i - \theta_{i-1}) - \partial_x\left(\alpha(\eta_i)\partial f_\varepsilon(\partial_x\theta_i) + \nu^2 \partial_x \theta_i\right) \ni v_i \text{ in } \Omega, \\ \partial_x \eta_i = \theta_i = 0 \text{ on } \Gamma, i = 1,2,3,\ldots,n, \\ \eta_0 \in H^1(\Omega),\ \theta_0 \in H_0^1(\Omega). \end{cases}$$

†. The one-dimensional embedding $H^1(\Omega) \subset C(\overline{\Omega})$ enables to remove the constraint for the temperature $u = \{u_i\}_{i=1}^n \in \mathbb{X}$

Proposition 3 (cf. [Rybka, Mucha](2013), [Kubota](2021))

Let us fix $0 \leq \beta_1 \in Y$ and $0 < \beta_2 \in Y$, and let us set the three convex functionals V_{β_1}, W_{β_2}, and Φ_{β_1,β_2}, defined as follows, respectively:

$$z \in X \mapsto V_{\beta_1}(z) := \sup \left\{ \int_\Omega z \partial_x \varphi \, dx \;\middle|\; \begin{array}{l} \varphi \in Y \cap C_c(\Omega), \text{ such that} \\ |\varphi| \leq \beta_1 \text{ on } \overline{\Omega} \end{array} \right\} \sim \int_\Omega \beta_1 |\partial_x z|,$$

$$z \in X \mapsto W_{\beta_2}(z) := \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega \beta_2 |\partial_x z|^2 dx, & \text{if } z \in Y, \\ \infty, & \text{otherwise,} \end{cases}$$

$$z \in X \mapsto \Phi_{\beta_1,\beta_2}(z) := V_{\beta_1}(z) + W_{\beta_2}(z).$$

Then, the subdifferential $\partial\Phi_{\beta_1,\beta_2} \subset X \times X$ of the convex function Φ_{β_1,β_2} is decomposed as follows:

$$\partial\Phi_{\beta_1,\beta_2} = \partial V_{\beta_1} + \partial W_{\beta_2} \text{ in } X \times X.$$

$\dagger_1$. Applying this Proposition to the case when $\beta_1 = \alpha(\eta_i)$, $\beta_2 \equiv \nu^2$,

$\theta_i \in H^2(\Omega)$, and $-\partial_x(\alpha(\eta_i)\omega_i^* + \nu^2 \partial_x \theta_i) = -\partial_x(\alpha(\eta_i)\omega_i^*) - \nu^2 \partial_x^2 \theta_i$ in X, with $\omega_i^* \in \partial f_\varepsilon(\partial_x \theta_i)$ a.e. in Ω.

Proposition 4 (H^2-regularity of the solution θ)

(I) Under (A0)–(A5), $\varepsilon \geq 0$ and $[u, v] \in [\mathbb{X}]^2$, the state-system $(\mathrm{S})_\varepsilon$ admits a unique solution $[\eta, \theta]$, defined as follows:

(S0) $\eta_i \in H^2(\Omega)$ and $\theta_i \in H^2(\Omega)$, $i = 1, 2, 3, \ldots, n$

(S1) $\dfrac{1}{\tau}(\eta_i - \eta_{i-1}) - \partial_x^2 \eta_i + g(\eta_i) + \alpha'(\eta_i) f_\varepsilon(\partial_x \theta_{i-1}) = u_i$ in Ω,

subject to $\partial_x \eta_i = 0$ on Γ, for any $i = 1, 2, 3, \ldots, n$, and $\eta_0 \in Y$

(S2) $\dfrac{1}{\tau}\alpha_0(\eta_{i-1})(\theta_i - \theta_{i-1}) - \partial_x \left(\alpha(\eta_i)\omega_i^*\right) - \nu^2 \partial_x^2 \theta_i = v_i$ in Ω,

with $\omega_i^* \in Y \cap L^\infty(\Omega)$ satisfying $\omega_i^* \in \partial f_\varepsilon(\partial_x \theta_i)$ a.e. in Ω,

subject to $\theta_i = 0$ on Γ, for any $i = 1, 2, 3, \ldots, n$, and $\theta_0 \in Y_0$

(II) Under (A0)–(A5), let us define:

$$\mathcal{S}_\varepsilon : [u, v] \in [\mathbb{X}]^2 \mapsto [\eta_\varepsilon, \theta_\varepsilon] := \mathcal{S}_\varepsilon[u, v] : \text{ the solution to } (\mathrm{S})_\varepsilon, \text{ for } \varepsilon \geq 0$$

Then,

$$\{\varepsilon_m\}_{m=1}^\infty \subset (0, 1],\ \varepsilon_m \to \varepsilon,\ [u_m, v_m] \to [u, v] \text{ weakly in } [\mathbb{X}]^2, \text{ as } m \to \infty$$
$$\Longrightarrow [\eta_m, \theta_m] := \mathcal{S}_{\varepsilon_m}[u_m, v_m] \to [\eta, \theta] := \mathcal{S}_\varepsilon[u, v] \text{ in } ([Y]^n \cap [C^1(\overline{\Omega})]^n) \times ([Y_0]^n \cap [C^1(\overline{\Omega})]^n),$$
$$\text{and weakly in } [H^2(\Omega)]^n \times [H^2(\Omega)]^n \ (\partial_x \theta_m \to \partial_x \theta \text{ in } C(\overline{\Omega})), \text{ as } m \to \infty$$

Theorem D (A precise characterization of the limiting necessary condition of optimality)

Under (A0)–(A5), there exist an optimal control $[u^\circ, v^\circ] \in [\mathbb{X}]^2$ for $(\mathrm{OP})_0$, $[\eta^\circ, \theta^\circ] = \mathcal{S}_0[u^\circ, v^\circ]$, and $[\xi^\circ, \zeta^\circ, \omega^\circ] = [\{\xi_i^\circ\}_{i=1}^n, \{\zeta_i^\circ\}_{i=1}^n, \{\omega_i^\circ\}_{i=1}^n] \in \mathbb{X} \times [H^{-1}(\Omega)]^n \times [L^\infty(\Omega)]^n$, s.t.:

$$[u^\circ, v^\circ] = -[p^\circ, z^\circ] \text{ in } [\mathbb{X}]^2 \text{ and } \ \omega_i^\circ \in \mathrm{Sgn}(\partial_x \theta_i^\circ) \text{ a.e. in } \Omega,$$

$$\begin{cases} \dfrac{1}{\tau}(p_i^\circ - p_{i+1}^\circ) - \partial_x^2 p_i^\circ + \big(g'(\eta_i^\circ) + \alpha''(\eta_i^\circ)|\partial_x \theta_{i-1}^\circ|\big) p_i^\circ + \alpha'(\eta_i^\circ)\xi_i^\circ \\ \qquad + \dfrac{1}{\tau}\alpha_0'(\eta_i^\circ)(\theta_{i+1}^\circ - \theta_i^\circ) z_{i+1}^\circ = \eta_i^\circ - \eta_{\mathrm{ad},i} \ \text{ in } \Omega, \\ \dfrac{1}{\tau}(\alpha_0(\eta_{i-1}^\circ) z_i^\circ - \alpha_0(\eta_i^\circ) z_{i+1}^\circ) + \zeta_i^\circ - \partial_x\big(\nu^2 \partial_x z_i^\circ + \alpha'(\eta_{i+1}^\circ)\omega_i^\circ p_{i+1}^\circ\big) = \theta_i^\circ - \theta_{\mathrm{ad},i} \ \text{ in } \Omega, \\ \partial_x p_i^\circ = z_i^\circ = 0 \ \text{ on } \Gamma, \text{ for any } i = n, \dots, 3, 2, 1, \quad p_{n+1}^\circ = z_{n+1}^\circ = 0, \text{ in } \Omega. \end{cases}$$

Keypoints: Under 1D-setting,

- the distribution ζ° is formally expressed by:

$$\zeta_i^\circ \sim -\partial_x\big[\alpha(\eta_i)\mathfrak{d}(\partial_x \theta_i^\circ)\partial_x v_i^\circ\big] \ \text{ in } \mathscr{D}'(\Omega), \text{ and } \mathrm{spt}\zeta_i^\circ \sim \{\partial_x \theta_i^\circ = 0\}, \text{ by using Dirac's delta } \mathfrak{d}$$

- as $\varepsilon \downarrow 0$, the limiting component $\partial_x \theta^\circ \in C(\overline{\Omega})$ is approached in the uniform topology on $\overline{\Omega}$
- the set $\{\partial_x \theta_i^\circ = 0\}$ corresponds to a closed region of locally constant parts (crystalline facets on grains)
  the set $\{\partial_x \theta_i^\circ \neq 0\}$ is an open set, corresponding to a neighborhood of grain boundary

Theorem D (A precise characterization of the limiting necessary condition of optimality)

Let us take any $\rho \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ with $\rho(0) = \rho'(0) = 0$. Then,

$$\rho(\partial_x \theta_i^\circ) \Bigl(\frac{1}{\tau}(p_i^\circ - p_{i+1}^\circ) - \partial_x^2 p_i^\circ + \bigl(g'(\eta_i^\circ) + \alpha''(\eta_i^\circ)|\partial_x \theta_{i-1}^\circ|\bigr) p_i^\circ + \alpha'(\eta_i^\circ)\omega_i^\circ \partial_x z_i^\circ$$
$$+ \frac{1}{\tau}\alpha_0'(\eta_i^\circ)(\theta_{i+1}^\circ - \theta_i^\circ) z_{i+1}^\circ - (\eta_i^\circ - \eta_{\mathrm{ad},i}) \Bigr) = 0 \text{ in } X,$$

$$\rho(\partial_x \theta_i^\circ) \left(\frac{1}{\tau}(\alpha_0(\eta_{i-1}^\circ) z_i^\circ - \alpha_0(\eta_i^\circ) z_{i+1}^\circ) - \nu^2 \partial_x^2 z_i^\circ - \alpha'(\eta_{i+1}^\circ)\omega_i^\circ p_{i+1}^\circ - (\theta_i^\circ - \theta_{\mathrm{ad},i}) \right) = 0 \text{ in } X,$$

$$\xi_i^\circ = \omega_i^\circ \partial_x z_i^\circ \text{ and } \zeta_i^\circ = 0 \text{ in } \mathscr{D}'(\{\partial_x \theta_i^\circ \neq 0\}), \text{ with } \omega_i^\circ \in \mathrm{Sgn}(\nabla \theta_i^\circ) \text{ in } \Omega, \text{ for } i = n, \ldots, 3, 2, 1.$$

Therefore,

$$\begin{cases} \frac{1}{\tau}(p_i^\circ - p_{i+1}^\circ) - \partial_x^2 p_i^\circ + \bigl(g'(\eta_i^\circ) + \alpha''(\eta_i^\circ)|\partial_x \theta_{i-1}^\circ|\bigr) p_i^\circ + \alpha'(\eta_i^\circ) \dfrac{\partial_x \theta_i^\circ}{|\partial_x \theta_i^\circ|} \partial_x z_i^\circ \\ \quad + \frac{1}{\tau}\alpha_0'(\eta_i^\circ)(\theta_{i+1}^\circ - \theta_i^\circ) z_{i+1}^\circ = \eta_i^\circ - \eta_{\mathrm{ad},i}, \\ \frac{1}{\tau}(\alpha_0(\eta_{i-1}^\circ) z_i^\circ - \alpha_0(\eta_i^\circ) z_{i+1}^\circ) - \nu^2 \partial_x^2 z_i^\circ - \alpha'(\eta_{i+1}^\circ) \dfrac{\partial_x \theta_i^\circ}{|\partial_x \theta_i^\circ|} p_{i+1}^\circ = \theta_i^\circ - \theta_{\mathrm{ad},i}, \end{cases} \quad \text{in } \{\partial_x \theta_i^\circ \neq 0\}$$

♢ Sketch of the proof (2nd eq. of the adjoint system):

$\forall i \in \{1, \dots, n\}$, let us take $\psi \in Y_0$, and test 2nd eq. by $\rho(\partial_x \theta_i^\circ)\psi \in Y_0$ $(\theta_i^\circ \in H^2(\Omega))$:

$$\text{(principal part) } I_\varepsilon^\circ := \int_\Omega \left(\alpha(\eta_{\varepsilon,i}^*) f_\varepsilon''(\partial_x \theta_{\varepsilon,i}^*) \partial_x z_{\varepsilon,i}^*\right) \cdot \partial_x \left(\rho(\partial_x \theta_i^\circ)\psi\right) dx$$

$$= \int_{\mathrm{spt}\rho(\partial_x \theta_i^\circ)} \partial_x z_{\varepsilon,i}^* \cdot \underbrace{\boxed{\alpha(\eta_{\varepsilon,i}^*) \frac{\varepsilon^2}{\sqrt{\varepsilon^2 + |\partial_x \theta_{\varepsilon,i}^*|^2}^3} \left(\rho'(\partial_x \theta_i^\circ)\partial_x^2 \theta_i^\circ \psi + \rho(\partial_x \theta_i^\circ)\partial_x \psi\right)}}_{(*1)_\varepsilon} dx$$

(Step 1): the case when $\mathbf{0 \notin K^\circ := \mathrm{spt}\rho}$, i.e. $\exists \delta^\circ > 0$ s.t. $K^\circ \cap (-\delta^\circ, \delta^\circ) = \emptyset$

- uniform convergence on $\overline{\Omega}$ of $\eta_{\varepsilon,i}^* \to \eta_i^\circ$, and $\partial_x \theta_{\varepsilon,i}^* \to \partial_x \theta_i^\circ$:

$$\exists \varepsilon^\circ > 0 \text{ s.t. } |\partial_x \theta_{\varepsilon,i}^*| \geq \delta^\circ/2, \text{ uniformly on } \mathrm{spt}\rho(\partial_x \theta_i^\circ), \ \forall \varepsilon \in (0, \varepsilon^\circ)$$

$$\Longrightarrow \left|(*1)_\varepsilon\right|_X \leq \text{Const.} \frac{\varepsilon^2}{\sqrt{\varepsilon^2 + |\partial_x \theta_{\varepsilon,i}|^2}^3} \left(|\partial_x^2 \theta_i^\circ|_X + |\partial_x \psi|_X\right) \leq \text{Const.} \frac{2}{\delta^\circ} \left(|\partial_x^2 \theta_i^\circ|_X + |\partial_x \psi|_X\right) \to 0, \text{ as } \varepsilon \downarrow 0$$

$$\Longrightarrow I_\varepsilon^\circ \to 0, \text{ as } \varepsilon \downarrow 0$$

(Step 2): the case when $\mathbf{0 \in K^\circ := \mathrm{spt}\rho}$ $\ (\rho(0) = \rho'(0) = 0)$

This case is obtained by means of approximating argument of ρ in $W^{1,\infty}(\mathbb{R})$ □

6. Future problems

(I) Optimal control problems for anisotropic Kobayashi–Warren–Carter system

Issue : 2D state-system with crystalline anisotropy

(II) Optimal control problems for WKLC system (cf. [Warren–Kobayashi–Lobkovsky –Carter] (2003))

Issue : state-system of "Fix–Caginalp model of phase transition" VS. "Kobayashi–Warren–Carter system"

(III) Generalization of boundary conditions

Issue : unification of the methods for nonhomogeneous Dirichlet / Neumann / Robin B.C., and dynamic B.C.

(IV) Issues for time-discrete state-systems in higher dimension

Issue : expression of ξ_i° and ζ_i°