Blowup solutions and vanishing estimates for singular Liouville equations

joint work with Juncheng Wei, UBC

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The singular Liouville equation

In this talk I only talk about a very short equation defined in two dimensional spaces:

$$\Delta v + h(x)e^{v(x)} = 4\pi\alpha\delta_0, \quad \text{in} \quad B_1 \subset \mathbb{R}^2.$$

where h is a positive smooth function and B_1 is the unit ball, δ_0 is a Dirac mass placed at the origin and $\alpha > -1$. Since

$$\Delta(\frac{1}{2\pi}\log|x|) = \delta_0,$$

Setting $u(x) = v(x) - 2\alpha\log|x|$ we have

$$\Delta u + |x|^{2\alpha}h(x)e^{u(x)} = 0.$$

Background

Nirenberg problem: which smooth functions K on $\mathbb{S}^2$ are realized as the Gauss curvature of a metric g on $\mathbb{S}^2$ pointwise conformal to the standard round metric g_0 of $\mathbb{S}^2 \subset \mathbb{R}^3$? For $g = e^{2u} g_0$ the equation for the Gauss curvature K of g is

$$\Delta u + K e^{2u} = 1 \tag{1}$$

so that the Nirenberg problem asks to characterize for which K is the nonlinear PDE (1) solvable.

If in a neighborhood of one point, the metric can be written as

$$g = e^h |z|^{2\alpha} |dz|^2,$$

we say at this point it has a conical singularity of order α. The corresponding PDE to study is

$$\Delta u + K(x)|x|^{2\alpha} e^u = 0.$$

classification of global solutions

Theorem

(Chen-Li 94) Let u be a solution of

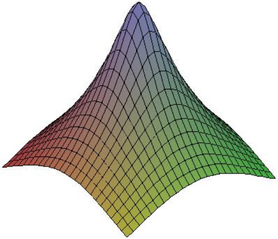
$$\Delta u + e^u = 0, \quad in \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u < \infty,$$

then

$$u(x) = \log \frac{e^\lambda}{(1 + \frac{e^\lambda}{8}|x - x_0|^2)^2}$$

for some $\lambda \in \mathbb{R}$ and $x_0 \in \mathbb{R}^2$.

$\int_{\mathbb{R}^2} e^u = 8\pi$. If $v(y) = u(\delta y) + 2\log\delta$, then $\int_{\mathbb{R}^2} e^v = \int_{\mathbb{R}^2} e^u$.



local blowup for regular equation

Let u_k be a sequence of bubbling solutions of

$$\Delta u_k + he^{u_k} = 0, \quad \text{in} \quad B_1,$$

where h is a positive smooth function. If

1. $$\max_x u_k(x) = u_k(0) \to \infty, \quad \text{and} \quad \max_{K \subset\subset B_1 \setminus \{0\}} u_k \leq C(K)$$

2. $$\int_{B_1} he^{u_k} \leq C,$$

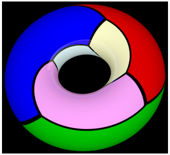
3. $$|u_k(x) - u_k(y)| \leq C, \quad \forall x, y \in \partial B_1,$$

Theorem

(Y.Y.Li, 95 CMP) Suppose $\lambda_k = u_k(0) = \max u_k \to \infty$*, then*

$$u_k(x) - \log \frac{e^{\lambda_k}}{(1 + \frac{e^{\lambda_k} h(0)}{8}|x|^2)^2} = O(1), \quad \forall x \in B_1.$$

Uniform Estimate



If we consider a mean field equation on a surface, say

$$\Delta_g u + \rho(\frac{he^u}{\int_M he^u} - 1) = 0. \quad vol(M) = 1.$$

The uniform estimate implies

1. Around each blowup point, there is only one bubble profile: $he_k^u \rightharpoonup 8\pi\delta_p$
2. The height of bubbles are roughly the same.
3. The energy ($\int_M he^{u_k}$) is concentrated around a few blowup points.
4. But this estimate did not tell the locations of the blowup points.
5. The error of the estimate is still too large for some applications.

We postulated a boundary oscillation finite-ness assumption:

$$|u_k(x) - u_k(y)| \leq C, \quad \text{for} \quad x, y \in \partial B_1.$$

This assumption is quite natural and important.

Theorem

(Xiuxiong Chen, 99) If the boundary oscillation assumption is removed, for any $m \in \mathbb{N}$, there exists u_k of solutions

$$\Delta u_k + e^{u_k} = 0, \quad in \quad B_1$$

such $e^{u_k} \rightharpoonup 8\pi m\delta_0$.

It is convenient to consider a harmonic function defined by the oscillation of u_k on the boundary:

$$\begin{cases} \Delta\psi_k = 0, \quad \text{in} \quad B_1, \\ \psi_k(x) = u_k(x) - \frac{1}{2\pi}\int_{\partial B_1} u_k dS, \quad x \in \partial B_1. \end{cases}$$

Then $\psi_k(0) = 0$ and all derivatives of ψ_k are bounded in $B_{1/2}$.
Let $\epsilon_k = e^{-\frac{1}{2}u_k(0)}$, where $u_k(0) = \max u_k$, then we have

Refined estimate

Theorem

(Chen-Lin 02, Z 06, Gluck 12)

$$\begin{aligned} u_k(x) &= \log \frac{e^{u_k(0)}}{(1+\frac{h(0)}{8}e^{u_k(0)}|x-q_k|^2)^2} + \psi_k \\ &\quad -8\frac{(\Delta \log h)(0)}{h(0)}\epsilon_k^2(\log(2+\epsilon_k^{-1}|x|))^2 + O(\epsilon_k^2 \log \epsilon_k^{-1}) \end{aligned}$$

where $q_k = 2\epsilon_k^2 \nabla h(0)/h^2(0) + O(\epsilon_k^3)$ *is the maximum point of* $u_k - \psi_k$,

$$|\nabla(\log h + \psi_k)(q_k)| = O(\epsilon_k^2(\log \epsilon_k^{-1})).$$

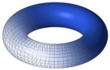
Application

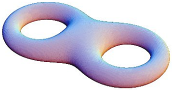
Theorem

(C.C.Chen-C.S.Lin CPAM 03) Suppose u is a solution of the following mean field equation on (M, g) (volume of $M = 1$)

$$\Delta_g u + \rho\left(\frac{he^u}{\int_M he^u dV_g} - 1\right) = 0$$

If $\rho > 0$ is not a multiple of 8π and the genus of M is greater than 0, then the equation has a solution.





- if $8\pi N < \rho < 8\pi(N+1)$ we have $|u| < C$
- $$T_\rho = -\rho\Delta_g^{-1}\left(\frac{he^u}{\int_M he^u} - 1\right)$$
- $$d_\rho := deg(I - T_\rho, B_R, 0)$$

  is well defined for $\rho \neq 8N\pi$.

Theorem

(Chen-Lin 02, 03, CPAM)

$$d_\rho = \begin{cases} 1 & \rho < 8\pi, \\ \frac{(-\chi_M+1)\dots(-\chi_M+N)}{N!} & 8N\pi < \rho < 8(N+1)\pi. \end{cases}$$

$\chi(M) = 2 - 2g_e$, the g_e is the genus of the manifold, which is the number of handles. $\rho = 8\pi$, Lin-Wang published a paper on Annals.

Classification Theorem

Theorem

(Prajapat-Tarantello 01) If $\alpha > -1$ *is not an integer, all solutions to*

$$\Delta u + |x|^{2\alpha} e^u = 0, \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{2\alpha} e^u < \infty,$$

are radially symmetric and can be written as

$$u(x) = \log \frac{e^\lambda}{(1 + \frac{e^\lambda}{8(1+\alpha)^2} |x|^{2+2\alpha})^2}$$

for some $\lambda \in \mathbb{R}$*. The total integration is*

$$\int_{\mathbb{R}^2} |x|^{2\alpha} e^u = 8\pi(1+\alpha).$$

Non-quantized singularity

Theorem

(Bartolucci-Chen-Lin-Tarantello 05) Let u_k be blowup solutions to

$$\Delta u_k + |x|^{2\alpha} h e^{u_k} = 0, \quad B_1$$

with $\alpha > -1$ and bounded oscillation on ∂B_1. Suppose 0 is the only blowup point in B_1, then

$$h e^{u_k} \rightharpoonup 8\pi(1+\alpha)\delta_0$$

and if α is not a positive integer

$$u_k(x) - \log \frac{e^{u_k(0)}}{(1 + \frac{h(0)}{8(1+\alpha)^2} e^{u_k(0)} |x|^{2\alpha+2})^2} = O(1) \quad B_1.$$

Non-quantized singularity

Theorem

(Z 09) Suppose $\alpha > 0$ *is not a positive integer then*

$$
\begin{aligned}
u_k(x) = & \log \frac{e^{u_k(0)}}{\left(1+\frac{h(0)}{8(1+\alpha)^2} e^{u_k(0)} |x|^{2\alpha+2}\right)^2} + \phi_k(x) \\
& - \frac{2(1+\alpha)}{\alpha h(0)} \frac{\nabla h(0) \cdot x}{1+\frac{h(0)}{8(1+\alpha)^2} e^{u_k(0)} |x|^{2\alpha+2}} \\
& + \left(\Lambda_1 \Delta h(0) + \Lambda_2 |\nabla h(0)|^2\right) \log\left(2 + e^{\frac{u_k(0)}{2(1+\alpha)}} |x|\right) e^{-\frac{u_k(0)}{1+\alpha}} \\
& + O(e^{-\frac{u_k(0)}{1+\alpha}}),
\end{aligned}
$$

where ϕ_k *is the harmonic function that eliminates the oscillation of* u_k *on* ∂B_1, $\Lambda_1 = -\frac{\pi}{h(0)\sin\left(\frac{\pi}{1+\alpha}\right)(1+\alpha)} \left(\frac{8(1+\alpha)^2}{h(0)}\right)^{\frac{1}{1+\alpha}}$,

$\Lambda_2 = \frac{\pi}{h^2(0)\sin(\frac{\pi}{1+\alpha})(1+\alpha)} \left(\frac{8(1+\alpha)^2}{h(0)}\right)^{\frac{1}{1+\alpha}}$.

Classification Theorem

Theorem

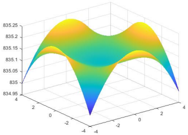
(Prajapat-Tarantello 01) All solutions of

$$\Delta u + |x|^{2N} e^u = 0, \quad \textit{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{2N} e^u < \infty,$$

are of the form

$$u(z) = \log \frac{e^\lambda}{\left(1 + \frac{e^\lambda}{8(1+N)^2} |z^{N+1} - \xi|^2\right)^2}$$

for some $\xi \in \mathbb{C}$. $\int_{\mathbb{R}^2} |x|^{2N} e^u = 8\pi(1+N)$.



Quantized singularity, Non-simple blowup

Let u_k be a sequence of solutions to

$$\Delta u_k + |x|^{2N} h(x) e^{u_k} = 0, \quad \text{in} \quad B_1 \subset \mathbb{R}^2,$$

where $h > 0$ is smooth. Suppose 0 is the only blowup point and N is a positive integer, u_k has bounded oscillation on ∂B_1.

Theorem

(Kuo-Lin 16, JDG, Bartolucci-Tarantello 18) If $N \in \mathbb{N}$, it is possible that

$$\max_{x \in B_1} u_k(x) + 2(1+N)\log|x| \to \infty.$$

When this happens, u_k has exactly $N+1$ local maximum points evenly distributed around 0.

Related questions

- Is it possible to approximate bubbling solutions by global solutions?
- Are there vanishing theorems? Especially the vanishing estimate of first derivatives of coefficient functions?

Vanishing Theorems

Theorem

(Wei-Z, 2022) Let u_k be non-simple blowup solutions under the usual assumptions. Then along a subsequence

$$\lim_{k\to\infty} \nabla(\log h_k + \phi_k)(0) = 0.$$

$$\lim_{k\to\infty} \Delta(\log h_k)(0) = 0.$$

The "non-simple" assumption cannot be removed.

Theorem

(Wu, 2022). There exists a sequence of blowup solutions that satisfies the spherical Harnack inequality around a blowup point with non-vanishing coefficients.

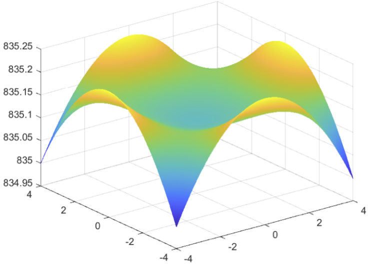
Local maximum points

Let p_0^k, ..., p_N^k be the $N+1$ local maximums of u_k

$$\Delta u_k + |x|^{2N} h_k(x) e^{u_k} = 0, \quad \text{in} \quad B_1.$$

Let

$$\delta_k = |p_0^k|, \quad \mu_k = u_k(p_0^k) + 2(1+N)\log\delta_k.$$



Trivial Observations

- The study of blowup solutions looks like that of a single Liouville equation near each local maximum point.
- The relations between these local maximums plays a crucial role.
- The blowup solutions look almost like a harmonic function away from the $N+1$ local maximums.
- If there is a perturbation on a global solution, there is a corresponding perturbation on each of its $N+1$ local maximumus:

$$V_k(x) = \log \frac{e^{\mu_k}}{(1+\frac{e^{\mu_k}}{8(N+1)^2}|x^{N+1}-(1+p_k)|^2)^2}.$$

Stage 1: First Vanishing Theorems

Theorem

(Wei-Z, Advances in Math, 21) Let ϕ_k be the harmonic function that eliminates the oscillation of u_k on ∂B_1, then

$$|\nabla(\log h_k + \phi_k)(0)| = O(\delta_k^{-1}\mu_k e^{-\mu_k}) + O(\delta_k).$$

$$\Delta(\log h_k)(0) = O(\delta_k^{-2}\mu_k e^{-\mu_k}) + O(\delta_k), \quad N \geq 2.$$

Obviously we don't know if $\nabla(\log h_k + \phi_k)(0) = o(1)$ when $\delta_k \leq C\mu_k e^{-\mu_k}$. We cannot tell if $\Delta h_k(0) = o(1)$ if $\delta_k \leq C\mu_k^{\frac{1}{2}} e^{-\mu_k/2}$ even for $N \geq 2$. The conclusion for $N = 1$ is even weaker.

Theorem

(Wei-Z, Advances 21) For $N = 1$,

$$(\partial_{e_k}(\log h_k)(0))^2 - (\partial_{e_k^\perp}(\log h_k)(0))^2 - 2\partial_{e_k e_k^\perp}(\log h_k)(0) = E_k,$$
$$\partial_{e_k}(\log h_k)(0)\partial_{e_k^\perp}(\log h_k)(0) - \partial_{e_k e_k^\perp}(\log h_k)(0) = E_k.$$

where $E_k = O(\delta_k^{-2}\mu_k e^{-\mu_k}) + O(\delta_k)$. e_k *is the direction determined by the two local maximum points.*

Theorem

(Wei-Z Advances 21) If $\delta_k \le Ce^{-\mu_k/4}$, *then there exists a sequence of global solutions* U_k *such that*

$$|u_k(x) - U_k(x)| \le C, \quad x \in B_1.$$

For $|x| \sim 1$, $u_k(x) = -u_k(p_0^k) + O(1)$.

Linearized equation

Let U be the solution of

$$\Delta U + 8e^U = 0, \quad \text{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^U < \infty,$$

with $\max_x U(x) = 1 = U(0)$. By Chen-Li, $U(x) = \log \frac{1}{(1+|x|^2)^2}$. Let ϕ be a solution of

$$\Delta \phi + 8e^U \phi = 0, \quad \text{in} \quad \mathbb{R}^2$$

with $\phi(x) = o(|x|)$ at infinity. Then $\phi(x) = c_0\phi_0 + c_1\phi_1 + c_2\phi_2$ where

$$\phi_0 = \frac{1 - |x|^2}{1 + |x|^2}, \quad \phi_1(x) = \frac{x_1}{1 + |x|^2}, \quad \phi_2 = \frac{x_2}{1 + |x|^2}.$$

key ideas of the proof

Step one: A lot of Pohozaev identities.

- A Pohozaev identity for $\Delta u_k + h_k e^{u_k} = 0$ on B_σ is

$$\int_{B_\sigma} (\nabla h_k \cdot x) e^{u_k} = \int_{\partial B_\sigma} \left(\frac{\sigma}{2}(|\partial_\nu u_k|^2 - |\partial_\tau u_k|^2) + \sigma h_k e^{u_k} + 2\partial_\nu u_k \right) dS.$$

-

$$\delta_k \nabla(\log h_k)(\delta_k Q_l^k) + 2N \frac{Q_l^k}{|Q_l^k|^2} + \nabla \phi_{l,k}(Q_l^k) = O(\mu_k e^{-\mu_k}).$$

-

$$\nabla \phi_l^k(Q_l^k) = -4 \sum_{m \neq l} \frac{Q_l^k - Q_m^k}{|Q_l^k - Q_m^k|^2} + O(\delta_k^2) + O(\mu_k e^{-\mu_k}).$$

- Denoting $Q_l^k = e^{i\frac{2\pi l}{N+1}}(1+m_l^k)$ and use this in the long computation of each Pohozaev identity, we have
- $$\begin{pmatrix} m_1^k \\ m_2^k \\ \vdots \\ m_N^k \end{pmatrix} = A^{-1}\delta_k \bar{\nabla}(\log h_k)(0) \begin{pmatrix} e^{i\beta_1} \\ e^{i\beta_2} \\ \vdots \\ e^{i\beta_N} \end{pmatrix} + O(\delta_k^2) + O(\mu_k e^{-\mu_k})$$

  where $\beta_l = 2\pi l/(N+1)$, $l = 0, ..., N$.

$$A = \begin{pmatrix} D & -d_1 & ... & -d_{N-1} \\ -d_1 & D & ... & -d_{N-2} \\ \vdots & \vdots & ... & \vdots \\ -d_{N-1} & -d_{N-2} & ... & D \end{pmatrix}$$

where

$$d_i = \frac{1}{\sin^2(\frac{i\pi}{N+1})}, \quad i = 1, ..., N, \quad D = d_1 + ... + d_N.$$

With the estimate of $(m_1^k, ..., m_N^k)$, $\nabla\phi_l^k(Q_l^k)$ can be improved to

$$\nabla\phi_l^k(Q_l^k) = -4\sum_{m\neq l}\frac{Q_l^k - Q_m^k}{|Q_l^k - Q_m^k|^2} + O(\delta_k^3) + O(\mu_k e^{-\mu_k}).$$

key ideas

1.Let $v_k(y) = u_k(\delta_k y) + 2\log\delta_k$. Since δ_k is the distance from a local maximum of v_k to the origin, and Δ is invariant under rotation of coordinates, we can assume that v_k has a local maximum at e_1. Then we use a global solution V_k that agrees with v_k at e_1:

$$\Delta V_k + h_k(\delta_k e_1)|y|^{2N} e^{V_k} = 0.$$

$$V_k(y) = \log \frac{e^{\mu_k}}{(1 + \frac{e^{\mu_k} h_k(\delta_k e_1)}{8(1+N)^2}|y^{N+1} - e_1|^2)^2}.$$

V_k has $N+1$ local maximums located at exactly $e^{2\pi i l/(N+1)}$ for $l = 0, ..., N$.

Let $w_k = v_k - V_k$. Then w_k is very small near e_1.

2. By Harnack inequality, this smallness will be passed to control all the regions away from the N other bubbling disks.

key ideas of the proof

3. The difference between the Pohozaev identities. Let Ω_s be the region about Q_s. Then the Pohozaev identity for v_k in this region is

$$\int_{\Omega_s} \partial_\xi(|y|^{2N} h_k(\delta_k y)) e^{v_k} - \int_{\partial\Omega_s} e^{v_k} |y|^{2N} h_k(\delta_k y)(\xi \cdot \nu) = \int_{\partial\Omega_s} (\partial_\nu v_k \partial_\xi v_k - \frac{1}{2}|\nabla v_k|^2 (\xi \cdot \nu)) dS.$$

$$\int_{\Omega_s} \partial_\xi(|y|^{2N} h_k(\delta_k e_1)) e^{V_k} - \int_{\partial\Omega_s} e^{V_k} |y|^{2N} h_k(\delta_k e_1)(\xi \cdot \nu) = \int_{\partial\Omega_s} (\partial_\nu V_k \partial_\xi V_k - \frac{1}{2}|\nabla V_k|^2 (\xi \cdot \nu)) dS.$$

Main ideas

Using these in the computation of the $N+1$ Pohozaev identities we have

$$\nabla(\log h_k + \phi_k)(0) = O(\delta_k^{-1}\mu_k e^{-\mu_k}) + O(\delta_k) \quad N \geq 1$$
$$\Delta \log h_k(0) = O(\delta_k^{-2}\mu_k e^{-\mu_k}) + O(\delta_k), \quad N \geq 2.$$

and a corresponding estimate for $N = 1$.

Stage 2, better first order estimates

Key ideas to prove the vanishing rate of $\nabla h_k(0)$ (for simplicity ϕ_k is ignored). Let $w_k = v_k - V_k$, then we have this key estimate:

$$|w_k(y)| \le C(|\nabla h_k|\delta_k + \delta_k^2 \mu_k).$$

Only need to consider $\delta_k \le o(\epsilon_k)$. The equation of w_k can be written as

$$\Delta w_k + h_k(\delta_k y)|y|^{2N} e^{\xi_k} w_k = \delta_k \nabla h_k(\delta_k e_1) \cdot (e_1 - y)|y|^{2N} e^{V_k} + E$$

where

$$E = O(\delta_k^2)|y - e_1|^2 |y|^{2N} e^{V_k}.$$

It is important to observe that the right hand side is zero when $y = e_1$. *The analysis is first carried out near e_1 and pass to other regions by Harnack inequality*

First order vanishing estimate

Let $M_k = \max|w_k|$ and let $\tilde{w}_k = w_k/M_k$. It is crucial to observe that we still have

$$\tilde{w}_k(e_1) = |\nabla \tilde{w}_k(e_1)| = 0.$$

This important information will make us obtain

$$\tilde{w}_k(e_1 + \epsilon_k z) \le C\epsilon_k^\sigma (1+|z|)^\sigma, \quad |z| < \epsilon_k^{-1}$$

where $\epsilon_k = e^{-\mu_k/2}$ and $\sigma \in (0,1)$. Because of the smallness of $|Q_s^k - e^{i\beta_s}|$, $\tilde{w}_k$ is supposed to converge to a kernel of

$$\Delta \phi + e^U \phi = 0$$

around each Q_s. The same argument can also be applied around each Q_s.

At Q_s^k, v_k is very close to another global solution V_s^k which agrees with v_k at Q_s^k and $\nabla V_s^k(Q_s^k)=0$. The expression of V_s^k, which satisfies

$$\Delta V_s^k + h_k(\delta_k Q_s^k)|y|^{2N} e^{V_s^k} = 0, \quad \text{in} \quad \mathbb{R}^2,$$

is

$$V_s^k(y) = \log \frac{e^{\mu_s^k}}{(1 + \frac{e^{\mu_s^k} h_k(\delta_k Q_s^k)}{8(1+N)^2}|y^{N+1} - (e_1 + p_s^k)|^2)^2}.$$

The function $\tilde{w}_k$ is supposed to converge

$$c_1 \frac{1-\frac{1}{8}|y|^2}{1+\frac{1}{8}|y|^2} + c_2 \frac{y_1}{1+\frac{1}{8}|y|^2} + c_3 \frac{y_2}{1+\frac{1}{8}|y|^2}.$$

All these coefficients are determined by $V_s^k - V_k$. It is standard to prove $c_1 = 0$. To prove c_2 and c_3 zero we need to use p_s.

If we take Q_s as a base and consider the kernel function around Q_l^k, then the limit function is supposed to be

$$c_{1,s,t}\frac{y_1}{1+\frac{1}{8}|y|^2}+c_{2,s,t}\frac{y_2}{1+\frac{1}{8}|y|^2}.$$

After some computations we have

$$c_{1,s,t}=\lim_{k\to\infty}\frac{|p_s^k-p_t^k|}{2(N+1)M_k\epsilon_k}\cos(\frac{2\pi s}{N+1}+\theta_{st}).$$

$$c_{2,s,t}=\lim_{k\to\infty}\frac{|p_s^k-p_t^k|}{2(N+1)M_k\epsilon_k}\sin(\frac{2\pi s}{N+1}+\theta_{st}).$$

where $p_s^k-p_t^k=|p_s^k-p_t^k|e^{i\theta_{ts}}$. If limit has to exist, p_1^k,....,p_N^k have to satisfy certain relations, which will lead to a contradiction if we observe the second order terms.

After proving

$$|w_k(y)| \leq C\delta_k|\nabla h(0)| + C\delta_k^2\mu_k,$$

we use this estimate in the computation of Pohozaev identities around each Q_s^k to obtain

$$|\nabla h_k(0)| \leq C\delta_k\mu_k.$$

Stage 3: Laplace Vanishing Theorem

The first order estimate leads to a better estimate on the difference function:

$$|w_k(y)| \le C\delta_k^2\mu_k,$$

Then we use Gluck's estimate for single Liouville equation around each Q_s^k ($s \neq 1$) to obtain the vanishing rate for $\Delta h_k(0)$. Recall the expansion of a blowup solution for a single Liouville equation:

$$\begin{aligned} u_k(x) &= \log \frac{e^{u_k(0)}}{(1+\frac{h(0)}{8}e^{u_k(0)}|x-q_k|^2)^2} + \psi_k \\ &\quad -8\frac{(\Delta \log h)(0)}{h(0)}\epsilon_k^2(\log(2+\epsilon_k^{-1}|x|))^2 + O(\epsilon_k^2 \log \epsilon_k^{-1}) \end{aligned}$$

Application to Toda systems

$$\Delta u_1 + 2\rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) - \rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0,$$
$$\Delta u_2 - \rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) + 2\rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0,$$

Theorem

(Lin-Wei-Z-Yang 18 APDE) For $(\rho_1, \rho_2) \in (4\pi m, 4\pi(m+1)) \times (4\pi n, 4\pi(n+1))$ *(*$n, m \in \mathbb{N}$*) and* $u = (u_1, u_2)$ *in certain Sobolev space, the following a priori estimate holds*

$$|u_i| \le C, \quad i = 1, 2.$$

This theorem leads to a huge degree counting program for Toda systems.

Theorem

(Wei-Wu-Z 22) If $u_k = (u_1^k, u_2^k)$ *is a sequence of blowup solutions corresponding to* $(\rho_1^k, \rho_2^k) \to (4\pi m, 4\pi n)$, *if one blowup point is a fully bubbling blowup point and*

$$\Delta_g \log h_i^k(x) - 2K(x) \notin 4\pi\mathbb{Z}, \quad i = 1, 2.$$

then the spherical Harnack inequality holds around each blowup point.

Impact

1. Toda system:

$$\Delta u_1 + 2\rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) - \rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0$$
$$\Delta u_2 - \rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) + 2\rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0.$$

2. Liouville system: Let $A = (a_{ij})_{n\times n}$ be a symmetric, non-negative matrix:

$$\Delta u_1 + a_{11}\rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) + a_{12}\rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0$$
$$\Delta u_2 + a_{12}\rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) + a_{22}\rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0.$$

More Impact

1. Fourth order equation: Q curvature equation on 4−manifold:

$$P_g u + 2Q_g = 2he^{4u} - 8\pi^2\gamma(\delta_q - \frac{1}{vol_g(M)})$$

$$P_g\phi = \Delta_g^2\phi + div_g((\frac{2}{3}R_g g - 2Ric_g)\nabla\phi)$$

$$Q_g = -\frac{1}{12}(\Delta_g R_g - R_g^2 + 2|Ric_g|^2)$$

Classification theorems were proved for

$$\Delta^2 u = 6e^{4u} - 8\pi^2\gamma\delta_0 \quad \text{in} \quad \mathbb{R}^4, \quad \int_{\mathbb{R}^4} e^{4u} < \infty.$$

If $\gamma = 0$ the classification theorem was proved by Chang-shou Lin. For $-1 < \gamma < 0$, the classification was done by Ahmedou-Wu-Z (22).

2. Many other equations and situations.

THANKS FOR YOUR ATTENTION