Nonlinear classification of Banach spaces based on Birkhoff-James orthogonality

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Backgrounds

This talk is concerned with

Geometric Nonlinear Functional Analysis,

or Nonlinear geometry of Banach spaces.

The main purpose of that area is to understand

what and **how** structures determine
Banach space properties

without the aid of linearity.

Backgrounds

There are many striking results in nonlinear geometry.

Theorem

The reflexivity, Radon-Nikodym property, and Asplundness are stable under Lipschitz homeomorphism.

Namely,

- *if there exists a bi-Lipschitz bijection between Banach spaces* X *and* Y*, and*
- *if* X *is reflexive, or has the Radon-Nikodym property, or Asplund,*

then Y *has the same property.*

Backgrounds

Theorem

The sequence space ℓ_p with $p \in (1, +\infty)$ is determined by its uniform structure.

Namely,

- *if there exists a uniformly bicontinuous bijection between ℓ_p and a Banach space X,*

then ℓ_p and X are isomorphic.

Corollary

Classical sequence spaces $\{\ell_p : p \in (1, +\infty)\}$ is classified by their uniform structure.

Backgrounds

As seen above, main goals of geometric nonlinear functional analysis are

(a) clarifying the stability of Banach space properties under nonlinear isomorphisms;
(b) finding essential structures that determine specific Banach spaces; and
(c) providing classification theories based on nonlinear equivalences.

The theme of this talk belongs to (c), nonlinear classification of Banach spaces.

Main tool: Birkhoff-James orthogonality

In this talk, we focus on nonlinear classification of Banach spaces based on Birkhoff-James (BJ) orthogonality.

BJ orthogonality is a generalization of orthogonality in Banach spaces inspired by the nearest point characterization of the usual orthogonality.

Definition (Birkhoff, 1935)

Let X be a Banach space over $\mathbb{K}$, and let $x, y \in X$. Then, x is BJ orthogonal to y, denoted by $x \perp_{BJ} y$, if $\|x + \lambda y\| \geq \|x\|$ for each $\lambda \in \mathbb{K}$.

Main tool: Birkhoff-James orthogonality

From its definition, BJ orthogonality is closely related to the geometric structure of Banach spaces.

For example, if $\|x\| = 1$, and if $x \perp_{BJ} y$, then the set $\{x + \lambda y : \lambda \in \mathbb{K}\}$ is tangent to the unit ball B_X of X.

This situation is generally explained by using support functionals for the unit ball.

Theorem (James, 1947)

Let X be a Banach space, and let $x, y \in X$. Then, $x \perp_{BJ} y$ if and only if there exists an $f \in X^*$ such that $\|f\| = 1$, $f(x) = \|x\|$, and $f(y) = 0$.

Main tool: Birkhoff-James orthogonality

The basic property of BJ orthogonality are as follows:

- $x \perp_{BJ} x$ implies $x = 0$ (non-degeneracy)
- $x \perp_{BJ} y$ implies $\alpha x \perp_{BJ} \beta y$ for each $\alpha, \beta \in \mathbb{K}$ (homogeneity)

Meanwhile, as a bad property of $\perp_{BJ}$, it is not symmetric in general.

Theorem (Day, 1947; James, 1947)

Let X be a Banach space. Suppose that $\dim X \geq 3$. If $\perp_{BJ}$ is symmetric in X (that is, $x \perp_{BJ} y$ always implies $y \perp_{BJ} x$ in X), then X is a Hilbert space.

Main tool: Birkhoff-James orthogonality

The existing results strongly indicate that BJ orthogonality structure contains much information about geometric features of Banach spaces.

For example, smoothness and strict convexity are characterized in terms of BJ orthogonality.

Hence, we anticipate that the large part of Banach space structure can be determined through the analysis on the behavior of BJ orthogonality.

Based on this idea, we develop the theory of an equivalence of Banach spaces based on BJ orthogonality.

Starting point

The starting point of our study is the following theorem.

Theorem (Blanco and Turnšek, 2006)

Let X and Y be Banach spaces, and let $T : X \to Y$ be linear. If T preserves BJ orthogonality in one direction (that is, $x \perp_{BJ} y$ implies that $Tx \perp_{BJ} Ty$), then T is a scalar multiple of an isometry.

(The real version was due to Koldobsky in 1993.)

The proof given by Blanco and Turnšek is valid both for the real and complex cases.

Starting point

As a consequence, we have the following result.

Corollary

Let X and Y be Banach spaces. If there exists a linear bijection $T : X \to Y$ that preserves BJ orthogonality (in one direction), then X and Y are isometrically isomorphic.

This means that the combination of linear and BJ orthogonality structures completely determines Banach spaces.

Starting point

Hence, as expected, BJ orthogonality structure is fine enough to classify Banach spaces.

Our next question is the following: What happens if the linearity is omitted in the preceding corollary?

To tackle this problem, we introduce a new nonlinear equivalence of Banach spaces.

Starting point

Definition

Let X and Y be Banach spaces, and let $T : X \to Y$. Then, T is called a BJ orthogonality preserver if it is bijective, and preserves $\perp_{BJ}$ in both direction (that is, $x \perp_{BJ} y$ if and only if $Tx \perp_{BJ} Ty$).

If there exists a BJ orthogonality preserver between X and Y, then we say that X and Y are BJ isomorphic, and write $X \sim_{BJ} Y$.

Starting point

BJ equivalence has the following properties.

Proposition 1

Let X, Y and Z be Banach spaces. Then, the following hold:

(i) $X \sim_{BJ} X$.

(ii) *If* $X \sim_{BJ} Y$, *then* $Y \sim_{BJ} X$.

(iii) *If* $X \sim_{BJ} Y$ *and* $Y \sim_{BJ} Z$, *then* $X \sim_{BJ} Z$.

(iv) *If* X *is a compex Banach space, then* $X \sim_{BJ} \overline{X}$,

where $\overline{X}$ *is the complex conjugate of* X.

Finite dimensional spaces

We proceed to developing the theory. We first consider the BJ classification in the finite dimensional case.

The following is key.

Lemma 1

Let X be an n-dimensional Banach space. Then, there exists a normalized basis $\{x_1, \ldots, x_n\}$ for X satisfying the following property:

If $x_j \perp_{BJ} y$ for each j, then $y = 0$.

(This is like a complete orthonormal basis.)

Finite dimensional spaces

Using the preceding lemma, we have the following result.

Theorem 1 (T., 2022a)

Let X and Y be Banach spaces such that $X \sim_{BJ} Y$. If X is finite dimensional, then Y is also finite dimensional, and $\dim X = \dim Y$.

The following corollary immediately follows.

Corollary 1

Let X and Y be Banach spaces such that $X \sim_{BJ} Y$. Suppose that either X or Y is finite dimensional. Then, $X \cong Y$, that is, X and Y are isomorphic.

Finite dimensional spaces

The converse to the preceding corollary is false in general.

Moreover, we will construct an example of a pair of two-dimensional Banach spaces (X, Y) such that

- $X \sim_{BJ} Y$; but
- $X \neq Y$, that is, X and Y are not isometrically isomorphic.

This will be done in the context of Hilbert spaces and Radon planes. Consequently, Corollary 1 is sharp.

Smooth spaces

Next, we consider smooth Banach spaces.

Recall that a Banach space X is smooth if, for each nonzero x, the set

$$\nu(x) = \{f \in X^* : \|f\| = 1,\ f(x) = \|x\|\}$$

is a singleton.

This is equivalent to the statement that the support functional for B_X at each unit vector is unique, that is, the norm is Gateaux differentiable at nonzero vectors.

Smooth spaces

In a smooth Banach space X, the singleton $\nu(x)$ is identified with its unique element.

A key sentence in the context of BJ orthogonality in smooth spaces is the following.

Lemma 2 (follows from James's characterization)

Let X be a smooth Banach space, and let

$$R_x = \{y \in X : x \perp_{BJ} y\}$$

for each $x \in X$. Then, $R_x = \ker \nu(x)$ for each $x (\neq 0)$.

Smooth spaces

The set R_x is preserved under BJ orthogonality preservers (regardless of the smoothness of the spaces).

Lemma 3

Let X and Y be Banach spaces, and let $T : X \to Y$ be a BJ orthogonality preserver. Then, $T(R_x) = R_{Tx}$ for each $x \in X$.

In particular, if X and Y are smooth, then

$$T(\ker \nu(x)) = \ker \nu(Tx)$$

for each $x(\neq 0)$.

Smooth spaces

The preceding lemma says that, in smooth Banach spaces, all support hyperplanes of the unit balls are preserved under BJ orthogonality preservers.

This bridges a gap between linear and nonlinear arguments in the smooth setting.

However, naturally, not all closed hyperplanes support the unit ball in general.

So, in this situation, the theory is (temporarily) developed by the assistance of the reflexivity.

Smooth spaces

Theorem 2 (T., 2022a)

Let X and Y be smooth Banach spaces, and let $T : X \to Y$ be a BJ orthogonality preserver. If M is a reflexive subspace of X, then $T(M)$ is a closed subspace of Y.

To show this, it is sufficient to notice that

$$M = \bigcap \{\ker \nu(x) : x \in X,\ \|x\| = d(x, M) = 1\}$$

whenever M is a reflexive subspace of X.

Smooth spaces

The preceding theorem has the following corollary.

Corollary 2

Let X and Y be reflexive smooth Banach spaces, and let $T : X \to Y$ be a BJ orthogonality preserver. Then, M is a closed subspace of X if and only if $T(M)$ is a closed subspace of Y.

This follows from the fact that each closed subspace of a reflexive Banach space is reflexive.

Smooth spaces

Now, we consider the classification of reflexive smooth Banach spaces.

In this direction, we know powerful tools for utilizing Corollary 2.

Let X be a Banach space. Then, the lattice of all closed subspaces of X is denoted by $\mathcal{C}(X)$.

Based on the fundamental theorem of projective geometry, we can see that an isomorphism between $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ induces a bounded linear isomorphism $T : X \to Y$ (or $T : X \to \overline{Y}$).

Smooth spaces

Theorem (Mackey, 1942)

Let X and Y be real Banach spaces. If there exists an isomorphism between $\mathcal{C}(X)$ and $\mathcal{C}(Y)$, then $X \cong Y$, that is, X and Y are isomorphic.

Theorem (Fillmore and Longstaff, 1984)

Let X and Y be complex infinite dimensional Banach spaces. If there exists an isomorphism between $\mathcal{C}(X)$ and $\mathcal{C}(Y)$, then $X \cong Y$ or $X \cong \overline{Y}$.

(The infinite-dimensionality is used for proving the continuity of induced mapping.)

Smooth spaces

If X and Y are reflexive smooth Banach spaces, and if $X \sim_{BJ} Y$, then there exists a BJ orthogonality preserver $T : X \to Y$.

By Corollary 2, T preserves closed subspaces of X and Y (and the inclusion relation of them).

Thus, we can define an isomorphism $\rho : \mathcal{C}(X) \to \mathcal{C}(Y)$ by $\rho(M) = T(M)$ for each $M \in \mathcal{C}(X)$.

Now, the theorems in the previous slide apply, and we have the following results.

Smooth spaces

Theorem 3 (T., 2022a)

Let X and Y be real reflexive smooth Banach spaces. If $X \sim_{BJ} Y$, then $X \cong Y$.

Theorem 4 (T., 2022a)

Let X and Y be complex reflexive smooth Banach spaces. If $X \sim_{BJ} Y$, then $X \cong Y$ or $X \cong \overline{Y}$.

(The finite dimensional case follows from Corollary 1.)

These results are usuful for classifying ℓ_p spaces.

Smooth spaces

It is well-known that ℓ_p spaces are reflexive and smooth for p with $p \in (1, +\infty)$, and $\ell_p \not\cong \ell_q$ whenever $p \neq q$.

Moreover, in the complex case, we have $\ell_p \cong \overline{\ell_p}$. Indeed, an isomorphism is given by $(a_n)_n \mapsto (\overline{a_n})_n$.

Therefore, we obtain the following classification.

Theorem 5 (T., 2022a)

The family $\{\ell_p : p \in (1, +\infty)\}$ is classified by BJ orthogonality structure. Namely, if $1 < p < q < \infty$, then $\ell_p \not\sim_{BJ} \ell_q$.

Smooth spaces

Before we leave this theme, we see some recent developments.

Theorem 6 (Ilišević and Turnšek, 2022)

Let X and Y be real smooth Banach spaces. Suppose that $\dim X \geq 3$ (or $\dim Y \geq 3$). Then, $X \sim_{BJ} Y$ if and only if $X = Y$.

Theorem 7 (Ilišević and Turnšek, 2022)

Let X and Y be complex infinite-dimensional smooth Banach spaces. Suppose that X (or Y) is reflexive. Then, $X \sim_{BJ} Y$ if and only if $X = Y$ or $X = \overline{Y}$.

Smooth spaces

Consequently, the BJ orthogonality structure is fine enough to determine Banach spaces by itself in the following cases:

- X and Y are real smooth Banach spaces, and $\dim X \geq 3$ (or $\dim Y \geq 3$).
- X and Y are complex infinite dimensional smooth Banach spaces, X (or Y is reflexive), and $X = \overline{X}$ (or $Y = \overline{Y}$).

Meanwhile, the non-smooth case is not so simple. It requires other tools and techniques.

Hilbert spaces

The next subject is about Hilbert spaces.

This case is expected to have the simplest conclusion.

That is true in the three or more dimensional case, but is not in the two-dimensional case.

We first consider the easier part. Then, the expected consequence is the following:

Hilbert spaces are determined by thier BJ orthogonality structure.

Hilbert spaces

In fact, the three or more dimensional case is almost finished by the following theorem.

Theorem (Day, 1947; James, 1947)

Let X be a Banach space. Suppose that $\dim X \geq 3$. Then, X is a Hilbert space if and only if BJ orthogonality is symmetric in X.

Since the symmetry of BJ orthogonality is stable under BJ equivalence, a three or more dimensional Banach space is BJ equivalent to a Hilbert space only if it is itself a Hilbert space.

Hilbert spaces

Therefore, it only remains to show that BJ equivalent Hilbert spaces have the same dimension.

To see this, we note that BJ orthogonality is equivalent to the usual orthogonality in Hilbert spaces.

Moreover, an orthogonal system $(e_\lambda)_\lambda$ in a Hilbert space is complete if and only if $x \perp_{BJ} e_\lambda$ for each λ implies that $x = 0$.

Combining this with the fact that BJ orthogonality preserver T satisfies $T0 = 0$, we have the following result.

Hilbert spaces

Theorem 8 (T., 2022a)

Let H be a Hilbert space, and let X be a Banach space. Suppose that $\dim H \geq 3$. Then, $H \sim_{BJ} X$ if and only if $H = X$.

Namely, three or more dimensional Hilbert spaces are isometrically determined by their BJ orthogonality structure.

Here, we remark that the assumption on $\dim H$ cannot be omitted (at least in the real case).

We next consider why this is so.

Hilbert spaces

The complexity in the two dimensional case is caused by the variety of Radon planes.

Definition

Let X be a real two dimensional Banach space. Then, X is called a Radon plane if BJ orthogonality is symmetric in X.

In three or more dimensional case, the symmetry of BJ orthogonality rarely happens.

However, in the two dimensional setting, we can construct various non-Hilbert Radon planes.

Hilbert spaces

Moreover, it turns out that smooth Radon planes cannot be classified by BJ equivalence.

Theorem 9 (T., 2022a)

Let X be a smooth Radon plane. Then, $X \sim_{BJ} \ell_2^2$, where ℓ_2^2 is the Euclidean 2-space.

Thus, all smooth Radon planes have the same BJ orthogonality structure.

A typical example of a non-Hilbert smooth Radon plane is as follows.

Hilbert spaces

Example 1

Suppose that $p \in (1, +\infty)$, and $p^{-1} + q^{-1} = 1$. Define a norm on $\mathbb{R}^2$ by

$$\|(a,b)\|_{p,q} = \begin{cases} (|a|^p + |b|^p)^{1/p} & (ab \geq 0) \\ (|a|^q + |b|^q)^{1/q} & (ab \leq 0) \end{cases} .$$

Then, $(\mathbb{R}^2, \|\cdot\|_{p,q})$ is a non-Hilbert smooth Radon plane.

The space $(\mathbb{R}^2, \|\cdot\|_{p,q})$ is called a Day-James space, and denoted by $\ell^2_{p,q}$.

Hilbert spaces

As a consequence, in the two-dimensional case, Hilbert spaces are not determined by their BJ orthogonality structure.

Moreover, the preceding example gives a pair of real Banach spaces (X, Y) such that $X \sim_{BJ} Y$ but $X \neq Y$.

Hence, BJ equivalence is not trivial even in the setting of real Banach spaces.

In fact, this phenomenon occurs also in higher dimensions...

To a non-smooth world

Analysis in the smooth setting finished almost in success.

Our next subject is a theory including non-smooth Banach spaces.

Then, new tools and techniques are needed.

In this section, we consider a closure space structure in Banach spaces which is stable under BJ equivalence.

As an application, we give a complete BJ classification of c_0 and ℓ_p spaces.

To a non-smooth world

Let X be a Banach space, and let

$$R_x = \{y \in X : x \perp_{BJ} y\}$$

for each $x \in X$.

We note that $R_x = X$ if and only if $x = 0$.

Define a binary relation "$\precsim$" on X by declaring that $x \precsim y$ if $R_x \subset R_y$.

It is easy to see that "$\precsim$" is a preorder on X.

To a non-smooth world

We borrow the notion of filters from the theory of posets.

Definition

Let X be a Banach space, and let $F \subset X$. Then, F is a filter on $(X, \precsim)$ if it satisfies

- $F \neq \emptyset$;
- If $x \in F$, and if $x \precsim y$, then $y \in F$; and
- F is downward directed in $(X, \precsim)$.

A filter F is said to be maximal if there is no filter G such that $F \subsetneq G$.

To a non-smooth world

We remark that each filter is automatically proper in $(X, \precsim)$ provided that $\dim X \geq 2$.

From this, by Zorn's lemma, we can always find a maximal filter containing a given filter.

Hence, there is a sufficient amount of maximal filters on $(X, \precsim)$.

We now introduce the geometric structure space of X.

To a non-smooth world

For a maximal filter U on $(X, \precsim)$, let

$$I_U = \bigcap_{x \in U} R_x = \bigcap_{x \in U} \bigcup_{f \in \nu(x)} \ker f.$$

The geometric structure space $\mathfrak{S}(X)$ is defined by

$$\mathfrak{S}(X) = \{I_U : U \text{ is a maximal filter on } (X, \precsim)\}.$$

For each $S \subset \mathfrak{S}(X)$, define its closure $S^=$ by

$$S^= = \left\{ I \in \mathfrak{S}(X) : \bigcap_{J \in S} J \subset I \right\}.$$

To a non-smooth world

The mapping $S \mapsto S^=$ has the following properties.

Proposition 2

Let X be a nontrivial Banach space. Then the following hold:

(i) $\emptyset^= = \emptyset$.

(ii) $S \subset S^=$.

(iii) $(S^=)^= = S^=$.

(iv) *If* $S_1 \subset S_2$, *then* $S_1^= \subset S_2^=$.

In other words, the mapping $S \mapsto S^=$ is a closure operator satisfying $\emptyset^= = \emptyset$.

To a non-smooth world

In the following, the closure space $\mathfrak{S}(X)$ is said to be topologizable if $\{S \subset \mathfrak{S}(X) : S^= = S\}$ satisfies the axioms of closed sets.

Naturally, not all $\mathfrak{S}(X)$ are topologizable.

Theorem 10 (T., 2022b)

Let X be a nontrivial smooth Banach space. Then,

$$\mathfrak{S}(X) = \{\ker \nu(x) : x \in X \setminus \{0\}\}.$$

Moreover, if X is reflexive and $\dim X \geq 2$, then $\mathfrak{S}(X)$ is not topologizable.

To a non-smooth world

It turns out that the closure space $\mathfrak{S}(X)$ and its topologizability is preserved under BJ orthogonality preservers.

Recall that a mapping $\Phi : \mathfrak{S}(X) \to \mathfrak{S}(Y)$ is said to be continuous if

$$\Phi(S^{=}) \subset \Phi(S)^{=}$$

for each $S \subset \mathfrak{S}(X)$.

If Φ is bijective and bicontinuous, then it is called a homeomorphism between $\mathfrak{S}(X)$ and $\mathfrak{S}(Y)$.

To a non-smooth world

Theorem 11 (T. 2022b)

Let X *and* Y *be Banach spaces such that* $X \sim_{BJ} Y$*. Then,* $\mathfrak{S}(X)$ *and* $\mathfrak{S}(Y)$ *are homeomorphic.*

Moreover, if $\mathfrak{S}(X)$ *is topologizable, then* $\mathfrak{S}(Y)$ *is also topologizable.*

This result allows us to use geometric structure spaces for classifying Banach spaces by their BJ orthogonality structure.

To a non-smooth world

A milestone is found in spaces of continuous functions.

Theorem 12 (T., 2022b)

Let K be a locally compact Hausdorff space. Then, $\mathfrak{S}(C_0(K))$ is topologizable, and is homeomorphic to K.

As a consequence, spaces of continuous functions are classified by their BJ orthogonality structure.

Corollary 3

Let K and L be locally compact Hausdorff spaces. Then, $C_0(K) \sim_{BJ} C_0(L)$ if and only if $C_0(K) = C_0(L)$.

To a non-smooth world

Since c_0 and ℓ_∞ are isometrically isomorphic to spaces of continuous functions, the results in the previous slide apply.

It follows that $c_0 \not\sim_{BJ} \ell_p \not\sim_{BJ} \ell_\infty$ whenever $p \in (1, +\infty)$, since $\mathfrak{S}(c_0)$ and $\mathfrak{S}(\ell_\infty)$ are topologizable but $\mathfrak{S}(\ell_p)$ is not.

Moreover, $c_0 \not\sim_{BJ} \ell_\infty$ by the preceding corollary.

Thus, to classify c_0 and ℓ_p, what only remains is to show that ℓ_1 is not BJ isomorphic to these spaces.

To a non-smooth world

First, we examine the topologizability of $\mathfrak{S}(\ell_1)$. Then, we have the following result.

Theorem 13 (T., 2022b)

$\mathfrak{S}(\ell_1)$ *is not topologizable.*

Hence, ℓ_1 is not BJ isomorphic to c_0 and ℓ_∞.

Our next aim is to show that $\ell_1 \not\sim_{BJ} \ell_p$ whenever $p \in (1, +\infty)$.

To this end, we introduce a homogeneity property of $\mathfrak{S}(X)$.

To a non-smooth world

Definition

Let X be a Banach space. Then, $\mathfrak{S}(X)$ is BJ homogeneous if, for each $I, J \in \mathfrak{S}(X)$, there exists a BJ orthogonality preserver $T : X \to X$ such that $T(I) = J$.

This property is stable under BJ equivalence.

Theorem 14 (T., 2022b)

Let X and Y be Banach spaces such that $X \sim_{BJ} Y$. Then, $\mathfrak{S}(X)$ is BJ homogeneous if and only if $\mathfrak{S}(Y)$ is.

To a non-smooth world

We have the following results on BJ homogeneity.

Theorem 15 (T., 2022b)

$\mathfrak{S}(\ell_1)$ *is BJ homogeneous.*

Theorem 16 (T., 2022b)

If $p \in (1, +\infty)$ *and* $p \neq 2$, *then* $\mathfrak{S}(\ell_p)$ *is not BJ homogeneous.*

Therefore, $\ell_1 \not\sim_{BJ} \ell_p$ if $p \in (1, +\infty)$ and $p \neq 2$, while $\ell_1 \not\sim_{BJ} \ell_2$ follows from the case of Hilbert spaces.

To a non-smooth world

As a summary, we have the following result.

Theorem 17 (T., 2022b)

The family $\{c_0\} \cup \{\ell_p : 1 \leq p \leq \infty\}$ *is classified by thier BJ orthogonality structure.*

This completes the first step of the study on nonlinear BJ classification of Banach spaces.

Conclusion

- BJ orthogonality structure is fine enough to develop the theory of nonlinear classification.
- In many cases, BJ equivalence coincides with isometric equivalence. So, BJ equivalence is rather strong.
- However, there are examples of pairs of Banach spaces (X, Y) such that $X \sim_{BJ} Y$ but $X \neq Y$ (even if $\dim X \geq 3$ or $\dim Y \geq 3$).
- To reveal an entire picture of BJ equivalence, a detalied research of BJ orthogonality structure in wider classes of Banach spaces are needed.

This talk is based on the following papers:

- [T., 2022a]

  R. Tanaka, *Nonlinear equivalence of Banach spaces based on Birkhoff-James orthogonality*, J. Math. Anal. Appl., **505** (2022), 125444.

- [T., 2022b]

  R. Tanaka, *Nonlinear equivalence of Banach spaces based on Birkhoff-James orthogonality, II*, J. Math. Anal. Appl., **514** (2022), 126307.

ご清聴ありがとうございました。

Thank you for your attention.