Quasiconvexity preserving property for fully nonlinear nonlocal parabolic equations

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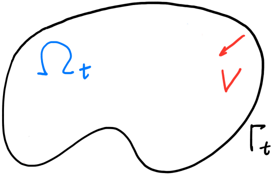
Introduction

The typical problem we are interested in is motion of a closed hypersurface Γ_t in $\mathbb{R}^n$:

$$V = H - m(\Omega_t),$$

where

- V is the inwardward normal velocity of Γ_t,
- H denotes the mean curvature of Γ_t,
- Ω_t is the set enclosed by Γ_t,
- m denotes the Lebesgue measure.



The level set formulation gives rise to

$$u_t - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u| m(\{u(\cdot, t) < u(x,t)\}) = 0.$$

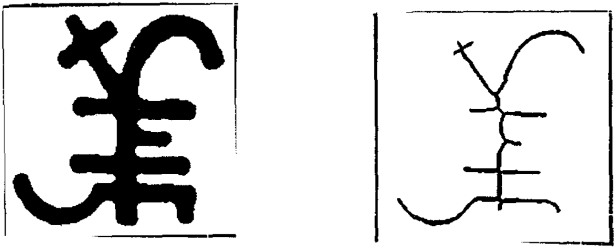
One may consider general geometric equations (u is sol. $\Rightarrow g(u)$ is sol. for g increasing)

$$u_t + F\left(\nabla u, \nabla^2 u, \{u(\cdot, t) < u(x,t)\}\right) = 0$$

See well-posedness results in [Chen-Hilhorst-Logak 97] [Cardaliaguet 00] [Slepčev 03].

Applications of level-set dependent nonlocal equations

- Image processing (thinning a shape)



Images: D. Pasquignon (1995), Computation of skeleton by partial differential equations, IEEE Comput. Soc. Press International Conference on Image Processing

- Plasma physics
  [Grad 79] [Temam 79] [Mossino-Temam 81] [Laurence-Stredulinsky 85]

$$-\Delta u + g(u, m(\{u < u(x)\})) = 0.$$

[Caffarelli-Tomasetti 21] studies regularity of viscosity solutions to fully nonlinear equations of the same type.

Convexity

We are interested in asymptotic behavior, singularity formation, control/game interpretation and

Convexity preserving

- Ω_0 is convex $\Rightarrow$ Ω_t is convex for all $t \geq 0$ ([Cardaliaguet 00] for geometric flows)
- $\{u(\cdot, 0) < h\}$ is convex $\Rightarrow \{u(\cdot, t) < h\}$ is convex for all $t \geq 0$

(Quasiconvexity of u is preserved.)

Quasiconvexity

We say $f \in C(\mathbb{R}^n)$ is quasiconvex if $\{f < h\}$ is convex for all $h \in \mathbb{R}$, or equivalently,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\} \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } \lambda \in (0,1).$$

See results on convexity preserving for various equations in [Korevaar 83] [Kawohl 85] [Kennington 85] [Giga-Goto-Ishii-Sato 91] [Alvarez-Lasry-Lions 97] [Cuoghi-Salani 06] ...

Objectives

Consider general nonlocal (degenerate) parabolic equations

$$\begin{cases} u_t + F(u, \nabla u, \nabla^2 u, K \cap \{u(\cdot, t) < u(x,t)\}) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \quad (1) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n, \quad (2) \end{cases}$$

where $K \subset \mathbb{R}^n$ is a compact set and u_0 is a given continuous function in $\mathbb{R}^n$.

Our aims

1. Provide quasiconvexity results for nonlocal equations that are possibly non-geometric.

   We do not assume that for all $c_1, c_2 \in \mathbb{R}$

   $$F(r_1, c_1 p, c_1 X + c_2 p \otimes p, A) = c_1 F(r_2, p, X, A) \quad (p \neq 0).$$

2. Give a direct PDE proof, avoiding the set-theoretic arguments in [Cardaliaguet 00].

3. Deepen our understanding about the local case.

   [Ishige-Salani 08] shows that the heat equation may fail to preserve quasiconvexity.

Power convexity

For $a, b > 0$, $q > 0$ and $\lambda \in (0,1)$, take the q-mean $M_q(a,b,\lambda) = (\lambda a^q + (1-\lambda) b^q)^{\frac{1}{q}}$.

A positive function $f \in C(\mathbb{R}^n)$ is q-convex if f^q is convex, i.e.,

$$f(\lambda x + (1-\lambda)y) \leq M_q(f(x), f(y), \lambda) \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } \lambda \in (0,1).$$

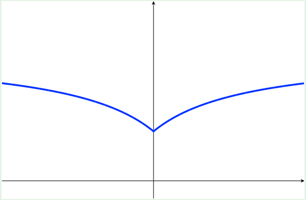
- If $q_1 \leq q_2$, q_1-convexity implies q_2-convexity.
- Quasiconvexity can be regarded as ∞-convexity.

Example

The radially symmetric solution of

$$\begin{cases} u_t + |\nabla u| m(\{u(\cdot,t) < u(x,t)\}) = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \\ u(x,0) = |x| + 1 & x \in \mathbb{R}^2 \end{cases}$$

is $\quad u(x,t) = \dfrac{|x|}{1 + \pi |x| t} + 1$, NOT convex in x for $t > 0$.



- $u^{1/q}$ is also a solution ($\Rightarrow$ q-convexity breaking)
- Without K, coercivity preserving may fail to hold.

Assumptions on $F : \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n \times \mathcal{B}_K \to \mathbb{R}$

($\mathbb{S}^n$ is the set of all $n \times n$ sym. matrices, $\mathcal{B}_K$ is the collection of all meas. subsets of K.)

(F1) $F(r_1, p, X_1, A) \leq F(r_2, p, X_2, A)$ if $r_1 \leq r_2$ and $X_1 \geq X_2$.

(F2) For each $R > 0$,

$$\sup\{|F(r, p, X, A)| : r \in \mathbb{R}, |p| \leq R \text{ with } p \neq 0, |X| \leq R, A \in \mathcal{B}_K\} < \infty.$$

(F3) F is continuous with topology of $\mathcal{B}_K$ given by $d(A_1, A_2) = m(A_1 \triangle A_2)$. Moreover, for any $R > 0$, $\exists$ a modulus ω_R such that

$$|F(r, p, X, A_1) - F(r, p, X, A_2)| \leq \omega_R\left(m(A_1 \triangle A_2)\right).$$

for all $p \in \mathbb{R}^n \setminus \{0\}$ with $|p| \leq R$.

(F4) $F(r, p, X, A_1) \leq F(r, p, X, A_2)$ if $A_1 \subset A_2$. (monotone)

(F5) $\exists$ a modulus ω such that

$$F(r, p_1, X_1, A) - F(r, p_2, X_2, A) \leq \omega\left(\frac{|Z||p_1 - p_2|}{\min\{|p_1|, |p_2|\}} + |p_1 - p_2| + \alpha\right)$$

for all $\alpha \geq 0$ if $X_1, X_2, Z \in \mathbb{S}^n$ satisfy

$$\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + \alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

(F6) $\exists \mu \in C(\mathbb{R})$ such that $\sup_{r \in \mathbb{R},\ A \in \mathcal{B}_K} |F(r, p, X, A) - \mu(r)| \to 0$ as $(p, X) \to (0, 0)$.

Definition of viscosity solutions

Recall

$$u_t + F(u, \nabla u, \nabla^2 u, K \cap \{u(\cdot, t) < u(x,t)\}) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1}$$

Denote by F_*, F^* the lower and upper semicontinuous envelopes of F.

Subsolution

A function $u \in USC(\mathbb{R}^n \times (0,\infty))$ is called a subsolution of (1) if whenever there exist $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$ and $\varphi \in C^2(\mathbb{R}^n \times (0,\infty))$ s.t. $u - \varphi$ attains a local maximum at (x_0, t_0),

$$\varphi_t(x_0, t_0) + F_*(u(x_0, t_0), \nabla\varphi(x_0, t_0), \nabla^2\varphi(x_0, t_0), K \cap \{u(\cdot, t_0) < u(x_0, t_0)\}) \le 0.$$

Supersolution

A function $u \in LSC(\mathbb{R}^n \times (0,\infty))$ is called a supersolution of (1) if whenever there exist $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$ and $\varphi \in C^2(\mathbb{R}^n \times (0,\infty))$ s.t. $u - \varphi$ attains a local minimum at (x_0, t_0),

$$\varphi_t(x_0, t_0) + F^*(u(x_0, t_0), \nabla\varphi(x_0, t_0), \nabla^2\varphi(x_0, t_0), K \cap \{u(\cdot, t_0) \le u(x_0, t_0)\}) \ge 0.$$

A function $u \in C(\mathbb{R}^n \times (0,\infty))$ is called a solution of (1) if it is both a sub- and supersolution.

Comparison Principle

Theorem 1

Assume that

- (F1)–(F6) hold;
- $u \in USC(\mathbb{R}^n \times [0,\infty))$ and $v \in LSC(\mathbb{R}^n \times [0,\infty))$ are resp. a subsol. and a supersol. of

$$u_t + F(u, \nabla u, \nabla^2 u, K \cap \{u(\cdot,t) < u(x,t)\}) = 0;$$

- for any $T > 0$, there exists $L_T > 0$ such that

$$u(x,t) \leq L_T(|x|+1), \quad v(x,t) \geq -L_T(|x|+1) \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0,T];$$

- there exists a modulus of continuity ω_0 such that

$$u(x,0) - v(y,0) \leq \omega_0(|x-y|) \quad \text{for all } x, y \in \mathbb{R}^n.$$

Then, $u \leq v$ holds in $\mathbb{R}^n \times [0,\infty)$.

Known comparison results: [Slepčev 03][Da Lio-Kim-Slepčev 04] (bdd domain)
[Srour 09] (bdd sub/supersol.) [Giga-Goto-Ishii-Sato 91] (local)

Uniqueness for non-monotone eqn.: [Alvarez-Cardaliaguet-Monneau 05] [Barles-Ley 06]
[Barles-Ley-Mitake 12] [Kim-Kwon 20] . . .

Concavity assumption on F

We consider positive solutions, i.e., $u > 0$. Let $v = u^q$ with $q >> 1$. Then v satisfies

$$v_t + qv^{\frac{q-1}{q}} F\left(v^{\frac{1}{q}}, \frac{1}{q} v^{\frac{1-q}{q}} \nabla v, \frac{1-q}{q^2} v^{\frac{1-2q}{q}} \nabla v \otimes \nabla v + \frac{1}{q} v^{\frac{1-q}{q}} \nabla^2 v, K \cap \{v(\cdot, t) < v(x,t)\}\right) = 0.$$

Letting $\beta = 1 - \frac{1}{q} \in (0,1)$, we get a transformed operator G_β

$$\begin{aligned} & G_\beta(r, p, X, A) \\ & = \frac{1}{1-\beta} r^\beta F\left(r^{1-\beta}, (1-\beta) r^{-\beta} p, (1-\beta) r^{-\beta} X + (\beta^2 - \beta) r^{-\beta-1} p \otimes p, A\right). \end{aligned}$$

(F7) For any $\beta < 1$ close to 1,

$$(r, X) \mapsto G_\beta(r, p, X, A) \quad \text{is concave in } (0, \infty) \times \mathbb{S}^n$$

and

$$r \mapsto r^\beta \mu(r^{1-\beta}) \quad \text{is concave in } (0, \infty),$$

where μ is given by (F6).

Main result

Theorem 2

Assume that

- (F1)–(F7);
- u_0 is uniformly continuous in $\mathbb{R}^n$;
- $u \in C(\mathbb{R}^n \times [0, \infty))$ be the unique viscosity solution of

$$\begin{cases} u_t + F(u, \nabla u, \nabla^2 u, K \cap \{u(\cdot, t) < u(x, t)\}) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \quad (1) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n; \quad (2) \end{cases}$$

- u satisfies

$$\inf_{\mathbb{R}^n \times (0,\infty)} u > 0;$$

- u satisfies

$$\inf_{|x| \geq R,\ t \leq T} u(x, t) \to \infty \quad \text{as } R \to \infty \text{ for any } T \geq 0;$$

- u_0 is quasiconvex in $\mathbb{R}^n$.

Then, $u(\cdot, t)$ is quasiconvex in $\mathbb{R}^n$ for all $t \geq 0$.

Such a solution does exist if additionally there is a subsolution $\phi \in C(\mathbb{R}^n \times (0, \infty))$ that is positive, coercive in space and satisfies $\phi(\cdot, 0) \leq u_0$ in $\mathbb{R}^n$.

Proof (cf. [Alvarez-Lasry-Lions 97] [Cuoghi-Salani 06] [Ishige-L-Salani 20])

Our goal is to prove the supersolution property of quasiconvex envelope

$$u_\star(x,t) = \min\left\{ \max\{u(y,t), u(z,t)\} : x = \lambda y + (1-\lambda)z \right\}.$$

1) Approximate $u_\star$ locally uniformly by

$$u_q(x,t) = \min\left\{ (\lambda u(y,t)^q + (1-\lambda)u(z,t)^q)^{\frac{1}{q}} : x = \lambda y + (1-\lambda)z \right\}.$$

2) Use the fact that $v = u^q$ is a supersolution and get

$$v_t(y,t) + G_\beta(v(y,t), \nabla v(y,t), \nabla^2 v(y,t), K \cap \{u(\cdot,t) \le u(y,t)\}) \ge 0,$$
$$v_t(z,t) + G_\beta(v(z,t), \nabla v(z,t), \nabla^2 v(z,t), K \cap \{u(\cdot,t) \le u(z,t)\}) \ge 0.$$

3) For $v_q = u_q^q$, notice

$$(y,z,t) \mapsto v_q(\lambda y + (1-\lambda)z, t) - \lambda v(y,t) - (1-\lambda)v(z,t)$$

attains a minimum and deduce at the minimizer $(y,z,t) = (y_q, z_q, t_q)$

$$(v_q)_t(x,t) = \lambda v_t(y,t) + (1-\lambda)v_t(z,t), \quad \nabla v_q(x,t) = \nabla v(y,t) = \nabla v(z,t),$$
$$\nabla^2 v_q(x,t) \ge \lambda \nabla^2 v(y,t) + (1-\lambda)\nabla^2 v(z,t), \quad v_q(x,t) = \lambda v(y,t) + (1-\lambda)v(z,t).$$

More about our proof

4) Verify that

$$m\big(K \cap \{u(\cdot,t) \le u(y,t)\} \setminus \{u_\star(\cdot,t) \le u_\star(x,t)\}\big) \to 0 \quad \text{as } q \to \infty.$$

Adopt (F3)

$$F(r,p,X,A_1) - F(r,p,X,A_2) \le \omega_R\left(m(A_1 \triangle A_2)\right)$$

to get

$$v_t(y,t) + G_\beta(v(y,t), \nabla v(y,t), \nabla^2 v(y,t), K \cap \{u_\star(\cdot,t) \le u_\star(x,t)\}) \ge \text{error},$$
$$v_t(z,t) + G_\beta(v(z,t), \nabla v(z,t), \nabla^2 v(z,t), K \cap \{u_\star(\cdot,t) \le u_\star(x,t)\}) \ge \text{error}.$$

5) Combine the inequalities and use (F7) to obtain

$$(v_q)_t(x,t) + G_\beta(v_q(x,t), \nabla v_q(x,t), \nabla^2 v_q(x,t), K \cap \{u_\star(\cdot,t) \le u_\star(x,t)\}) \ge \text{error}.$$

6) Rewrite the equation

$$(u_q)_t(x,t) + F(u_q(x,t), \nabla u_q(x,t), \nabla^2 u_q(x,t), K \cap \{u_\star(\cdot,t) \le u_\star(x,t)\}) \ge \text{error}.$$

and adopt the stability arguments to conclude

$$(u_\star)_t(x,t) + F^*(u_\star(x,t), \nabla u_\star(x,t), \nabla^2 u_\star(x,t), K \cap \{u_\star(\cdot,t) \le u_\star(x,t)\}) \ge 0.$$

7) By the comparison principle, $u_\star \ge u$. On the other hand, by definition $u_\star \le u$.

Examples

Example 1. Level-set nonlocal curvature flow equations

Let $a \in \mathbb{R}$, $b \geq 0$, $c \geq 0$. Consider in $\mathbb{R}^n \times (0, \infty)$

$$u_t + a|\nabla u| + b|\nabla u| m(K \cap \{u(\cdot, t) < u(x, t)\}) - c|\nabla u| \operatorname{tr}\left(\nabla^2 \gamma\left(\nabla u\right) \nabla^2 u\right) = 0,$$

where the energy density $\gamma \in C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\})$ satisfies

- $\gamma > 0$ in $\mathbb{R}^n \setminus \{0\}$, $\gamma(0) = 0$,
- $\gamma(\alpha x) = \alpha \gamma(x)$ for $x \in \mathbb{R}^n$ and $\alpha > 0$.

Then

$$G_\beta(r, p, X, A) = F(p, X, A) = a|p| + b|p| m(A) - |p| \operatorname{tr}\left(\nabla^2 \gamma(p) X\right)$$

satisfies (F7).

Our proof is PDE-based, in contrast to the set-theoretic arguments in [Cardaliaguet 00].

Examples

Example 2. Nonlocal evolution equations with u-dependence

Consider in $\mathbb{R}^n \times (0, \infty)$

$$u_t + V(u) + |\nabla u| m(K \cap \{u(\cdot, t) < u(x, t)\}) - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0,$$

in $\mathbb{R}^n \times (0, \infty)$, where $V \in C^2(\mathbb{R})$ is a given bounded function satisfying

$$V(0) \geq 0, \;\; V' \geq 0 \text{ and } V'' \leq 0 \;\text{ in } (0, \infty).$$

Then

$$G_\beta(r, p, X, A) = \frac{1}{1-\beta} r^\beta V(r^{1-\beta}) + |p| m(A) - \operatorname{tr}\left(\left(I - \frac{p \otimes p}{|p|^2}\right) X\right)$$

satisfies (F7).

See [Tsai-Giga 03] for applications of the local counterpart in crystal growth.

The Laplacian

The heat equation

Consider in $\mathbb{R}^n \times (0, \infty)$
$$u_t - \Delta u = 0.$$
It is known [Ishige-Salani 08] that in general the quasiconvexity of u in space is not preserved.

Note that $F(p, X) = -\operatorname{tr} X$ and
$$G_\beta(r, p, X) = -\operatorname{tr} X + \frac{\beta}{r}|p|^2.$$
In this case, G_β fails to satisfy the concavity assumption (F7).

Decompose Δu into
$$\begin{aligned}\Delta u &= \operatorname{tr}\left[\left(I - \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\right)\nabla^2 u\right] + \operatorname{tr}\left[\left(\frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\right)\nabla^2 u\right]\\ &= \Delta_1^N u + \Delta_\infty^N u.\end{aligned}$$

Summary

Conclusion

- We provide a sufficient condition to guarantee the quasiconvexity preserving property.
- Our PDE-based approach applies to a general class of nonlocal evolution equations.
- The infinity-Laplacian part may cause quasiconvexity breaking.

Further problems

- How can we get a sufficient and necessary condition for quasiconvexity preserving?
- How about general non-monotone evolution equations?

  It seems that quasiconvexity is still preserved by

$$u_t - |\nabla u| m(\{u(\cdot, t) < u(x, t)\}) = 0.$$

Thank you for your kind attention!