Asia-Pacific Analysis and PDE Seminar

Online, July 26, 2021

Joint work with Nader Masmoudi & Tong Yang

Gevrey well-posedness of the 3D Prandtl equations without Structural Assumption

Wei-Xi LI (Wuhan University)

Inviscid limit

Navier-Stokes equation: $\boldsymbol{u}^\varepsilon = (u_1^\varepsilon, \cdots, u_n^\varepsilon), \quad n = 2, 3$

$$\begin{cases} \partial_t \boldsymbol{u}^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon - \varepsilon \Delta \boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = 0, \\ \nabla \cdot \boldsymbol{u}^\varepsilon = 0, \\ \boldsymbol{u}^\varepsilon|_{t=0} = \boldsymbol{u}_0, \end{cases}$$

Question: $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}^E$ as $\varepsilon \to 0$? Here $\boldsymbol{u}^E$ solves Euler equation

$$\partial_t \boldsymbol{u}^E + (\boldsymbol{u}^E \cdot \nabla)\boldsymbol{u}^E + \nabla p^E = 0, \quad \nabla \cdot \boldsymbol{u}^E = 0$$

Inviscid limit

Navier-Stokes equation: $\boldsymbol{u}^\varepsilon = (u_1^\varepsilon, \cdots, u_n^\varepsilon), \quad n = 2, 3$

$$\begin{cases} \partial_t \boldsymbol{u}^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon - \varepsilon\Delta\boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = 0, \\ \nabla \cdot \boldsymbol{u}^\varepsilon = 0, \\ \boldsymbol{u}^\varepsilon|_{t=0} = \boldsymbol{u}_0, \end{cases}$$

Question: $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}^E$ as $\varepsilon \to 0$? Here $\boldsymbol{u}^E$ solves Euler equation

$$\partial_t \boldsymbol{u}^E + (\boldsymbol{u}^E \cdot \nabla)\boldsymbol{u}^E + \nabla p^E = 0, \quad \nabla \cdot \boldsymbol{u}^E = 0$$

- $\mathbb{R}^n$ or **torus**: well-developed
- **a domain with boundary**: an outstanding open problem

inviscid limit for fluid domain with boundary

Mismatched **boundary conditions:** $\Omega = \{x_n > 0\}$,

Navier-Stokes: $\boldsymbol{u}^{\varepsilon}|_{x_n=0} = \mathbf{0}$

Euler: $\boldsymbol{u}^{E} \cdot \boldsymbol{n}|_{x_n=0} = u^{E}_{\boldsymbol{n}}|_{x_n=0} = 0$

inviscid limit for fluid domain with boundary

Mismatched **boundary conditions:** $\Omega = \{x_n > 0\}$,

Navier-Stokes: $\boldsymbol{u}^\varepsilon|_{x_n=0} = \mathbf{0}$

Euler: $\boldsymbol{u}^E \cdot \boldsymbol{n}|_{x_n=0} = u^E_{\boldsymbol{n}}|_{x_n=0} = 0$

Prandtl' postulate (1908):

$$\boldsymbol{u}^\varepsilon = \begin{cases} \boldsymbol{u}^E + O(\sqrt{\varepsilon}), & \text{outside boundary layer} \\ \boldsymbol{u}^P + O(\sqrt{\varepsilon}), & \text{inside boundary layer} \end{cases}$$

Euler + Prandtl expansion of NS

Question: Euler + Prandtl expansion of Navier-Stokes?

cf. Sammartino-Caflisch, Gérard-Varet, Guo, Iyer, Maekawa, Masmoudi, Nguyen, Wang, Zhang, Yang, Xie, Xin, $\cdots$

Goal: Well-posedness for Prandtl system

Derivation of boundary layer equations

Prandtl's ansatz (2D e.g.): $\tilde{y} = y/\sqrt{\varepsilon}$

$$\begin{cases} u^\varepsilon(t,x,y) = u^{\mathbf{0}}(t,x,y) + u^b(t,x,\tilde{y}) + O(\sqrt{\varepsilon}), \\ v^\varepsilon(t,x,y) = v^{\mathbf{0}}(t,x,y) + \sqrt{\varepsilon} v^b(t,x,\tilde{y}) + O(\sqrt{\varepsilon}), \\ p^\varepsilon(t,x,y) = p^{\mathbf{0}}(t,x,y) + O(\sqrt{\varepsilon}), \end{cases}$$

- Far from the boundary: Euler flow $(u^{\mathbf{0}}, v^{\mathbf{0}}, p^{\mathbf{0}})$
- Inside boundary layer: described by Prandtl equations (u^b, v^b).

2D Prandtl (scalar) equation

Governing equation for boundary layer: writting $\tilde{y} = y$ for short,

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u + \partial_x p = 0, \ x \in \mathbb{R}, \ y > 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y\to+\infty} u = U(t,x), \\ u|_{t=0} = u_0 , \end{cases}$$

where

$U(t,x)$, $p(t,x)$: givens functions, the trace at $z = 0$ of the tangential velocity and pressure of the Euler flow

3D Prandtl system

(u, v, w): velocity fields

$$\begin{cases} \partial_t u + \big(u\partial_x + v\partial_y + w\partial_z\big)\, u - \partial_z^2 u + \partial_x p = 0, \\ \partial_t v + \big(u\partial_x + v\partial_y + w\partial_z\big)\, v - \partial_z^2 v + \partial_y p = 0, \\ \partial_x u + \partial_y v + \partial_z w = 0 \\ (u, v, w)|_{z=0} = (0, 0, 0), \qquad \lim\limits_{z\to+\infty} (u, v) = \big(U, V\big), \\ u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \end{cases}$$

Main difficulty for Prandtl equation

Loss of derivative coupled with **nonlocal property**:

$$\mathbf{2D}: \qquad \begin{cases} \partial_t u + (u\partial_x + v\partial_y)u - \partial_y^2 u + \partial_x p = 0, \\ v(t,x,y) = -\int_0^y \partial_x u(t,x,r)dr. \end{cases}$$

Main difficulty for Prandtl equation

Loss of derivative coupled with **nonlocal property**:

$$\mathbf{2D}: \qquad \begin{cases} \partial_t u + (u\partial_x + v\partial_y)u - \partial_y^2 u + \partial_x p = 0, \\ v(t,x,y) = -\int_0^y \partial_x u(t,x,r)dr. \end{cases}$$

$$\mathbf{3D}: \qquad \begin{cases} \partial_t u + \left(u\partial_x + v\partial_y + w\partial_z\right) u - \partial_z^2 u + \partial_x p = 0, \\ \partial_t v + \left(u\partial_x + v\partial_y + w\partial_z\right) v - \partial_z^2 v + \partial_y p = 0, \\ w(t,x,y,z) = -\int_0^z \left(\partial_x u(t,x,y,r) + \partial_y v(t,x,y,r)\right)dr. \end{cases}$$

Outlines

Sobolev well-posedness

2D well-posedness under **Oleinik's monotonicity condition ($\partial_y u > 0$)**,

- Local-in-time solutions: Oleinik, Alexandre-Wang-Xu-Yang(2015), Masmoudi-Wong (2015)
- Global-in-time Solutions: Zhang-Xin
- ...

Sobolev well-posedness

2D well-posedness under **Oleinik's monotonicity condition** ($\partial_y u > 0$),

- Local-in-time solutions: Oleinik, Alexandre-Wang-Xu-Yang(2015), Masmoudi-Wong (2015)
- Global-in-time Solutions: Zhang-Xin
- ...

Sobolev well-posedness for 3D : much less is known!

- Liu-Wang-Yang (2017): $v(t, x, y, z) = k(t, x, y)u(t, x, y, z)$.
- weak solutions; Luo-Xin (2018)
- Global-in-time solution ?

Well-posedness for general data

Function spaces: $x \mapsto f(x)$,

Analytic functions: $$\sum_{\alpha} \frac{\partial^{\alpha} f(x_0)}{\alpha!}(x - x_0)^{\alpha} < +\infty$$

Gevrey functions: $$\sum_{\alpha} \frac{\partial^{\alpha} f(x_0)}{(\alpha!)^{\sigma}}(x - x_0)^{\alpha} < +\infty$$

Analytic space $\subset$ Gevrey class space $\subset C^{\infty}$

Gevrey (analytic) well-posedness

Without structural assumption: critical Gevrey index 2.

Gevrey ill-posedness for index > 2:

- Gérard-Vare and Dormy(2010, 2D), Liu-Wang-Yang (2016, 3D)

Gevrey (analytic) well-posedness

Without structural assumption: critical Gevrey index 2.

Gevrey ill-posedness for index > 2:

- Gérard-Vare and Dormy(2010, 2D), Liu-Wang-Yang (2016, 3D)

Gevrey well-posedness for index ≤ 2:

- Analytic space: Sammartino-Caflisch(1998), 2D & 3D ,
- Sharp Gevery space: Dietert-Gérard-Vare (2019, 2D); 3D?
- Global solutions: Paicu-Zhang (analytic,2D,2020), Wang-Wang-Zhang (Gevrey, 2D, 2021); 3D?

Outlines

3D Prandtl

3D Prandtl :
$$\begin{cases} \partial_t u + \big(u\partial_x + v\partial_y + w\partial_z\big)\, u - \partial_z^2 u = 0, \\ \partial_t v + \big(u\partial_x + v\partial_y + w\partial_z\big)\, v - \partial_z^2 v = 0, \\ \partial_x u + \partial_y v + \partial_z w = 0 \\ (u, v, w)|_{z=\mathbf{0}} = (0, 0, 0), \qquad \lim\limits_{z\to+\infty} (u, v) = \big(0, 0\big), \\ u\big|_{t=\mathbf{0}} = u_{\mathbf{0}}, \quad v\big|_{t=\mathbf{0}} = v_{\mathbf{0}}, \end{cases}$$

Assumption

Suppose $(u_0, v_0) \in X_{\rho_0,\sigma}$ for $1 < \sigma \leq 2$, where we say $(u, v) \in X_{\rho,\sigma}$ if

$$\|(u,v)\|_{\rho,\sigma} = \sup_{\substack{0\leq j\leq 5\\ |\alpha|+j\geq 7}} \frac{\rho^{|\alpha|+j-7}}{[(|\alpha|+j-7)!]^{\sigma}} \left\| \langle z\rangle^{\ell+j}\, \partial_{x,y}^{\alpha}\partial_z^{j}(u,v)\right\|_{L^2} + \sup_{\substack{0\leq j\leq 5\\ |\alpha|+j\leq 6}} \left\| \langle z\rangle^{\ell+j}\, \partial_{x,y}^{\alpha}\partial_z^{j}(u,v)\right\|_{L^2}.$$

where $\ell > 1/2$ and $\langle z\rangle = (1+|z|^2)^{1/2}$

Well-poseness in Gevrey class

Theorem (L.-Masmoudi-Yang)

Suppose $(u_{\mathbf{0}}, v_{\mathbf{0}}) \in X_{\rho_0,\sigma}$ *for* $1 < \sigma \leq 2$*, compatible with boundary condtions. Then the 3D Prandtl system admits a unique solution* $(u, v) \in L^\infty\big([0, T];\ X_{\rho,\sigma}\big)$ *for some* $T > 0$ *and some* $0 < \rho < \rho_{\mathbf{0}}$.

Outlines

Methodology

Cancellation mechanism + Abstract Cauchy-Kowalewski (ACK) theorem

Outlines

Intrinsic cancellation law

Observe, applying ∂_x to eq. $\left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)u = 0$,

$$\left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)\partial_x u = -(\partial_y u)\partial_x v(t,x,y) - (\partial_x u)^2$$

with bad term $\partial_x v$

Auxilliary function $\mathcal{U}$, inspired by Dietert and Gérard-Varet [Annals of PDE (2019)],

$$\begin{cases} \left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right) \displaystyle\int_0^y \mathcal{U}(t,x,r)dr = -\partial_x v(t,x,y), \\ \mathcal{U}|_{t=0} = 0, \quad \partial_y \mathcal{U}|_{y=0} = \mathcal{U}|_{y\to+\infty} = 0. \end{cases}$$

Intrinsic cancellation law

$$\begin{cases} \left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right) \int_0^y \mathcal{U}(t,x,r)dr = -\partial_x v(t,x,y), \\ \left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)\partial_x u = -(\partial_y u)\partial_x v(t,x,y) - (\partial_x u)^2. \end{cases}$$

$$\xrightarrow{\text{cancellation}} \quad \begin{aligned} &\lambda = \partial_x u - (\partial_y u) \int_0^y \mathcal{U}(t,x,r)dr; \\ &\left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)\lambda = l.o.t. \end{aligned}$$

Estimate on $\lambda \Longrightarrow$ Estimate on $\partial_x u$?

$\int_0^y \mathcal{U}(t,x,r)dr$ by estimating $\mathcal{U}$

Outlines

Abstract Cauchy-Kowalewski theorem

A toy model:
$$\begin{cases} \partial_t \vec{h} = F(t, \partial_x \vec{h}), \\ \vec{h}|_{t=0} = \vec{h}_0. \end{cases} \tag{1}$$

A key estimate: $\forall\ \rho < \tilde{\rho}$,

$$|\vec{h}(t)|_\rho^2 \leq |\vec{h}_0|_\rho^2 + C \int_0^t \frac{|\vec{h}(s)|^2_{\tilde{\rho}(s)}}{\tilde{\rho}(s) - \rho}\, ds,$$

where $|\vec{h}|_\rho = \sup_{k\geq 0} \frac{\rho^k}{k!} \|\partial_x^k \vec{h}\|_{L^2}$.

Conclusion: The Cauchy problem (1) admits a unique local solution in $L^\infty([0,T]; X_\rho)$, provided $\vec{h}_0 \in X_{\rho_0}$

ACK Theorem for Gevery space

Model:
$$\begin{cases} \partial_t^2 \vec{h} = F(t, \partial_x \vec{h}), \\ \vec{h}|_{t=0} = \vec{h}_0, \quad \partial_t \vec{h}|_{t=0} = \vec{h}_1. \end{cases} \tag{2}$$

A key estimate: $\forall\ \rho < \tilde{\rho}$,

$$|\vec{b}(t)|_\rho^2 \le |\vec{b}_0|_\rho^2 + C \int_0^t \frac{|\vec{b}(s)|_{\tilde{\rho}(s)}^2}{\tilde{\rho}(s) - \rho}\, ds, \quad \vec{b} := (\vec{h}, \partial_t \vec{h})$$

where $|\vec{b}|_\rho = \sup_{k\ge 0} \frac{\rho^k}{(k!)^2} \|\partial_x^k \vec{h}\|_{L^2} + \sup_{k \ge 0} \frac{\rho^{k+1}}{(k+1)!^2} k \|\partial_x^k \partial_t \vec{h}\|_{L^2}$.

Conclusion: The Cauchy problem (2) admits a unique local solution in $L^\infty([0,T]; X_\rho)$, provided $\vec{h}_0, \vec{h}_1 \in X_{\rho_0}$

Proof of ACK for Gevrey space

Denote $\vec{\xi} = \partial_t \vec{h}$.

$$\begin{cases} \partial_t^2 \vec{h} = F(t, \partial_x \vec{h}), \\ \vec{h}|_{t=0} = \vec{h}_0, \quad \partial_t \vec{h}|_{t=0} = \vec{h}_1. \end{cases} \Longrightarrow \begin{cases} \partial_t \vec{\xi} = F(t, \partial_x \vec{h}), \\ \partial_t \vec{h} = \vec{\xi}, \\ (\vec{h}, \vec{\xi})|_{t=0} = (\vec{h}_0, \vec{h}_1). \end{cases}$$

This suggests $\vec{\xi} \approx |D_x|^{1/2} h$. Recall

$$|\vec{b}|_\rho = \sup_{k \geq 0} \frac{\rho^k}{k!^2} \|\partial_x^k \vec{h}\|_{L^2} + \sup_{k \geq 0} \frac{\rho^{k+1}}{[(k+1)!]^2} k \|\partial_x^k \vec{\xi}\|_{L^2}.$$

ACK for general model

$$\partial_t^2 \vec{h} = F(t, \partial_x \vec{h}) \xrightarrow{\text{more general}} (\underbrace{\partial_t + u\partial_x + v\partial_y - \partial_y^2}_{\text{Prandtl operator}})^2 \vec{h} = F(t, \partial_x \vec{h}).$$

Estimate on auxillary function $\mathcal{U}$

$$\begin{cases} \left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right) \int_0^y \mathcal{U}(t,x,r)dr = -\partial_x v(t,x,y), \\ \left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)\partial_x u = -(\partial_y u)\partial_x v(t,x,y) - (\partial_x u)^2. \end{cases}$$

$$\xRightarrow{\text{cancellation}} \quad \begin{aligned} &\lambda = \partial_x u - (\partial_y u) \int_0^y \mathcal{U}(t,x,r)dr; \\ &\left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)\lambda = l.o.t. \end{aligned}$$

Estimate on $\lambda \Longrightarrow$ Estimate on $\partial_x u$? $\int_0^y \mathcal{U}(t,x,r)dr$ by estimating $\mathcal{U}$

Estimate on auxillary function $\mathcal{U}$

$$\left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right) \int_0^y \mathcal{U}(t,x,r)dr = -\partial_x w.$$

$$\Longrightarrow \quad \begin{cases} (\partial_t + u\partial_x + v\partial_y - \partial_y^2)\mathcal{U} = \partial_x \lambda + l.o.t. \\ (\partial_t + u\partial_x + v\partial_y - \partial_y^2)\partial_x \lambda = \text{terms of order } \leq 2 \end{cases}$$

$$\Longrightarrow \quad \left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)^2 \mathcal{U} = \text{terms of order } \leq 2 + l.o.t.$$

Conclusion: Estimate $\mathcal{U}$ in Gevrey space ≤ 2, applying ACK theorem for $\left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)^2 \vec{h} = F(t, \partial_x \vec{h})$

A priori estimate

$\forall\ (\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$ and $\forall\ t \in [0, T]$.

$$\left|\vec{a}(t)\right|^2_{\rho,\sigma} \leq C\,\|u_0\|^2_{\rho_0,\sigma} + C\bigg(\int_0^t \left(\left|\vec{a}(s)\right|^2_{\rho,\sigma} + \left|\vec{a}(s)\right|^4_{\rho,\sigma}\right) ds + \int_0^t \frac{\left|\vec{a}(s)\right|^2_{\tilde{\rho},\sigma}}{\tilde{\rho}-\rho}\, ds\bigg),$$

where $\vec{a} = (u, \mathcal{U}, \lambda)$ and

$$\left|\vec{a}\right|_{\rho,\sigma} = \sup_{\substack{0\leq j\leq 5\\ m+j\geq 7}} \frac{\rho^{m+j-7}}{[(m+j-7)!]^\sigma} \left\| \langle y\rangle^{\ell+j}\, \partial_x^m \partial_y^j u \right\|_{L^2} + \sup_{m\geq 6} \frac{\rho^{m-6}}{[(m-6)!]^\sigma} \left\|\partial_x^m \mathcal{U}\right\|_{L^2} + \sup_{m\geq 6} \frac{\rho^{m-6}}{[(m-6)!]^\sigma}\, m \|\partial_x^m \lambda\|_{L^2}.$$

3D counterparts: auxilliary functions

$$\begin{cases} \left(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\right) \int_0^z \mathcal{U}(t,x,y,\tilde{z})d\tilde{z} = -\partial_x w(t,x,y,z), \\ \mathcal{U}|_{t=0} = 0, \quad \partial_z \mathcal{U}|_{z=0} = \mathcal{U}|_{z\to+\infty} = 0. \end{cases}$$

and

$$\begin{cases} \left(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\right) \int_0^z \widetilde{\mathcal{U}}(t,x,y,\tilde{z})d\tilde{z} = -\partial_y w(t,x,y,z), \\ \widetilde{\mathcal{U}}|_{t=0} = 0, \quad \partial_z \widetilde{\mathcal{U}}|_{z=0} = \widetilde{\mathcal{U}}|_{z\to+\infty} = 0. \end{cases}$$

3D counterparts: auxialliary functions

$$\begin{cases} \lambda = \partial_x u - (\partial_z u) \displaystyle\int_0^z \mathcal{U} d\tilde{z}, & \tilde{\lambda} = \partial_y u - (\partial_z u) \displaystyle\int_0^z \widetilde{\mathcal{U}} d\tilde{z}, \\ \delta = \partial_x v - (\partial_z v) \displaystyle\int_0^z \mathcal{U} d\tilde{z}, & \tilde{\delta} = \partial_y v - (\partial_z v) \displaystyle\int_0^z \widetilde{\mathcal{U}} d\tilde{z}, \end{cases}$$

Thanks for your attention!