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A pathwise regularization by noise phenomenon for the evolutionary p-Laplace equation

Jörn Wichmann

joint work with Florian Bechtold (Bielefeld)

Asia-Pacific – Analysis and PDE Seminar

SFB 1283

UNIVERSITÄT BIELEFELD

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The p-Laplace equation

$$\mathcal{J}(u) = \int_{\mathcal{O}} \frac{1}{p}|\nabla u|^p - B(u)\,\mathrm{d}x$$

$$\partial_t u = -D\mathcal{J}(u) \quad \Leftrightarrow \quad \partial_t u - \operatorname{div} S(\nabla u) = b(u), \quad S(A) = |A|^{p-2}A$$

Conditions for b (resp. B) to get well-posedness?

Example

- $b(u) \equiv f \in L^{p'}W^{-1,p'} + L^1L^2$
- $b(u) = |u|^{r-2}u$, $r \in \mathbb{R}$
- $b(u) = \tilde{b}(u - w)$, w regularizing path

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Potentials and Literature

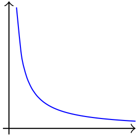
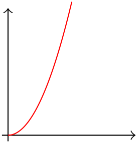


Figure: $b(u) = |u|^{-\alpha}$



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Monotone operator theory

- elliptic: [Minty '62], [Browder '63], [Leray, Lions '65]
- parabolic: [Kato '67], [Brézis '73], [Barbu '76], [Otani '77], [Coulhon, Hauer '16], [Arendt, Hauer '20]
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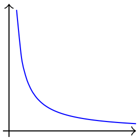

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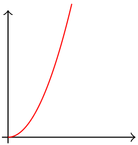

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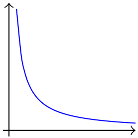

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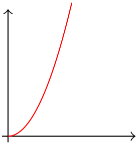

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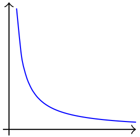

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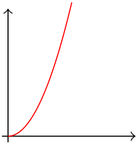

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Phenomena of regularization by noise

Regularization by transport

- B. Gess. "Regularization and well-posedness by noise for ordinary and partial differential equations". In: *Stochastic partial differential equations and related fields*. Vol. 229. Springer Proc. Math. Stat. Springer, Cham, 2018, pp. 43–67. DOI: 10.1007/978-3-319-74929-7_3
- T. Lange. "Regularization by noise of an averaged version of the Navier-Stokes equations". In: *arXiv e-prints*, arXiv:2205.14941 (May 2022), arXiv:2205.14941. arXiv: 2205.14941 [math.PR]
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Application in numeric

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Local smoothing - Freezing

$$\begin{aligned}\int_s^t b(u-w)\,\mathrm{d}\tau &\approx \int_s^t b(\langle u\rangle - w)\,\mathrm{d}\tau \\ &= \int_{\mathbb{R}} b(\langle u\rangle - x)\big|\{\tau\in[s,t]\,|\,w_\tau \in \mathrm{d}x\}\big| \\ &= \int_{\mathbb{R}} b(\langle u\rangle - x) L_{s,t}(x)\,\mathrm{d}x = \big(b * L_{s,t}\big)(\langle u\rangle)\end{aligned}$$

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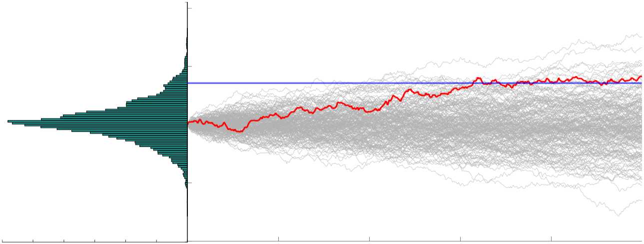

Global reconstruction - Sewing Lemma

$$\int_0^t b(u-w)\,\mathrm{d}\tau = \sum_k \int_{t_{k-1}}^{t_k} b(u-w)\,\mathrm{d}\tau = \sum_k B_{t_{k-1},t_k}$$

$$\approx \sum_k \int_{t_{k-1}}^{t_k} b(\langle u\rangle_k - w)\,\mathrm{d}\tau = \sum_k A_{t_{k-1},t_k}$$

Assume $\delta A_{srt} = A_{st} - (A_{sr} + A_{rt})$ *with*

$$\sup_{srt} |\delta A_{srt}||t-s|^{-(1+\alpha)} = [\delta A]_{1+\alpha} < \infty.$$

Then exists $\mathcal{I}(A) = \lim_{|\pi|\to 0} \sum_{[st]\in\pi} A_{s,t}$ *with*

$$|\mathcal{I}(A)_t - \mathcal{I}(A)_s - A_{st}| \lesssim [\delta A]_{1+\alpha} |t-s|^{1+\alpha}.$$

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Energy bounds – open

$$\partial_t u - \operatorname{div} S(\nabla u) = b(u - w)$$

$$\int_t \int_x |\partial_t u - \operatorname{div} S(\nabla u)|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_t \int_x |b(u-w)|^2 \,\mathrm{d}x\,\mathrm{d}t$$

Strongly finite energy solutions

$$\|\partial_t u\|^2_{L^2_t L^2_x} + \|\nabla u\|^p_{L^\infty_t L^p_x} + \|\operatorname{div} S(\nabla u)\|^2_{L^2_t L^2_x} \lesssim \|\nabla u_0\|^p_{L^p_x} + \int_t \int_x |b(u-w)|^2 \,\mathrm{d}x\,\mathrm{d}t$$

Time regularity

$$[u]^2_{C^{0,1/2}_t L^2_x} \le \|\partial_t u\|^2_{L^2_t L^2_x}$$

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A germ estimate

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Local representation

$$A_{st} = \int_x \left(|b|^2 * L_{st}\right)(u(s))\,\mathrm{d}x$$

Local error

$$\begin{aligned}\delta A_{srt} &= \int_x \left(|b|^2 * L_{st}\right)(u(s)) - \left(|b|^2 * L_{st}\right)(u(r))\,\mathrm{d}x \\ &\le \left[|b|^2 * L_{st}\right]_{C^{0,1}(\mathbb{R})} \|u(s)-u(r)\|_{L^1_x} \\ &\le \|b\|^2_{L^2(\mathbb{R})} [L_{st}]_{C^{0,1}(\mathbb{R})} \|u(s)-u(r)\|_{L^1_x}\end{aligned}$$

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Energy bounds – closed

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$$\lesssim \|\nabla u_0\|^p_{L^p_x} + \int_t \int_x |b(u-w)|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \|\nabla u_0\|^p_{L^p_x} + \|b\|^2_{L^2(\mathbb{R})} \|L\|_{C_t^{0,1/2+\delta} C^{0,1}(\mathbb{R})} \|\partial_t u\|_{L^2_t L^2_x}$$

Definition (robustified solution)

- $u \in \left\{ v \in C_t^{0,1/2} L^2_x \cap L^\infty_t W^{1,p}_{0,x} \,\middle|\, \partial_t v, \operatorname{div} S(\nabla v) \in L^2_t L^2_x \right\}$
- for all t

$$u_t - u_0 - \int_0^t \operatorname{div} S(\nabla u_r)\,\mathrm{d}r = (\mathcal{I}A^u)_{0,t}$$

where $(\mathcal{I}A^u)_{0,t}$ is sewing of $A^u_{s,t} = (b * L_{s,t})(u_s)$

Energy bounds – closed

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\begin{aligned}
\|\partial_t u\|_{L_t^2 L_x^2}^2 &+ \|\nabla u\|_{L_t^\infty L_x^p}^p + \|\operatorname{div} S(\nabla u)\|_{L_t^2 L_x^2}^2 \\
&\lesssim \|\nabla u_0\|_{L_x^p}^p + \int_t \int_x |b(u-w)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim \|\nabla u_0\|_{L_x^p}^p + \|b\|_{L^2(\mathbb{R})}^2 \|L\|_{C_t^{0,1/2+\delta} C^{0,1}(\mathbb{R})} \|\partial_t u\|_{L_t^2 L_x^2}
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Main result

$$\partial_t u - \operatorname{div} S(\nabla u) = \partial_t \mathcal{I} A^u, \quad u|_\partial = 0, \quad u(0) = u_0$$

(Theorem 1.1 in [Bechtold, Wichmann (22+)])

Let $r \in [1, \infty]$ and

- $u_0 \in L^2_x \cap W^{1,p}_x$
- *w be a regularizing path with localtime* $L \in C_t^{0,1/2+\delta} W^{1,r'}(\mathbb{R})$
- $b \in L^{2r}(\mathbb{R})$

Then exists robustified solution u with

$$\|u\|^2_{L^\infty_t L^2_x} + \|\nabla u\|^p_{L^\infty_t L^p_x} + \|\partial_t u\|^2_{L^2_t L^2_x} + \|\operatorname{div} S(\nabla u)\|^2_{L^2_t L^2_x}$$
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Comments on the proof

1. approximate potential

$$b^\varepsilon \in C_b^\infty \text{ such that } b^\varepsilon \to b \in L^{2r}$$

2. classical theory

$$b^\varepsilon \in C_b^\infty \Rightarrow \exists!\, u^\varepsilon$$

3. new a priori bounds

$$\|u^\varepsilon\|^2_{L^\infty_t L^2_x} + \|\nabla u^\varepsilon\|^p_{L^\infty_t L^p_x} + \|\partial_t u^\varepsilon\|^2_{L^2_t L^2_x} + \|\operatorname{div} S(\nabla u^\varepsilon)\|^2_{L^2_t L^2_x}$$

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$$u^\varepsilon \to u \in C_t L^2_x$$

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Non-triviality and example

(Theorem 3.1 in [Harang, Perkowski '21])

Let w^H be a fBM with $H < 1/d$. Then exists localtime L and

$$\|L\|_{C_t^{0,\alpha} H^\beta} < \infty \quad \text{for} \quad \beta < 1/(2H) - d/2 \,\&\, \alpha < 1 - (\beta + d/2)H$$

Example

$$b(x) = |x|^{-\alpha} e^{-x^2} \text{ for } \alpha < 1/2$$

Alternative formulation $v = u - w$

$$\begin{cases} \mathrm{d}v - \mathrm{div} S(\nabla v)\, \mathrm{d}t & = b(v)\, \mathrm{d}t - \mathrm{d}w \\ v|_\partial & = -w \\ v(0) & = u(0) - w(0) \end{cases}$$

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Summary

- sewing justifies freezing
- freezing implies local smoothing via localtimes

⇝ stable energy bounds for singular potentials

- applicable to wider class of equations

(Bechtold, Wichmann (22+))

For singular but shifted potentials $b(u-w)$ one can construct robustified solutions u such that

$$\partial_t u - \mathrm{div} S(\nabla u) = \partial_t \mathcal{I} A^u \approx b(u-w).$$

F. Bechtold and J. Wichmann. "A pathwise regularization by noise phenomenon for the evolutionary p-Laplace equation". In: *arXiv e-prints*, arXiv:2209.13448 (Sept. 2022). arXiv: 2209.13448 [math.AP]

Thank you for your attention! Questions?

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