On Principal Eigenvalues for Elliptic and Parabolic Operators

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Asia-Pacific Analysis and PDEs Seminar
2021.11.01

Talk Outline

1. Introduction
2. Large drift limit
3. Small diffusion limit
4. Frequency and eigenvalue

Time-periodic parabolic operators

$$\begin{cases} \tau\partial_t\varphi - D\Delta\varphi + \alpha\mathbf{v}\cdot\nabla\varphi + c\varphi = \lambda\varphi, & x\in\Omega, t\in[0,1], \\ b\varphi + (1-b)\nabla\varphi\cdot\mathbf{n} = 0 & x\in\partial\Omega, t\in[0,1], \\ \varphi(x,0) = \varphi(x,1), & x\in\Omega. \end{cases}$$

- **v**: C^1 vector field, time-periodic with unit period
- c: continuous, time-periodic with unit period; $b \in [0,1]$
- Principal eigenvalue $\lambda = \lambda(D,\alpha,\tau)$: It is real and has the smallest real part among all eigenvalues (Hess, 1991)
- Q. How does principal eigenvalue depend on D, α, τ?

Principal eigenvalue for elliptic operator

$$\begin{cases} -D\Delta\varphi + \alpha\mathbf{v}\cdot\nabla\varphi + c(x)\varphi = \lambda(\alpha)\varphi & \text{in } \Omega, \\ b\varphi + (1-b)\nabla\varphi\cdot\mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

- $\mathbf{v}(x)$: C^1 vector field; $c(x) \in C(\bar{\Omega})$; $b \in [0,1]$
- Principal eigenvalue $\lambda(D,\alpha)$: It is real, simple and has the smallest real part among all eigenvalues
- Q. How does principal eigenvalue depend on D, α?

Dirichlet boundary condition

$$\begin{cases} -\Delta\varphi + \alpha\mathbf{v}\cdot\nabla\varphi = \lambda(\alpha)\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

- Wentzell (1975):

$$\lim_{\alpha\to+\infty} \frac{\lambda(\alpha)}{\alpha^2} = \frac{1}{4} \lim_{T\to+\infty} \inf_{X} \frac{\int_0^T |X'(t) + \mathbf{v}(X(t))|^2\, dt}{T} < +\infty,$$

where $X \in C^1([0, T]; \bar{\Omega})$.

- $\lambda(\alpha) = O(\alpha^2)$

Devinatz, Ellis, Friedman (1973-75)

- $\mathbf{v} \cdot \nabla\phi < 0$ in $\bar{\Omega}$ for some $\phi \in C^1(\bar{\Omega})$: $\exists c > 0$ s.t. for $\alpha \gg 1$,

$$\lambda(\alpha) \geq c\alpha^2;$$

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- $\mathbf{v} = -\nabla U(x)$ with $|\nabla U| > 0$ in $\bar{\Omega}$:

$$\lim_{\alpha\to\infty} \frac{\lambda(\alpha)}{\alpha^2} = \frac{1}{4}\min_{\bar{\Omega}} |\nabla U|^2;$$

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- $\mathbf{v}$ vanishes at some point in Ω: $\lambda(\alpha) = O(\alpha)$

- $\mathbf{v}\cdot\mathbf{n} < 0$ on $\partial\Omega$: $\lambda(\alpha) = O(\alpha e^{-c\alpha})$

Divergence free vector field

Theorem 1 (Berestycki-Hamel-Nadirashvili 2005)

If div$(\mathbf{v}) = 0$ *in* Ω*, then*

$$\lim_{\alpha\to\infty} \lambda(\alpha) = \inf_{\varphi\in\mathcal{I}_1} \frac{\int_\Omega(|\nabla\varphi|^2 + c(x)\varphi^2)}{\int_\Omega \varphi^2} \leq +\infty,$$

where

$$\mathcal{I}_1 = \{\varphi \in H_0^1(\Omega) : \varphi \neq 0, \mathbf{v}\cdot\nabla\varphi = 0 \text{ a.e. in } \Omega\}$$

- $\lim_{\alpha\to\infty} \lambda(\alpha) = +\infty$ iff $\mathcal{I}_1$ is empty.

Neumann boundary condition

$$\begin{cases} -\Delta\varphi + \alpha\mathbf{v}\cdot\nabla\varphi + c(x)\varphi = \lambda(\alpha)\varphi \quad \text{in } \Omega, \\ \nabla\varphi\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega, \\ \varphi > 0 \quad \text{in } \Omega. \end{cases}$$

- For any α,

$$\min_{\bar{\Omega}} c \le \lambda(\alpha) \le \max_{\bar{\Omega}} c.$$

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- For any α,

$$\min_{\bar{\Omega}} c \le \lambda(\alpha) \le \max_{\bar{\Omega}} c.$$

- Question: $\lim_{\alpha\to\infty} \lambda(\alpha) = ?$

Divergent free drift

[Berestycki-Hamel-Nadirashvili, CMP 2005]

- Assume $div(\mathbf{v}) = 0$ in Ω and $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$, then

$$\lim_{\alpha\to\infty} \lambda(\alpha) = \inf_{\varphi\in\mathcal{I}} \frac{\int_\Omega (|\nabla\varphi|^2 + c(x)\varphi^2)}{\int_\Omega \varphi^2} < +\infty,$$

where

$$\mathcal{I} = \{\varphi \in H^1(\Omega) : \varphi \neq 0, \mathbf{v} \cdot \nabla\varphi = 0 \text{ a.e. in } \Omega\}$$

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- If $\mathcal{I}$ consists of constants only,

$$\lim_{\alpha \to \infty} \lambda(\alpha) = \frac{1}{|\Omega|} \int_\Omega c$$

Potential flow

$$\begin{cases} -\Delta\varphi - \alpha\nabla m \cdot \nabla\varphi + c(x)\varphi = \lambda(\alpha)\varphi \quad \text{in } \Omega, \\ \nabla\varphi \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

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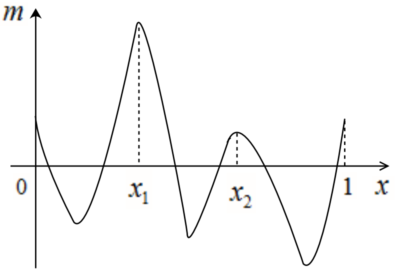
Theorem 2 (Chen-L, IUMJ 2008)

Suppose that $m \in C^2(\bar{\Omega})$ and all critical points of m are non-degenerate, $c \in C(\bar{\Omega})$. Then

$$\lim_{\alpha\to\infty} \lambda(\alpha) = \min_{x\in\mathcal{M}} c(x),$$

where $\mathcal{M}$ is the set of points of local maximum of m.

An illustration of Theorem 2



$$\boxed{\lim_{\alpha\to\infty} \lambda(\alpha) = \min\left\{c(0),\ c(x_1),\ c(x_2),\ c(1)\right\}.}$$

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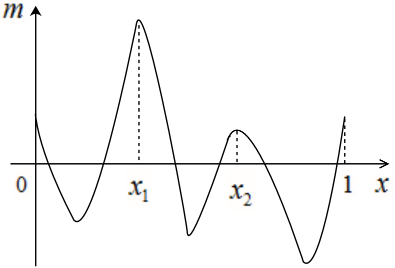

$$\lim_{\alpha\to\infty} \lambda(\alpha) = \min\{c(0),\ c(x_1),\ c(x_2),\ c(1)\}.$$

- Concentration of mass: As $\alpha \to \infty$, principal eigenfunction $\varphi(\alpha,\cdot)$ concentrates on some local maximum of m

Time-periodic parabolic operator

$$
\begin{cases}
\varphi_t - \varphi_{xx} - \alpha m_x(x,t)\varphi_x + c(x,t)\varphi = \lambda(\alpha)\varphi & \text{in } (0,1)\times[0,1], \\
\varphi_x(0,t) = \varphi_x(1,t) = 0 & \text{on } [0,1], \\
\varphi(x,t) = \varphi(x,t+1) & \text{on } (0,1).
\end{cases}
$$

- What is the limit of $\lambda(\alpha)$ when $\alpha \to \infty$?

Time-periodic parabolic operator

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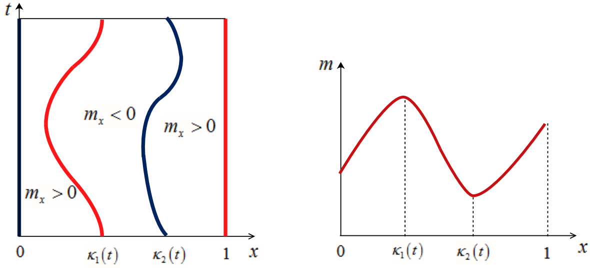
- What is the limit of $\lambda(\alpha)$ when $\alpha \to \infty$?
- The limit relies on spatially critical points of m:

$$\{(x,t) \in [0,1]\times[0,1] : m_x(x,t) = 0\}$$

Example

Assumption on m: $\exists \kappa_1(t), \kappa_2(t) \in (0,1)$ such that

$$\begin{cases} m_x > 0, & x \in (0, \kappa_1(t)) \cup (\kappa_2(t), 1)\,, \;\; t \in [0,1], \\ m_x = 0, & x \in \{\kappa_1(t),\, \kappa_2(t)\}, \;\; t \in [0,1], \\ m_x < 0, & x \in (\kappa_1(t), \kappa_2(t))\,, \;\; t \in [0,1]. \end{cases}$$



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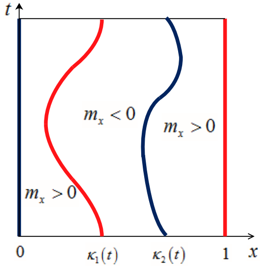

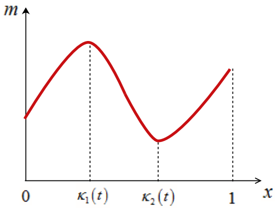

$$\lim_{\alpha\to\infty} \lambda(\alpha) = \min\left\{ \int_0^1 c(\kappa_1(s), s)\,ds, \ \int_0^1 c(1, s)\,ds \right\}.$$

Asymptotic with large advection

Theorem 3 (Liu-L-Peng-Zhou, SIMA 2021)

Suppose that all spatially critical points of m are non-degenerate. Let $\{(\kappa_i(t), t) : t \in [0, 1], 1 \le i \le N\}$ *be the set of points of local maximum of m. Then*

$$\lim_{\alpha \to \infty} \lambda(\alpha) = \min_{1 \le i \le N} \left\{ \int_0^1 c\,(\kappa_i(s), s)\, \mathrm{d}s \right\}.$$

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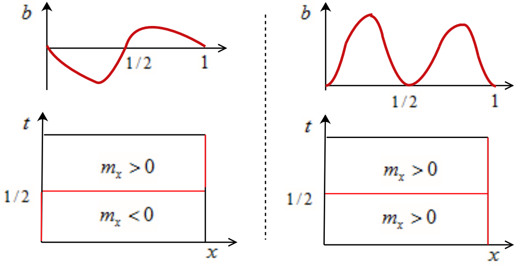
$$\lim_{\alpha\to\infty} \lambda(\alpha) = \min_{1\leq i\leq N} \left\{ \int_0^1 c\,(\kappa_i(s), s)\, \mathrm{d}s \right\}.$$

- Peng-Zhao, CVPDE 2015: If $m_x > 0$, (resp. $m_x < 0$), then

$$\lim_{\alpha\to\infty} \lambda(\alpha) = \int_0^1 c\,(1, s)\, \mathrm{d}s \quad \left(\textit{resp.} \lim_{\alpha\to\infty} \lambda(\alpha) = \int_0^1 c\,(0, s)\, \mathrm{d}s \right).$$

An example: $m(x,t) = b(t)x$

- Assumption on b:



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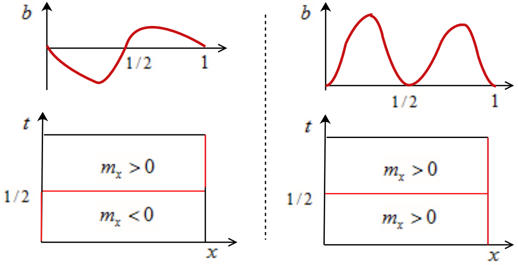
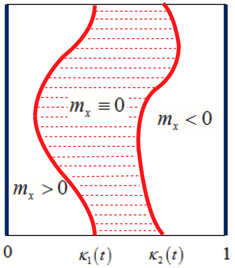


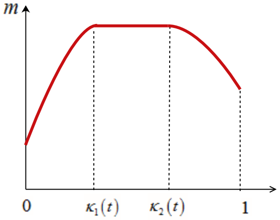
- Left figure: $\lim_{\alpha\to\infty} \lambda(\alpha) = \int_0^{\frac{1}{2}} c(0,s)\mathrm{d}s + \int_{\frac{1}{2}}^1 c(1,s)\mathrm{d}s$;
- Right figure: $\lim_{\alpha\to\infty} \lambda(\alpha) = \int_0^1 c(1,s)\mathrm{d}s$.

Case 3: Spatial degeneracy

Assumption on m: $\exists \kappa_1(t), \kappa_2(t) \in (0,1)$ such that

$$\begin{cases} m_x > 0, & x \in (0, \kappa_1(t)) \cup (\kappa_2(t), 1), \ \ t \in [0,1], \\ m_x \equiv 0, & x \in \{\kappa_1(t), \ \kappa_2(t)\}, \ \ t \in [0,1], \\ m_x < 0, & x \in (\kappa_1(t), \kappa_2(t)), \ \ t \in [0,1]. \end{cases}$$





Case 3: Spatial degeneracy

Proposition 4 (Liu-L-Peng-Zhou)

$$\boxed{\lim_{\alpha\to\infty} \lambda(\alpha) = \lambda^{\mathcal{N}\mathcal{N}}((\kappa_1, \kappa_2)),}$$

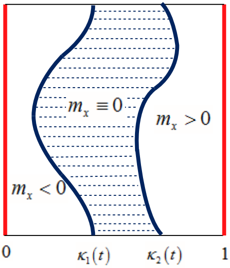
where $\lambda^{\mathcal{N}\mathcal{N}}((\kappa_1, \kappa_2))$ *denotes the principal eigenvalue of*

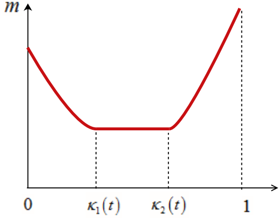
$$\begin{cases} \psi_t - \psi_{xx} + c(x,t)\psi = \lambda\psi, & x \in (\kappa_1(t),\ \kappa_2(t)),\ t \in [0,1], \\ \psi_x(\kappa_1(t), t) = \psi_x(\kappa_2(t), t) = 0, & t \in [0,1]\,, \\ \psi(x,0) = \psi(x,1), & x \in [\kappa_1(t), \kappa_2(t)]. \end{cases}$$

Case 4: Spatial degeneracy

Assumption on m: $\exists \kappa_1(t), \kappa_2(t) \in (0,1)$ such that

$$\begin{cases} m_x < 0, & x \in (0, \kappa_1(t)) \cup (\kappa_2(t), 1), \ \ t \in [0,1], \\ m_x \equiv 0, & x \in \{\kappa_1(t), \ \kappa_2(t)\}, \ \ t \in [0,1], \\ m_x > 0, & x \in (\kappa_1(t), \kappa_2(t)), \ \ t \in [0,1]. \end{cases}$$





Case 4: Spatial degeneracy

Proposition 5 (Liu-L-Peng-Zhou)

$$\boxed{\lim_{\alpha\to\infty}\lambda(\alpha)=\min\left\{\lambda^{\mathcal{DD}}((\kappa_1,\kappa_2)),\ \int_0^1 c(0,s)\,\mathrm{d}s,\ \int_0^1 c(1,s)\,\mathrm{d}s\right\},}$$

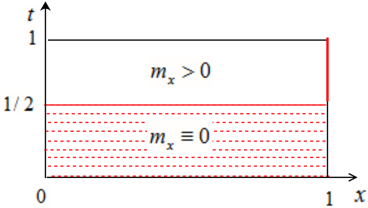
where $\lambda^{\mathcal{DD}}((\kappa_1,\kappa_2))$ *denotes the principal eigenvalue of*

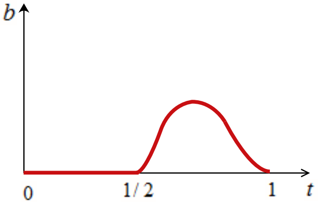
$$\begin{cases}\psi_t-\psi_{xx}+c(x,t)\psi=\lambda\psi, & x\in(\kappa_1(t),\ \kappa_2(t)),\ t\in[0,1],\\ \psi(\kappa_1(t),t)=\psi(\kappa_2(t),t)=0, & t\in[0,1],\\ \psi(x,0)=\psi(x,1), & x\in[\kappa_1(t),\kappa_2(t)].\end{cases}$$

Case 5: Temporal degeneracy

$$\begin{cases} \varphi_t - \varphi_{xx} - \alpha b(t)\varphi_x + c(x,t)\varphi = \lambda(\alpha)\varphi & \text{in } (0,1)\times[0,1], \\ \varphi_x(0,t) = \varphi_x(1,t) = 0 & \text{on } [0,1], \\ \varphi(x,0) = \varphi(x,1) & \text{on } (0,1). \end{cases}$$

- Assume $b \equiv 0$ for $t \in [0, \frac{1}{2}]$ and $b > 0$ for $t \in (\frac{1}{2}, 1)$.





Case 5: Temporal degeneracy

Proposition 6 (Liu-L-Peng-Zhou)

$$\boxed{\lim_{\alpha\to\infty}\lambda(\alpha)=\lambda^*,}$$

where λ^* *is the principal eigenvalue of*

$$\begin{cases}\psi_t-\psi_{xx}+c(x,t)\psi=\lambda\psi & \text{in } (0,1)\times(0,\frac{1}{2}],\\ \psi_t+c(1,t)\psi=\lambda\psi & \text{on } (\frac{1}{2},1],\\ \psi(x,\frac{1}{2}+)\equiv\psi(1,\frac{1}{2}-) & \text{on } (0,1),\\ \psi_x(0,t)=\psi_x(1,t)=0 & \text{on } [0,1],\\ \psi(x,0)=\psi(x,1) & \text{on } (0,1).\end{cases}$$

Small diffusion limit

$$\begin{cases} -D\Delta\varphi + c(x)\varphi = \lambda(D)\varphi & \text{in } \Omega, \\ \nabla\varphi \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

- Concentration of mass:

$$\lim_{D\to 0} \lambda(D) = \min_{x\in\bar{\Omega}} c(x).$$

- Cooperative elliptic system: Dancer (2009); Lam-L (2016)
- Time-periodic cooperative system: Shen et al. (2017, 2018, 2019); Liang-Zhang-Zhao (2017, 2019); Bai-He (2020)

Small diffusion limit

$$\begin{cases} -D\Delta\varphi - \nabla m \cdot \nabla\varphi + c(x)\varphi = \lambda(D)\varphi & \text{in } \Omega, \\ \nabla\varphi \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 7 (Chen-L, IUMJ 2012)

Assume $|\nabla m| \neq 0$ *on* $\partial\Omega$*, all critical points of* m *are non-degenerate.*

$$\lim_{D\to 0} \lambda(D) = \inf_{x\in\Sigma_1\cup\Sigma_2} \left\{ c(x) + \frac{1}{2}\sum_{i=1}^{n} \Big(|\kappa_i(x)| + \kappa_i(x) \Big) \right\},$$

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where $\kappa_i(x)$ *are eigenvalues of* $D^2 m(x)$*, and*

$$\Sigma_1 = \{x \in \Omega : |\nabla m| = 0\},$$
$$\Sigma_2 = \{x \in \partial\Omega : |\nabla m| = \nabla m \cdot \mathbf{n} > 0\}$$

Dirichlet condition

$$\begin{cases} -D\Delta\varphi - \nabla m \cdot \nabla\varphi + c(x)\varphi = \lambda(D)\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 8 (Peng-Zhang-Zhou, SIMA 2019)

Assume $|\nabla m| \neq 0$ on $\partial\Omega$, critical points of m are non-degenerate.

(i) *If $\Sigma_1 = \emptyset$, then $\lim_{D\to 0} \lambda(D) = +\infty$;*

(ii) *If $\Sigma_1 \neq \emptyset$, then*

$$\lim_{D\to 0} \lambda(D) = \inf_{x\in\Sigma_1} \left\{ c(x) + \frac{1}{2}\sum_{i=1}^{n} \Big(|\kappa_i(x)| + \kappa_i(x) \Big) \right\}.$$

Robin condition

$$\begin{cases} -D\Delta\varphi - \nabla m\cdot\nabla\varphi + c(x)\varphi = \lambda(D)\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial n} + \beta(x)\varphi = 0 & \text{on } \partial\Omega \end{cases}$$

Theorem 9 (Peng-Zhang-Zhou, SIMA 2019)

Assume $|\nabla m| \neq 0$ *on* $\partial\Omega$*, critical points of* m *are non-degenerate.*

$$\begin{aligned} &\lim_{D\to 0}\lambda(D) \\ &= \min\Bigg\{ \inf_{x\in\Sigma_1}\Big\{c(x) + \frac{1}{2}\sum_{i=1}^{n}(|\kappa_i(x)| + \kappa_i(x))\Big\}, \\ &\qquad \inf_{x\in\Sigma_2}\Big\{c(x) + \frac{1}{2}\sum_{i=1}^{n}(|\kappa_i(x)| + \kappa_i(x)) + \beta(x)|\nabla m(x)|\Big\}\Bigg\}. \end{aligned}$$

Time-periodic parabolic operators

$$\begin{cases} \varphi_t - D\varphi_{xx} - m_x(x,t)\varphi_x + c(x,t)\varphi = \lambda(D)\varphi & \text{in } (0,1)\times[0,1], \\ \varphi_x(0,t) = \varphi_x(1,t) = 0 & \text{on } [0,1], \\ \varphi(x,0) = \varphi(x,1) & \text{on } (0,1). \end{cases}$$

- Set $p_+(x,t) := \max\{p(x,t),0\}$, $\hat{p}(x) = \int_0^1 p(x,t)\,dt$.
- Redefine $\hat{m}_{xx}(0)$ and $\hat{m}_{xx}(1)$ as

$$\hat{m}_{xx}(0) = \begin{cases} -\infty & \text{if } \hat{m}_x(0+) < 0, \\ +\infty & \text{if } \hat{m}_x(0+) > 0, \end{cases}$$

$$\hat{m}_{xx}(1) = \begin{cases} +\infty & \text{if } \hat{m}_x(1-) < 0, \\ -\infty & \text{if } \hat{m}_x(1-) > 0. \end{cases}$$

Small diffusion

$$\begin{cases} \frac{dP}{dt} = -m_x\left(P(t), t\right), \quad t > 0, \\ P(t) = P(t+1). \end{cases} \qquad \text{(ODE)}$$

Theorem 10 (Liu-L-Peng-Zhou, TAMS 2021)

Assume $m_x(0,t) \neq 0$ and $m_x(1,t) \neq 0$ for $t \in [0,1]$.

(i) *If (ODE) has finite number of periodic solutions $\{P_i(t)\}_{i=1}^N$, satisfying $0 = P_0 < P_1(t) < \ldots < P_N(t) < P_{N+1} = 1$, and $m_{xx}\left(P_i(t), t\right) \neq 0$ for $1 \leq i \leq N$ and $t \in [0,1]$, then*

$$\lim_{D\to 0} \lambda(D) = \min_{0\leq i\leq N+1} \left\{ \int_0^1 \Big[c\left(P_i(s), s\right) + [m_{xx}]_+\left(P_i(s), s\right) \Big] \mathrm{d}s \right\}.$$

Small diffusion limit

$$\begin{cases} \frac{dP}{dt} = -m_x\left(P(t), t\right), \\ P(t) = P(t+1). \end{cases} \qquad \text{(ODE)}$$

Theorem 11 (Liu-L-Peng-Zhou)

Assume $m_x(0,t) \neq 0$ *and* $m_x(1,t) \neq 0$ *for* $t \in [0,1]$.

(ii) *If* (ODE) *has no periodic solutions, then*

$$\lim_{D\to 0} \lambda(D) = \min\left\{ \hat{c}(0) + [\hat{m}_{xx}]_+(0),\ \hat{c}(1) + [\hat{m}_{xx}]_+(1) \right\}.$$

Special case: $m = \alpha b(t)x$

$$
\begin{cases}
\varphi_t - D\varphi_{xx} - \alpha b(t)\varphi_x + c(x,t)\varphi = \lambda(D)\varphi & \text{in } (0,1)\times[0,1],\\
\varphi_x(0,t) = \varphi_x(1,t) = 0 & \text{on } [0,1],\\
\varphi(x,0) = \varphi(x,1) & \text{on } [0,1].
\end{cases}
$$

Theorem 12 (Liu-L-Peng-Zhou)

(i) *If $\hat{b} \neq 0$, then for all $\alpha > 0$,*

$$
\lim_{D\to 0} \lambda(D) = \begin{cases}
\hat{c}(1) & \text{for } \hat{b} > 0,\\
\hat{c}(0) & \text{for } \hat{b} < 0.
\end{cases}
$$

Special case: $m = \alpha b(t)x$

Theorem 13 (Liu-L-Peng-Zhou)

(ii) *If* $\hat{b} = 0$, *set* $P(t) = -\int_0^t b(s)\mathrm{d}s$, $\overline{P} = \max P$, $\underline{P} = \min P$. *Then*

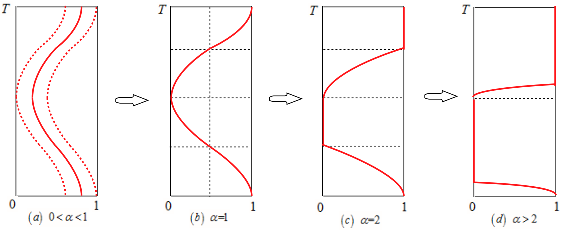
$$\lim_{D\to 0} \lambda(D) = \begin{cases} \min\limits_{y\in[-\alpha\underline{P}, 1-\alpha\overline{P}]} \int_0^1 c(\alpha P(s)+y, s)\mathrm{d}s & 0<\alpha\le \frac{1}{\overline{P}-\underline{P}}, \\ \int_0^1 c(\tilde{P}_\alpha(s), s)\mathrm{d}s, & \alpha > \frac{1}{\overline{P}-\underline{P}}, \end{cases}$$

where $\tilde{P}_\alpha$ *is the unique periodic solution of* $\dot{\tilde{P}}(t) = -\alpha F\left(\tilde{P}(t), t\right)$ *with*

$$F(x,t) = \begin{cases} b(t) & 0<x<1,\ t\in[0,1], \\ \min\{b(t),0\}, & x=0,\ t\in[0,1], \\ \max\{b(t),0\}, & x=1,\ t\in[0,1]. \end{cases} \quad (1)$$

An explicit example

- $b(t) = -\pi \sin(2\pi t)$, $P(t) = \frac{1}{2}\cos(2\pi t) - \frac{1}{2}$, $\overline{P} = 0$, $\underline{P} = -1$



(a) $\alpha < 1$: $\exists y_\alpha \in [\alpha, 1]$ s.t. $\lambda(D) \to \int_0^1 c(\alpha P(s) + y_\alpha, s)\mathrm{d}s$;

(b) $\alpha = 1$: $\lambda(D) \to \int_0^1 c(P(s) + 1, s)\mathrm{d}s$;

(c,d) $\alpha > 1$: $\lambda(D) \to \int_0^1 c(\tilde{P}_\alpha(s), s)\mathrm{d}s$.

Frequency and principal eigenvalue

$$\begin{cases} \tau\partial_t\varphi - \Delta\varphi - \nabla m\cdot\nabla\varphi + c\varphi = \lambda(\tau)\varphi, & x\in\Omega, t\in[0,1], \\ b\varphi + (1-b)\nabla\varphi\cdot\mathbf{n} = 0, & x\in\partial\Omega, t\in[0,1], \\ \varphi(x,0)=\varphi(x,1), & x\in\Omega. \end{cases}$$

- $\tau > 0$: frequency; $b\in[0,1]$.
- $m(x,t), c(x,t)$: periodic function in t with unit period.
- Given any periodic function $p(x,t)$, set $\hat{p}(x) := \int_0^1 p(x,s)\mathrm{d}s$.

Asymptotics of $\lambda(\tau)$

Theorem 14 (Liu-L-Peng-Zhou, PAMS 2019)

- *For fixed* $t \in [0,1]$, *let* $\lambda^0(t)$ *be the principal eigenvalue of*

$$\begin{cases} -\Delta\varphi - \nabla m(x,t)\cdot\nabla\varphi + c(x,t)\varphi = \lambda(t)\varphi, & x \in \Omega, \\ b\varphi + (1-b)\nabla\varphi\cdot\mathbf{n} = 0, & x \in \partial\Omega. \end{cases}$$

Then $\lim_{\tau\to 0} \lambda(\tau) = \int_0^1 \lambda^0(s)\mathrm{d}s$.

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  Then $\lim_{\tau\to 0}\lambda(\tau) = \int_0^1 \lambda^0(s)\mathrm{d}s$.

- *Let* λ^∞ *be the principal eigenvalue of*

$$\begin{cases} -\Delta\varphi - \nabla\hat{m}(x)\cdot\nabla\varphi + \hat{c}(x)\varphi = \lambda\varphi, & x \in \Omega, \\ b\varphi + (1-b)\nabla\varphi\cdot\mathbf{n} = 0, & x \in \partial\Omega. \end{cases}$$

  Then $\lim_{\tau\to\infty}\lambda(\tau) = \lambda^\infty$.

Case $\nabla m(x,t) = 0$:

- If $\nabla m(x,t) = 0$, then

$$\lim_{\tau\to 0} \lambda(\tau) = \int_0^1 \lambda^0(s)\mathrm{d}s \le \lambda^\infty = \lim_{\tau\to\infty} \lambda(\tau).$$

Case $\nabla m(x,t)=0$:

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- Hutson, Shen & Vickers (PAMS 2000)

  If $\nabla m(x,t)=\nabla\hat{m}(x)$, then

$$\lambda(\tau)\le\lim_{\tau\to\infty}\lambda(\tau)\text{ for all }\tau>0.$$

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- Hutson, Shen & Vickers (PAMS 2000)

  If $\nabla m(x,t) = \nabla \hat{m}(x)$, then

$$\lambda(\tau) \le \lim_{\tau\to\infty} \lambda(\tau) \text{ for all } \tau > 0.$$

- Conjecture: Hutson, Michaikow & Poláčik (JMB 2001)

$$\text{If } \nabla m(x,t) = 0, \text{ then } \lambda(\tau) \text{ is monotonic in } \tau.$$

Case $\nabla m(x,t) = 0$:

Theorem 15 (Liu-L-Peng-Zhou)

Assume $\nabla m = 0$. Then $\lambda(\tau)$ is non-decreasing in $\tau > 0$.

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Assume $\nabla m = 0$. Then $\lambda(\tau)$ is non-decreasing in $\tau > 0$. Furthermore,

- *If $c(x,t) = \hat{c}(x) + g(t)$, then $\lambda(\tau)$ is constant for $\tau > 0$;*
- *Otherwise $\frac{\partial \lambda}{\partial \tau}(\tau) > 0$ for every $\tau > 0$.*

Case $\nabla m(x,t) = 0$:

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- *If $c(x,t) = \hat{c}(x) + g(t)$, then $\lambda(\tau)$ is constant for $\tau > 0$;*
- *Otherwise $\frac{\partial \lambda}{\partial \tau}(\tau) > 0$ for every $\tau > 0$.*

Example for $\nabla m \neq 0$:

$$\begin{cases} \tau \partial_t \varphi - a(t)\Delta\varphi - \partial_{x_1}\varphi + \frac{-x_1 a'(t)}{2a^2(t)}\varphi = \lambda(\tau)\varphi, & x \in \Omega, t \in [0,1], \\ \nabla\varphi \cdot \mathbf{n} = 0, & x \in \partial\Omega, t \in [0,1], \\ \varphi(x,0) = \varphi(x,1), & x \in \Omega. \end{cases}$$

If $a'(t) \not\equiv 0$, then $\lim_{\tau\to\infty} \lambda(\tau) = 0 < \lambda(1)$.

Case $\nabla m(x,t) \neq 0$:

Theorem 16 (Liu-L-Peng-Zhou)

Assume $\nabla m(x,t) = \nabla \hat{m}(x)$. *Then* $\lambda(\tau)$ *is non-decreasing in* $\tau > 0$ *and*

- *If* $c = \hat{c}(x) + g(t)$, *then* $\lambda(\tau)$ *is constant for* $\tau > 0$;
- *Otherwise* $\frac{\partial \lambda}{\partial \tau}(\tau) > 0$ *for every* $\tau > 0$.

Case $\nabla m(x,t) \neq 0$:

Theorem 16 (Liu-L-Peng-Zhou)

Assume $\nabla m(x,t) = \nabla \hat{m}(x)$. Then $\lambda(\tau)$ is non-decreasing in $\tau > 0$ and

- *If $c = \hat{c}(x) + g(t)$, then $\lambda(\tau)$ is constant for $\tau > 0$;*
- *Otherwise $\frac{\partial \lambda}{\partial \tau}(\tau) > 0$ for every $\tau > 0$.*

Counterexample for $\nabla m(x,t) \neq \nabla \hat{m}(x)$:

$$\begin{cases} \tau \partial_t \varphi - \partial_{xx}\varphi + \sin x \sin t \partial_x \varphi + c(x,t)\varphi = \lambda(\tau)\varphi, & x, t \in (0, 2\pi), \\ \partial_x \varphi(0,t) = \partial_x \varphi(2\pi, t) = 0, & t \in [0, 2\pi], \\ \varphi(x,0) = \varphi(x, 2\pi), & x \in (0, 2\pi). \end{cases}$$

If $c(x,t) = \frac{1}{2}\cos x(\sin t + \cos t)$, then $\lim\limits_{\tau \to \infty} \lambda(\tau) = 0 < \lambda(1)$.

Proof: Monotonicity of $\lambda(\tau)$

$$
\begin{cases}
L_\tau \varphi_\tau := \tau \partial_t \varphi_\tau - \Delta \varphi_\tau + c(x,t)\varphi_\tau = \lambda(\tau)\varphi_\tau, & x \in \Omega, t \in [0,1], \\
b\varphi_\tau + (1-b)\nabla \varphi_\tau \cdot \mathbf{n} = 0, & x \in \partial\Omega, t \in [0,1], \\
\varphi_\tau(x,0) = \varphi_\tau(x,1), & x \in \Omega.
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$$

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- Adjoint operator: $L_\tau^* := -\tau\partial_t - \Delta + c(x,t)$.
- ψ_τ: principal eigenfunction of L_τ^*.

Proof: Monotonicity of $\lambda(\tau)$

$$\begin{cases} L_\tau\varphi_\tau := \tau\partial_t\varphi_\tau - \Delta\varphi_\tau + c(x,t)\varphi_\tau = \lambda(\tau)\varphi_\tau, & x \in \Omega, t \in [0,1], \\ b\varphi_\tau + (1-b)\nabla\varphi_\tau \cdot \mathbf{n} = 0, & x \in \partial\Omega, t \in [0,1], \\ \varphi_\tau(x,0) = \varphi_\tau(x,1), & x \in \Omega. \end{cases}$$

- Adjoint operator: $L_\tau^* := -\tau\partial_t - \Delta + c(x,t)$.
- ψ_τ: principal eigenfunction of L_τ^*.
- Let $\mathcal{C} = \Omega \times (0,1)$. Introduce functional J_τ:

$$J_\tau(\zeta) = \int_{\mathcal{C}} \varphi_\tau\psi_\tau \left(\frac{L_\tau\zeta}{\zeta}\right) \mathrm{d}x\mathrm{d}t \quad (\zeta > 0)$$

Prove $\frac{\partial \lambda}{\partial \tau} \geq 0$

Lemma 17

$$J_\tau(\varphi_\tau) - J_\tau(\zeta) = \int_{\mathcal{C}} \varphi_\tau \psi_\tau \left| \nabla \log \left(\frac{\zeta}{\varphi_\tau} \right) \right|^2$$

Prove $\frac{\partial\lambda}{\partial\tau} \geq 0$

Lemma 17

$$J_\tau(\varphi_\tau) - J_\tau(\zeta) = \int_{\mathcal{C}} \varphi_\tau\psi_\tau \left|\nabla \log\left(\frac{\zeta}{\varphi_\tau}\right)\right|^2$$

- For each $\tau > 0$, $\frac{\partial\lambda}{\partial\tau}(\tau) \geq 0$:

$$\begin{aligned}
\frac{\partial\lambda}{\partial\tau} = \int_{\mathcal{C}} \psi_\tau\partial_t\varphi_\tau = \frac{1}{2\tau}\int_{\mathcal{C}} \psi_\tau(L_\tau - L_\tau^*)\varphi_\tau \\
=&\frac{1}{2\tau}\left[\int_{\mathcal{C}} \psi_\tau L_\tau\varphi_\tau - \int_{\mathcal{C}} \varphi_\tau L_\tau\psi_\tau\right] \\
=&\frac{1}{2\tau}\left[J_\tau(\varphi_\tau) - J_\tau(\psi_\tau)\right] \\
=&\frac{1}{2\tau}\int_{\mathcal{C}} \varphi_\tau\psi_\tau\left|\nabla \log\left(\frac{\zeta}{\varphi_\tau}\right)\right|^2 \\
\geq&0.
\end{aligned}$$

Summary

- Large drift limit: Concentration at local spatial max
- Small diffusion limit: Concentration at periodic solutions (stable or unstable) of transport equation
- Frequency: Drift rate in time
- Level set of $\lambda(D, \alpha, \tau)$: topological/geometrical structure; applications to biology and infectious disease (Liu-L, 2021)

Thank you!