Singular solutions for divergence-form elliptic equations involving regular variation theory[1]

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[1] This is joint work with Florica C. Cîrstea.

Let $p > 1$ and consider nonlinear elliptic equations in divergence form

$$-\operatorname{div}\left(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u\right) + b(x)\,h(u) = 0 \quad \text{in } B^* := B_1 \setminus \{0\}, \tag{1}$$

where B_1 denotes the open unit ball centred at 0 in $\mathbb{R}^N$ ($N \geq 2$).
Let $\mathcal{A} \in C^1(0, 1]$ be a positive function such that

$$\lim_{t \to 0^+} \frac{t\mathcal{A}'(t)}{\mathcal{A}(t)} = \vartheta \in \mathbb{R}. \tag{2}$$

This means that $L_{\mathcal{A}}(t) = \mathcal{A}(t)/t^{\vartheta}$ is a positive $C^1(0, 1]$ function satisfying $\lim_{t \to 0^+} tL'(t)/L(t) = 0$. In particular, L is a slowly varying function at 0.

Assumption A. *Let* $b \in C(\overline{B_1} \setminus \{0\})$ *be positive with* $\lim_{|x| \to 0} \frac{b(x)}{b_0(|x|)} = 1$ *and* $h \in C[0, \infty)$ *be a positive non-decreasing function on* $(0, \infty)$ *such that* $h(t)/t^{p-1}$ *is bounded for small* $t > 0$.

Definition 1

A positive measurable function L defined on an interval $(0, c]$ for some $c > 0$ is called *slowly varying at (the right of) zero* if

$$\lim_{t \to 0} \frac{L(\lambda t)}{L(t)} = 1 \quad \text{for every } \lambda > 0.$$

A function f is called *regularly varying at* 0 *with real index* ρ, or $f \in RV_\rho(0+)$ in short, if $f(t)/t^\rho$ is slowly varying at 0.

Example 2

Non-trivial examples of slowly varying functions L for small $t > 0$:

(a) the logarithm $\log(1/t)$, its m iterates $\log_m(1/t)$ defined as $\log \log_{m-1}(1/t)$ and powers of $\log_m(1/t)$ for any integer $m \geq 1$;

(b) $\exp((\log(1/t))^\alpha)$ with $\alpha \in (0, 1)$.

(c) $\exp\left(-(\log(1/t))^{1/3} \cos((\log(1/t))^{1/3})\right)$.

Definition 3

A function $u \in C^1(B^*)$ is said to be a *solution* (*sub-solution*) of (1) if for all functions (non-negative functions) $\psi \in C_c^1(B^*)$, we have

$$\int_{B_1} \mathcal{A}(|x|)|\nabla u|^{p-2}\nabla u \cdot \nabla\psi \,\mathrm{d}x + \int_{B_1} b(x)h(u)\,\psi\,\mathrm{d}x = 0 \quad (\leq 0). \tag{3}$$

Let $\omega_N = \mathrm{vol}(B_1)$ and Φ be given by

$$\Phi(x) := \frac{1}{(N\omega_N)^{1/(p-1)}} \int_{|x|}^{1} \left(\frac{t^{1-N}}{\mathcal{A}(t)}\right)^{\frac{1}{p-1}} dt \quad \text{for every } x \in B^*. \tag{4}$$

Assumption B. *Let* (2) *and Assumption A hold. Let* $\lim_{r\to 0}\Phi(r) = \infty$, $b_0 \in RV_\sigma(0+)$ *and* $h \in RV_q(\infty)$ *with* $q+1 > p > \vartheta - \sigma$.

We can see Φ as the fundamental solution of

$$-\Delta_{\mathcal{A},p}\Phi := -\mathrm{div}\left(\mathcal{A}(|x|)\,|\nabla\Phi|^{p-2}\nabla\Phi\right) = \delta_0 \quad \text{in } \mathcal{D}'(B_1) \tag{5}$$

with homogeneous Dirichlet boundary condition.

A positive solution of (1) is said to have a *weak singularity* at 0 if $u(x)/\Phi(|x|)$ converges to a positive number as $|x| \to 0$.

Theorem 4 (Existence of weak singularities, C.–Cîrstea)

Let Assumption B hold. Eq. (1) admits a positive solution with a weak singularity at 0 if and only if $b(x)h(\Phi) \in L^1(B_{1/2})$, or in other words,

$$\int_{0^+} r^{N-1} b_0(r) h(\Phi(r))\, \mathrm{d}r < \infty. \tag{6}$$

From Assumption B, we have $p \le N + \vartheta$. We set

$$q_* := \frac{(N+\sigma)(p-1)}{N+\vartheta-p} \text{ if } p < N+\vartheta \text{ and } q_* := \infty \text{ if } p = N+\vartheta. \tag{7}$$

1. If $p = N + \vartheta$, then (6) holds automatically for any $q < \infty$.
2. If $p < N + \vartheta$ and $q \neq q_*$, then (6) holds iff $q < q_*$. If $L_{\mathcal{A}} = L_b = 1$ and $h(t) = t^{q_*}(\ln t)^{\alpha}$ for $t > 0$ large, then (6) holds iff $\alpha < -1$.

Theorem 5 (Removability, C.–Cîrstea)

Let Assumption B hold. If $b(x)h(\Phi) \notin L^1(B_{1/2})$*, then* $p < N + \vartheta$*,* $q \geq q_*$ *and every positive solution of* (1) *can be extended as a positive continuous solution of* (1) *in* B_1*.*

Remark 1

1. By applying Theorem 5 with $\mathcal{A} = b = 1$ and $h(t) = t^q$, then we recover the removability result of Brezis–Véron (1980) (for $p = 2$) and Vázquez–Véron (1980/1981) (for $1 < p < N$).
2. Theorem 5 in the case $\mathcal{A} = 1$ gives a sharp version of Theorem 1.3 in Cîrstea–Du (2010).
3. The proof of Theorem 5 is crucially based on understanding the solutions with strong singularities and it uses techniques in Cîrstea (Memoirs AMS, 2014).

If (6) and Assumption B hold, we prove that $\exists$ positive solutions of (1) satisfying $\lim_{|x|\to 0} u(x)/\Phi(x) = \infty$.

Case 1: $q < q_*$. We define $\tilde{u}(r)$ for $r > 0$ small by

$$\int_{\tilde{u}(r)}^{\infty} \frac{\mathrm{d}t}{[th(t)]^{\frac{1}{p}}} = \int_0^r \left[M_1 \frac{b_0(\tau)}{\mathcal{A}(\tau)} \right]^{\frac{1}{p}} \mathrm{d}\tau, \tag{8}$$

where M_1 is given by

$$M_1 := \frac{p + \sigma - \vartheta}{(N + \sigma)(p - 1) - (N + \vartheta - p)q}.$$

Case 2: $q = q_* < \infty$ (for $p < N + \vartheta$). We need extra information:

$$\begin{cases} \text{either (a) } t \longmapsto L_h(e^t) \text{ is regularly varying at } \infty, \\ \text{or (b) } t \longmapsto \left[L_{\mathcal{A}}(e^{-t}) \right]^{-\frac{q_*}{p-1}} L_b(e^{-t}) \text{ is regularly varying at } \infty. \end{cases} \tag{9}$$

We introduce $F_1 : (0,\infty) \to (0,\infty)$ and $M_2 > 0$ as follows

$$\begin{cases} F_1(s) := \displaystyle\int_0^{\Phi^{-1}(s)} \xi^{N-1} b_0(\xi)\, h(\Phi(\xi))\, \mathrm{d}\xi \quad \text{for } s > 0, \\ M_2 := \dfrac{N\omega_N(\sigma - \vartheta + p)}{N + \vartheta - p} > 0. \end{cases} \tag{10}$$

For any $r > 0$ small, we define $\tilde{u}(r)$ of the following form

$$\begin{cases} \tilde{u}(r) := \Phi(r)\,[M_2 F_1(\Phi(r))]^{-\frac{1}{q_* - p + 1}} \quad \text{if (9)}(a) \text{ holds,} \\ \displaystyle\int_c^{\tilde{u}(r)} [M_2 F_1(t)]^{\frac{1}{q_* - p + 1}}\, \mathrm{d}t := \Phi(r) \quad \text{if (9)}(b) \text{ holds.} \end{cases} \tag{11}$$

Theorem 6 (Classification, C.–Cîrstea)

Let Assumption B and (6) *hold. Then for every positive solution* u *of* (1), *exactly one of the following cases occurs:*

(i) u *can be extended as a positive continuous solution of* (1) *in* B_1*;*

(ii) $\lim_{|x|\to 0} u(x)/\Phi(x) = \lambda \in (0,\infty)$ *and, moreover,* u *verifies*

$$-\Delta_{\mathcal{A},p} u + b(x)h(u) = \lambda^{p-1}\delta_0 \quad \text{in } \mathcal{D}'(B_1). \tag{12}$$

(iii) $u(x) \sim \tilde{u}(|x|)$ *as* $|x| \to 0$, *where* $\tilde{u}$ *is given by* (8) *if* $q < q_*$ *and by* (11) *when* $q = q_* < \infty$ *and* (9) *holds.*

Remark 2

1. Theorem 6 gives a sharp version of Theorem 1.1 in Cîrstea–Du (2010) (where $\mathcal{A} = 1$).
2. Theorems 4, 5 and 6 extend the optimal results in Brandolini–Chiacchio–Cîrstea–Trombetti (2013) ($p = 2$, $b = 1$, $h(t) = t^q$).

Crucial ingredients

Lemma 7 (A priori estimates)

Let $H(t) = \int_0^t h(s)\, ds$. *For any* $r_0 \in (0, 1/2)$, *there exists a constant* $c = c(r_0) > 0$ *s.t. for every positive (sub-)solution of* (1), *we have*

$$\int_{u(x)}^{\infty} \frac{\mathrm{d}t}{\sqrt[p]{H(t)}} \geq c|x| \left(\frac{b(x)}{\mathcal{A}(|x|)} \right)^{\frac{1}{p}} \quad \text{for all } 0 < |x| \leq r_0. \qquad (13)$$

Lemma 8 (A spherical Harnack-type inequality)

Fix $r_0 \in (0, 1/2)$. *There exists a positive constant* K *(depending on* p, N *and* r_0*) such that for every positive solution* u *of* (1), *we have*

$$\max_{|x|=r} u(x) \leq K \min_{|x|=r} u(x) \quad \text{for all } 0 < r \leq r_0/2. \qquad (14)$$

Lemma 9 (A regularity result)

Fix $r_0 \in (0, 1/4)$ *and* $\delta \geq 0$. *Let* g *be a positive continuous function on* $(0, 1)$ *such that* $g \in RV_{-\delta}(0+)$. *Suppose that* u *is a positive solution of* (1) *and* C_0 *is a positive constant such that*

$$0 < u(x) \leq C_0\, g(|x|) \quad \text{for } 0 < |x| < 2r_0. \tag{15}$$

Then there exist positive constants $C > 0$ *and* $\alpha \in (0, 1)$ *such that*

$$|\nabla u(x)| \leq C\,\frac{g(|x|)}{|x|} \quad \text{and} \quad |\nabla u(x) - \nabla u(x')| \leq C\,\frac{g(|x|)}{|x|^{1+\alpha}}|x - x'|^{\alpha} \tag{16}$$

for any x, x' *in* $\mathbb{R}^N$ *satisfying* $0 < |x| \leq |x'| < r_0$.

Corollary 10

Assume that u *is a positive solution of* (1) *such that* $\lim_{|x|\to 0} u(x) = \infty$. *Then, for every* $\epsilon \in (0,1)$, *there exists* $r_\epsilon \in (0,1)$ *such that the equation*

$$-\Delta_{\mathcal{A},p} v + b_0(|x|)L_h(v)v^q = 0 \quad \text{in } B^*_{r_\epsilon} := B_{r_\epsilon} \setminus \{0\} \tag{17}$$

has a positive solution v_ϵ *satisfying*

$$(1-\epsilon)u \le v_\epsilon \le (1+\epsilon)u \quad \text{in } B^*_{r_\epsilon}.$$

Corollary 11

Let $r_\epsilon \in (0,1)$ *be arbitrary and* v *be a positive solution of* (17). *Then there exist two positive radial solutions of* (17) *in* $B^*_{r_\epsilon/2}$, *say* v_* *and* v^*, *such that*

$$K^{-1}v \le v_* \le v \le v^* \le Kv \quad \text{in } B^*_{r_\epsilon/2}, \tag{18}$$

where $K > 1$ *is a sufficiently large constant.*

Theorem 12 (Strong singularities)

Let Assumption B and (6) hold. If u is any positive solution of (1) with a strong singularity at 0, then $u(x) \sim \tilde{u}(|x|)$ *as* $|x| \to 0$*, where* $\tilde{u}$ *is given by (8) if* $q < q_*$ *and by (11) when* $q = q_* < \infty$ *and and (9) holds.*

Proposition 1 (Case $q < q_*$)

For any positive radial solution v of (17) with a strong singularity at 0, we have $v(r) \sim \tilde{u}(r)$ *as* $r \to 0^+$*, where* $\tilde{u}$ *is defined by (8).*

We adapt ideas from Cîrstea–Du (2010, JFA). We first show the following.

Lemma 13 (Case $q < q_*$)

Let f be a regularly varying function at 0 with index μ*.*

(a) *If* $\mu < -(p+\sigma-\vartheta)/(q-p+1)$*, then we have* $\lim_{r\to 0^+} v(r)/f(r) = 0$*.*

(b) *If* $\mu > -(p+\sigma-\vartheta)/(q-p+1)$*, then* $\lim_{r\to 0^+} v(r)/f(r) = \infty$*.*

We next construct a local family of sub-super-solutions of (17). Let $\theta = M_1(p-1)$. Fix $\eta_0 \in (0,1)$ small. For each $\eta \in [0, \eta_0]$, we define

$$v_{\pm\eta}(r) = C_{\pm\eta}[\tilde{u}(r)]^{1\pm\eta} \text{ for } r \in (0,1),$$

where $C_{\pm\eta}$ is a positive constant given by

$$C_{\pm\eta} := \left[(1 \pm \eta)^{p-1}(1 \pm \eta\theta)\right]^{\frac{1}{q-p+1}}. \tag{19}$$

Lemma 14 (Case $q < q_*$)

For every $\epsilon \in (0,1)$ small, there exists $r_\epsilon \in (0,1)$ such that $(1-\epsilon)v_{-\eta}$ and $(1+\epsilon)v_\eta$ is a sub-solution and super-solution of (17) in $B^*_{r_\epsilon}$, respectively, for every $\eta \in [0, \eta_0]$.

By Lemma 13, we find that

$$\lim_{r\to 0^+} \frac{v(r)}{v_\eta(r)} = 0 \quad \text{and} \quad \lim_{r\to 0^+} \frac{v(r)}{v_{-\eta}(r)} = \infty. \tag{20}$$

Notice that $(1+\epsilon)v_\eta(r) + v(r_\epsilon)$ and $v(r) + \tilde{u}(r_\epsilon)$ are super-solutions of (17) in $B^*_{r_\epsilon}(0)$. Then by the comparison principle,

$$v(r) \leq (1+\epsilon)v_\eta(r) + v(r_\epsilon) \quad \text{and} \quad v(r) + \tilde{u}(r_\epsilon) \geq (1-\epsilon)v_{-\eta}(r) \tag{21}$$

for all $0 < r \leq r_\epsilon$. By letting $\eta \to 0^+$ in (21), we have

$$v(r) \leq (1+\epsilon)\tilde{u}(r) + v(r_\epsilon) \quad \text{and} \quad v(r) + \tilde{u}(r_\epsilon) \geq (1-\epsilon)\tilde{u}(r) \tag{22}$$

for all $0 < r \leq r_\epsilon$. By letting $r \to 0^+$ in (22), we conclude that

$$1-\epsilon \leq \liminf_{r\to 0^+} \frac{v(r)}{\tilde{u}(r)} \leq \limsup_{r\to 0^+} \frac{v(r)}{\tilde{u}(r)} \leq 1+\epsilon. \tag{23}$$

Finally, we pass to the limit with $\epsilon \to 0$ in (23).

Proposition 2 (Critical case $q = q_*$ for $p < N + \vartheta$)

If v is a positive radial solution of (17) *with a strong singularity at* 0 *and* (9) *holds, then $v(r) \sim \tilde{u}(r)$ as $r \to 0^+$, where $\tilde{u}$ is defined by* (11).

Main ideas in the proof:

We apply the change of variable $y(s) = v(r)$ with $s = \Phi(r)$ and arrive at

$$\left|\frac{dy}{ds}\right|^{p-2} \frac{d^2y}{ds^2} = \frac{(N\omega_N)^{\frac{p}{p-1}}}{p-1} r^{\frac{p(N-1)}{p-1}} [\mathcal{A}(r)]^{\frac{1}{p-1}} b_0(r)\, L_h(y(s))\, [y(s)]^q \tag{24}$$

for $s > 0$. After many hidden analyses, we have that

$$\frac{1}{2} \leq \frac{s(dy/ds)}{y(s)} \leq C'' + 2 \quad \forall s \geq s_0 \text{ large.} \tag{25}$$

Step 1: *Show that* $0 < \liminf_{r\to 0^+} \frac{v(r)}{\tilde{u}(r)} \le \limsup_{r\to 0^+} \frac{v(r)}{\tilde{u}(r)} < \infty.$

Define $E_1(r)$ and $E_2(r)$ for $r \in (0,1)$ as follows

$$E_1(r) := r^{\frac{p(N-1)}{p-1}} [\mathcal{A}(r)]^{\frac{1}{p-1}} b_0(r) \quad \text{and} \quad E_2(r) := [L_{\mathcal{A}}(r)]^{-\frac{q_*}{p-1}} L_b(r). \quad (26)$$

Using (25) into (24), we find positive constants c_1 and c_2 so that

$$\boxed{c_1 E_1(\Phi^{-1}(s))\, L_h(y)\, s^{q_*} \le \left[\frac{dy}{ds}\right]^{-q_*+p-2} \frac{d^2y}{ds^2} \le c_2\, E_1(\Phi^{-1}(s))\, L_h(y)\, s^{q_*}} \quad (27)$$

for all $s \ge s_0$. For some $\ell > 0$, we obtain that

$$E_1(r) \sim \ell\, [\Phi(r)]^{-q_*-1}\, E_2(r) \quad \text{as } r \to 0^+. \quad (28)$$

Hence, using (28), $\exists$ positive constants c_3 and c_4 s.t. $\forall s \ge s_0$

$$\boxed{\frac{c_3}{s} E_2(\Phi^{-1}(s))\, L_h(y) \le \left[\frac{dy}{ds}\right]^{-q_*+p-2} \frac{d^2y}{ds^2} \le \frac{c_4}{s} E_2(\Phi^{-1}(s))\, L_h(y).} \quad (29)$$

Case 1: *Assume that* (9)(*a*) *holds.*
Then, using $\ln y(s) \sim \ln s$, we get that

$$L_h(y(s)) \sim L_h(s) \sim h(s)/s^{q_*} \quad \text{as } s \to \infty. \tag{30}$$

So, from (27) and (30), there exist positive constants $\tilde{c}_1$ and $\tilde{c}_2$ such that

$$\tilde{c}_1 E_1(\Phi^{-1}(s))\, h(s) \le \left[\frac{dy}{ds}\right]^{-q_*+p-2} \frac{d^2y}{ds^2} \le \tilde{c}_2 E_1(\Phi^{-1}(s))\, h(s) \quad \text{for } s \ge s_0. \tag{31}$$

Using that $y'(s) \to \infty$ as $s \to \infty$ and integrating (31), we obtain that

$$c_5\, F_1(s) \le \left[\frac{dy}{ds}\right]^{-q_*+p-1} \le c_6\, F_1(s) \quad \text{for all } s \ge s_0, \tag{32}$$

where c_5 and c_6 are positive constants, whilst $F_1(s)$ is defined by

$$F_1(s) := \int_s^\infty E_1(\Phi^{-1}(t))\, h(t)\, \mathrm{d}t = \int_0^{\Phi^{-1}(s)} \xi^{N-1} b_0(\xi)\, h(\Phi(\xi))\, \mathrm{d}\xi. \tag{33}$$

From (25) and (32), $\exists$ positive constants d_1 and d_2 such that

$$d_1\left[F_1(s)\right]^{-\frac{1}{q_*-p+1}} \leq \frac{y(s)}{s} \leq d_2\left[F_1(s)\right]^{-\frac{1}{q_*-p+1}} \quad \text{for all } s \geq s_0,$$

or, equivalently, for every $r \in (0, \Phi^{-1}(s_0))$, it holds

$$d_1\left[F_1(\Phi(r))\right]^{-\frac{1}{q_*-p+1}} \leq \frac{v(r)}{\Phi(r)} \leq d_2\left[F_1(\Phi(r))\right]^{-\frac{1}{q_*-p+1}}.$$

Hence, using the definition of $\tilde{u}$ in (11), we conclude Step 1.

Case 2: *Assume that (9)(b) holds.*

Then, using that $\ln \Phi^{-1}(s) \sim \ln \Phi^{-1}(y(s))$ as $s \to \infty$, we obtain that

$$[L_{\mathcal{A}}(\Phi^{-1}(s))]^{-\frac{q_*}{p-1}} L_b(\Phi^{-1}(s)) \sim [L_{\mathcal{A}}(\Phi^{-1}(y(s)))]^{-\frac{q_*}{p-1}} L_b(\Phi^{-1}(y(s)))$$

as $s \to \infty$. This, jointly with (28), gives that

$$E_2(\Phi^{-1}(s)) \sim E_2(\Phi^{-1}(y(s))) \sim \frac{E_1(\Phi^{-1}(y(s)))}{\ell\,[y(s)]^{-q_*-1}} \quad \text{as } s \to \infty, \tag{34}$$

where E_1 and E_2 are defined by (26). From (25), (29) and (34), $\exists$ positive constants d_3 and d_4 such that

$$d_3 E_1(\Phi^{-1}(y))\, h(y)\, \frac{dy}{ds} \leq \left[\frac{dy}{ds}\right]^{-q_*+p-2} \frac{d^2y}{ds^2} \leq d_4 E_1(\Phi^{-1}(y))\, h(y)\, \frac{dy}{ds}$$

for all $s \geq s_0$. With F_1 as defined in (33), this gives that

$$[d_4(q_* - p + 1)]^{-\frac{1}{q_*-p+1}} \leq \frac{d}{ds}\left(\int_{y(s_0)}^{y(s)} [F_1(t)]^{\frac{1}{q_*-p+1}}\,\mathrm{d}t\right) \leq [d_3(q_* - p + 1)]^{-\frac{1}{q_*-p+1}}$$

for every $s > s_0$. Jointly with the definition of $\tilde{u}$ in (11), we thus conclude Step 1.

Step 2: *Construction of sub-super-solutions for* (17).

Fix $\eta_0 \in (0,1)$ small. Using M_2 in (10), we define $C_{\pm\eta}$ by

$$C_{\pm\eta} := \left(\frac{M_2}{1\pm\eta}\right)^{\frac{1}{1\pm\eta}} = \left[\frac{(q_* - p + 1)N\omega_N}{q-1}\right]^{\frac{1}{1\pm\eta}} \quad \text{for all } \eta \in [0,\eta_0]. \quad (35)$$

If (9)(a) holds, then for any $\eta \in [0,\eta_0]$, we define $v_{\pm\eta}$ as follows

$$v_{\pm\eta}(r) := \int_{s_0}^{\Phi(r)} [C_{\pm\eta}F_1(t)]^{-\frac{1\pm\eta}{q_*-p+1}}\,\mathrm{d}t \quad \text{for any } r \in (0,\Phi^{-1}(s_0)), \quad (36)$$

where $s_0 > 0$ is fixed large enough and F_1 is given by (33).
If, in turn, (9)(b) is satisfied, we introduce $v_{\pm\eta}$ in the next identity

$$\int_{c}^{v_{\pm\eta}(r)} [C_{\pm\eta}F_1(t)]^{\frac{1\pm\eta}{q_*-p+1}}\,\mathrm{d}t = \Phi(r) \quad \text{for any } r > 0 \text{ small}, \quad (37)$$

where $c > 0$ is a large constant such that $\Phi^{-1}(c) < 1$.

Lemma 15

For every $\epsilon \in (0,1)$ *small, there exists* $r_\epsilon \in (0,1)$ *such that* $(1-\epsilon)v_{-\eta}$ *and* $(1+\epsilon)v_\eta$ *is a sub-solution and super-solution of* (17) *in* $B^*_{r_\epsilon}$*, respectively, for every* $\eta \in [0, \eta_0]$*.*

Step 3: *Proof of Proposition 2 concluded.*

In either Case 1 (that is, (9)(a) holds) or Case 2 (when (9)(b) holds), by using the definitions of $\tilde{u}$ and $v_{\pm\eta}$, we infer that

$$\lim_{r\to 0^+} \frac{\tilde{u}(r)}{v_\eta(r)} = 0 \quad \text{and} \quad \lim_{r\to 0^+} \frac{\tilde{u}(r)}{v_{-\eta}(r)} = \infty \quad \text{for every } \eta \in (0, \eta_0]. \tag{38}$$

From Step 1 and (38), we regain (20). Following the proof of Proposition 1, we obtain (21)–(23), proving that $v(r) \sim \tilde{u}(r)$ as $r \to 0^+$.

Assume that

$$\begin{cases} \mathcal{A}(t) \sim t^{\vartheta}(\ln(1/t))^{\alpha} \quad \text{as } t \to 0 \quad \text{for some } \alpha \in \mathbb{R} \\ b(x) \sim |x|^{\sigma}(\ln(1/|x|))^{\beta} \quad \text{as } |x| \to 0 \quad \text{for some } \beta \in \mathbb{R} \\ h(t) \sim t^{q} \exp(-(\log t)^{\nu}) \quad \text{as } t \to \infty \quad \text{for some } q > p-1, \nu \in (0,1). \end{cases} \tag{39}$$

Let u be any positive solution of (1).

(A) If $p-1 < q < q^*$, then exactly one of the following occurs as $|x| \to 0$:

(i) u can be extended as a positive continuous solution of (1) in the whole ball B_1, that is $\lim_{|x|\to 0} u(x) \in (0,\infty)$ and (3) holds for every $\phi \in C_c^1(B_1)$.

(ii) u has a weak singularity at 0, that is $\lim_{|x|\to 0} u(x)/\Phi(x) = \lambda \in (0,\infty)$ and, moreover, u verifies

$$-\Delta_{\mathcal{A},p} u + b(x)h(u) = \lambda^{p-1}\delta_0 \quad \text{in } \mathcal{D}'(B_1). \tag{40}$$

(iii) u has a strong singularity at 0 and moreover, we have

$$u(x) \sim \left[M_1 M_3^p \left(\log \frac{1}{|x|} \right)^{-\alpha+\beta} \exp\left(- \left(M_3^{-1} \log \frac{1}{|x|} \right)^{\nu} \right) |x|^{p+\sigma-\vartheta} \right]^{-\frac{1}{q-p+1}} \quad \text{as } |x| \to 0. \quad (41)$$

where $M_3 = \left(\frac{q-p+1}{p+\sigma-\vartheta} \right)$.

(B) If $q = q_*$, then the conclusions above hold except for (41) which is replaced by

$$u(x) \sim \left[\frac{M_3^{p-1+\nu}}{\nu} \frac{p+\sigma-\vartheta}{N+\vartheta-p} \left(\log \frac{1}{|x|} \right)^{-\alpha+\beta-\nu+1} \exp\left(- \left(M_3^{-1} \log \frac{1}{|x|} \right)^{\nu} \right) |x|^{p+\sigma-\vartheta} \right]^{-\frac{1}{q_*-p+1}} \quad \text{as } |x| \to \quad (42)$$

(C) If $q > q_*$, then only case (A)(i) occurs.

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