Curvature contraction of convex hypersurfaces

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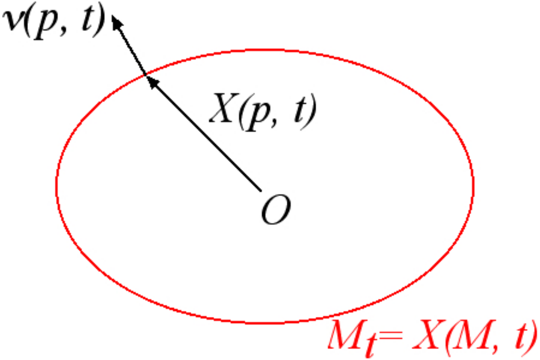
Talk outline

We begin with curvature contraction flows of the form

$$\frac{\partial X}{\partial t}(p,t) = -F\left(\mathcal{W}(p,t)\right)\nu(p,t) \tag{1}$$

with a smooth, compact, strictly convex initial hypersurface $X(\mathbb{S}^n, 0) = X_0(\mathbb{S}^n) = M_0$ without boundary.



Definition

The ***principal curvatures*** κ_i, $i = 1, \ldots, n$, are the eigenvalues of the Weingarten map $\mathcal{W}$ of M_t.

Write $\kappa = \{\kappa_1, \ldots, \kappa_n\}$.

Basic properties of the speed:

- $F(\mathcal{W}) = f(\kappa)$, f symmetric on the positive cone

$$\Gamma_+ = \{\kappa : \kappa_i > 0 \text{ for all } i = 1, \ldots, n\}.$$

- $f > 0$ (contraction flow), $f(1, \ldots, 1) = 1$ (normalised)
- f strictly increasing in each argument, everywhere on Γ_+
- f is homogeneous of degree $\alpha > 0$

Definition (Homogeneous functions)

A function $f(\kappa) = f(\kappa_1, \ldots, \kappa_n)$ is ***homogeneous of degree*** α if for every κ and for all $k > 0$,

$$f(k\kappa) = k^{\alpha} f(\kappa).$$

Theorem (Euler's homogeneous function theorem)

If f is differentiable and homogeneous of degree α then

$$\sum_{i=1}^{n} \dot{f}^{i} \kappa_i := \sum_{i=1}^{n} \frac{\partial f}{\partial \kappa_i} \kappa_i = \alpha f.$$

Idea of proof: Take $\frac{\partial}{\partial k}$ of definition then set $k = 1$. □

Note: $f(k\kappa) = k^{\alpha} f(\kappa) \Leftrightarrow F(k\mathcal{W}) = k^{\alpha} F(\mathcal{W})$.

For any smooth strictly convex initial hypersurface M_0 and speed f smooth and homogeneous of degree 1:

- (Huisken, ’84) Under **mean curvature flow**, $F = H$
- (Chow, ’85) Under flow by $F = K^{1/n}$
- (Andrews, ’94) Under flows by f either
  - convex, or
  - concave and
    - $n = 2$
    - $f \to 0$ as $\kappa \to \partial\Gamma_+$, or
    - $\sup_{M_0} \frac{H}{F} < \liminf_{\kappa \in \partial\Gamma_+} \frac{\sum_i \kappa_i}{f(\kappa)}$.
- (Andrews, ’07) Under flows by f concave and *inverse concave*, that is, the function f_* is concave, where

$$f_* (x_1, \ldots, x_n) = \frac{1}{f\left(\frac{1}{x_1}, \ldots \frac{1}{x_n}\right)}.$$

- (Andrews, ’10) Under flows by f when $n = 2$ (surfaces).

And more recently

- (Andrews, M., Zheng, '13) Under flows with f inverse concave and either
  - $f_* \to 0$ on $\partial\Gamma_+$, or
  - matrix of inverse Weingarten map (r_{ij}) of M_0 satisfies

$$\sup_{\nu \in T_z\mathbb{S}^n, \|\nu\|=1} \left(\frac{r(\nu,\nu)|_z}{F_*(r(z))} \right) < \liminf_{\partial\Gamma_+} \inf_{\nu \in T_z\mathbb{S}^n, \|\nu\|=1} \frac{r(\nu,\nu)}{f_*(r)}.$$

- (M., Mofarreh, V-M Wheeler, '14) M_0 axially symmetric

More results for smooth speeds f homogeneous of positive degree with M_0 suitably pointwise 'curvature pinched':

- (Chow, '85) Flows by K^β, $\beta \geq \frac{1}{n}$
- (Schulze, '06) $F = H^k$, $k \geq 1$
- (Alessandroni-Sinestrari, '10), $F = R^k$, $k \geq \frac{1}{2}$
- (Andrews, M., '12) f homogeneous of degree $\alpha \geq 1$ (rather tight pinching).

If $n = 2$, more results are possible, without curvature pinching, or convexity of f, by exploiting symmetries of Codazzi equations:

- (Andrews, '99) $F = K$ (Firey's conjecture)
- (Schnürer, '05) various speeds of integer homogeneity between 1 and 6
- (Schnürer-Schulze, '06) $F = H^k$, $k = 1, 2, 3, 4, 5$
- (Andrews, '10) f homogeneous of degree $\alpha \geq 1$ (α-dependent pinching if $\alpha > 1$)
- (Andrews-Chen, '12) $F = K^{\frac{\alpha}{2}}$, $\alpha \in [1, 2]$

Even more results on convergence of smooth convex hypersurfaces to a point (or not) without roundness.
See, eg (Andrews, M., Zheng '13).

Definition

Let $X : \mathbb{S}^n \to \mathbb{R}^{n+1}$ represent a suitably smooth convex hypersurface M in $\mathbb{R}^{n+1}$. The ***support function*** of M is given by

$$u(x) = \langle X(x), \nu(x) \rangle .$$

Note: The image of M in $\mathbb{R}^{n+1}$ can be reconstructed via

$$\hat{X}(x) = u(x)\, x + \overline{\nabla} u(x)\,, \tag{2}$$

where $\overline{\nabla}$ and $\overline{\sigma}$ are the standard gradient and metric on $\mathbb{S}^n$.

Solutions of (1) remain convex and correspond to solutions of the scalar parabolic equation

$$\frac{\partial u}{\partial t} = -F\left(\left(\overline{\nabla}_i \overline{\nabla}_j u + \overline{\sigma}_{ij} u\right)^{-1}\right) = -F_*\left(r_{ij}\right)^{-1} =: \Psi\left(r_{ij}\right). \tag{3}$$

Suppose now $F(\mathcal{W})$ satisfies the following

- $F(\mathcal{W}) = f(\kappa)$, f symmetric on Γ_+
- $f > 0$, $f(1, \ldots, 1) = 1$
- for all $\kappa \in \Gamma_+$, $f(\kappa + \delta e_i) > f(\kappa)$ for all $\delta > 0$ and each i
- f is homogeneous of degree 1
- f is convex, ie. for all $x, y \in \Gamma_+$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda y)) \leq \lambda f(x) + (1 - \lambda) f(y) .$$

Properties:

- f is almost everywhere in Γ_+ twice differentiable (Aleksandrov).
- f is Lipschitz on compact subsets of Γ_+.

Example ($n = 2$, $f(\kappa) = \alpha\,\kappa_{\min} + \beta\,\kappa_{\max}$, $\beta \geq \alpha > 0$, $\alpha + \beta = 1$)

- f defined on $\mathbb{R}^2$ and symmetric, since

$$\kappa_{\min} = \left(\kappa_1 + \kappa_2 - |\kappa_1 - \kappa_2|\right)/2$$

$$\kappa_{\max} = \left(\kappa_1 + \kappa_2 + |\kappa_1 - \kappa_2|\right)/2$$

  We can rewrite $f(\kappa) = \frac{1}{2}H + \left(\frac{\beta-\alpha}{2}\right)|\kappa_1 - \kappa_2|$.
- $f > 0$ on Γ_+, not differentiable if $\beta > \alpha$ wherever $\kappa_1 = \kappa_2$
- f is everywhere increasing (by triangle inequality)
- f degree 1 homogeneous, convex (triangle inequal, $\beta \geq \alpha$)

Example (Maxima of convex functions)

$$F = \max\left(\frac{H}{n}, \eta\,|A|\right), \; \frac{1}{n} < \eta < \frac{1}{\sqrt{n}}.$$

For a mollifier $j_\varepsilon(x) = \varepsilon^{-n} j\left(\frac{x}{\varepsilon}\right)$, such as

$$j(x) = \begin{cases} c_n\, e^{\frac{1}{|x|^2-1}} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

where c_n is chosen such that $\int_{\mathbb{R}^n} j(x)\, dx = 1$, set

$$f_\varepsilon(\kappa) = \frac{H}{\hat{f}^1_\varepsilon} \int_{\mathbb{R}^n} j_\varepsilon(y)\, f\left(\frac{\kappa}{H} - y\right) dy = \frac{1}{\hat{f}^1_\varepsilon} \int_{\mathbb{R}^n} j_\varepsilon(y)\, f(\kappa - Hy)\, dy, \tag{4}$$

where $\hat{f}^1_\varepsilon$ is a normalisation factor.

Lemma

1. *For each $\varepsilon > 0$, f_ε is smooth; $f_\varepsilon \to f$ uniformly on $\tilde{\Gamma} \subset\subset \Gamma_+$.*
2. *For each $\varepsilon \in \left(0, \min\left(\frac{1}{n}, \varepsilon_0\right)\right)$, f_ε satisfies the same properties as f.*
3. $f(\kappa) - \varepsilon H \leq f_\varepsilon(\kappa) \leq f(\kappa) + \varepsilon H$.

Theorem (Andrews, Holder, M., G. Wheeler, V.-M. Wheeler, Williams, '14)

Given M_0 compact, strictly convex $C^{1,\beta}$ hypersurface and a convex function f on Γ_+ satisfying the above properties, a solution $u \in C^{2,\alpha}\left(\mathbb{S}^n \times (0,T)\right)$ to (3) exists for $T < \infty$. The M_t contract to a point as $t \to T$. Under rescaling, $\tilde{M}_t$ approaches $\mathbb{S}^n$ exponentially in $C^{2,\alpha'}$ for $0 < \alpha' < \alpha$.

Remarks:

1. Cannot estimate curvature derivatives via Schauder estimates. So to obtain contraction to a point by contradiction, we need short time existence for $C^{1,\beta}$ initial hypersurfaces. Modification of Lieberman, Chapter 14.
2. If the speed is more regular, then the solution is correspondingly more regular, by boot-strapping.

We obtain estimates independent of ε for the flows

$$\frac{\partial u^{\varepsilon}}{\partial t} = -F^{\varepsilon}\left(\left(\overline{\nabla}_i\overline{\nabla}_j u^{\varepsilon} + \overline{\sigma}_{ij}u^{\varepsilon}\right)^{-1}\right) =: F_*^{\varepsilon}\left(r_{ij}\right), \qquad (5)$$

all with initial hypersurface M_0. The speeds are given by

$$F^{\varepsilon}\left(\mathcal{W}\right) := f^{\varepsilon}\left(\kappa\left(\mathcal{W}\right)\right);$$

we will denote by κ_i^{ε} the curvatures of the M_t^{ε}, etc.

Let ρ_-, ρ_+ denote the inner and outer radius of M_0.

Lemma (maximal time estimate)

$$\frac{\rho_-^2}{2} \leq T \leq \frac{\rho_+^2}{2}.$$

Proof: The radius $r^{\varepsilon}(t)$ of a sphere evolving under (5) satisfies

$$\frac{d}{dt} r^{\varepsilon}(t) = -f^{\varepsilon}\left(\frac{1}{r^{\varepsilon}}, \ldots, \frac{1}{r^{\varepsilon}}\right) = -f^{\varepsilon}(1, \ldots, 1)\frac{1}{r^{\varepsilon}} = -\frac{1}{r^{\varepsilon}}.$$

With a condition $r^{\varepsilon}(t_0) = r_0$, independent of ε, the ODE has solution

$$r^{\varepsilon}(t) = \sqrt{r_0^2 - 2(t - t_0)}.$$

The sphere shrinks to a point at time $t = t_0 + \frac{r^2(t_0)}{2}$.

Since M_0 encloses B_{ρ_-} and is enclosed by B_{ρ_+},

$$\frac{\rho_-^2}{2} \leq T_{\varepsilon} \leq \frac{\rho_+^2}{2}$$

for all ε. $\square$

Lemma (lower bound on speed and mean curvature)

Under (3), $H^\varepsilon \geq H_0 > 0$ and $F^\varepsilon \geq \frac{1}{n} H_0$ remain true, independent of ε.

Proof: Solutions of (5) satisfy

$$\frac{\partial}{\partial t} H^\varepsilon = \mathcal{L}^\varepsilon H^\varepsilon + \ddot{F}_\varepsilon^{kl,rs} \nabla_\varepsilon^i h_{kl}^\varepsilon \nabla_i^\varepsilon h_{rs}^\varepsilon + \dot{F}_\varepsilon^{kl} h_{km}^\varepsilon h_\varepsilon^m{}_l H^\varepsilon ,$$

where $\mathcal{L}^\varepsilon = \dot{F}_\varepsilon^{ij} \nabla_i^\varepsilon \nabla_j^\varepsilon$. Since the F_ε are convex,

$$\min_{M_t} H^\varepsilon \geq \min_{M_0} H =: H_0 > 0$$

independent of ε.

Any convex F satisfies $F \geq \frac{1}{n} H$, the result follows. □

Using Hamilton's maximum principle for tensors,

Lemma (preservation of convexity, curvature pinching)

Under (3)*,*

1. $h_{\varepsilon\ j}^{i} > 0$,
2. $h_{\varepsilon\ j}^{i} - \eta F^{\varepsilon}\delta_j^i \geq 0$, *for any constant* $\eta \leq \min_{M_0} \frac{n\,\kappa_{\min}}{n\,f(\kappa) + H}$.

Proof: For any constant η,

$$\frac{\partial}{\partial t}\left(h_{\varepsilon\ j}^{i} - \eta F^{\varepsilon}\delta_j^i\right) = \mathcal{L}^{\varepsilon}\left(h_{\varepsilon\ j}^{i} - \eta F^{\varepsilon}\delta_j^i\right) + \ddot{F}_{\varepsilon}^{kl,rs}\nabla_{\varepsilon}^{i} h_{kl}^{\varepsilon}\nabla_j^{\varepsilon} h_{rs}^{\varepsilon} + \dot{F}_{\varepsilon}^{kl} h_{km}^{\varepsilon} h_{\varepsilon\ l}^{m}\left(h_{\varepsilon\ j}^{i} - \eta F^{\varepsilon}\delta_j^i\right).$$

so if the inequality holds initially, then it is preserved under the flow (5). Choice of η follows from estimate of f_{ε} in terms of f and H. □

Remark: Taking $\varepsilon \leq \varepsilon_0 = \frac{\eta}{n}$, the argument of the convolution f_ε remains within Γ_+.

Lemma (upper speed bound, while inradius remains positive)

Fix $t_0 < T$. Then for any $r > 0$ such that $u \geq r$ at time t_0, we have on $[0, t_0]$:

$$F^\varepsilon(x,t) \leq 2\rho_+ \max\left\{ \frac{\max_{M_0} F}{r}, \frac{2}{r^2} \right\}.$$

Idea of proof: A method of Chou ('85); choose origin such that $B_r(O)$ is enclosed by $M^\varepsilon_{t_0}$. Then $u^\varepsilon(x,t) \geq r$ for all $x \in \mathbb{S}^n$ and $t \in [0, t_0]$. The function $Q^\varepsilon = \frac{F^\varepsilon}{2u^\varepsilon - r}$ satisfies

$$\frac{\partial}{\partial t} Q^\varepsilon \leq \mathcal{L}^\varepsilon Q^\varepsilon + 4\dot{F}^{kl}_\varepsilon \frac{\nabla^\varepsilon_k u^\varepsilon}{2u^\varepsilon - r} \nabla^\varepsilon_l Q^\varepsilon + Q^2_\varepsilon \left(2 - r^2 Q^\varepsilon\right).$$

□

Lemma (equation (5) is uniformly parabolic)

$$\underline{C}\,\delta \leq f\,(\kappa + \delta e_i) - f\,(\kappa) \leq \overline{C}\,\delta.$$

Idea of proof: Upper and lower bounds on F and curvature pinching give that κ remains within a compact $K \subset \Gamma_+$. □

We have now shown that the solutions u^ε to (5) are bounded in $C^{2,\alpha}$, independent of ε, as long as the inradius is positive. Taking $\varepsilon \to 0$ we obtain a C^2 solution to (3).

Specifically, in view of our estimates independent of ε, using the method of continuity and mollification as in Lieberman Chapter 14, we obtain

Theorem (short-time existence of solution to (1))

For any $u_0 \in C^{1,\alpha}\,(\mathbb{S}^n)$ there exists a $\delta > 0$ and a unique solution $u \in C^{2,1}\,(\mathbb{S}^n \times (0,\delta)) \cap C\,(\mathbb{S}^n \times [0,\delta))$.

Theorem (contraction to a point)

The images M_t shrink to a point as $t \to T < \infty$.

Proof:

1. Suppose $\rho_- \not\to 0$ as $t \to T$.
2. Speed bounds and curvature pinching imply bounds above and below on the principal curvatures.
3. These imply convergence to $u(\cdot, T)$, generating a $C^{1,1}$ hypersurface $\tilde{M}_T$.
4. $\tilde{M}_T$ could then be used as initial data in the short-time existence theorem, contradicting the maximality of T.
5. Therefore $\rho_- \to 0$ as $t \to T$ and, via curvature pinching, $\rho_+ \to 0$ also. □

Moreover, by standard arguments we get $u \in C^{2,\alpha}(\mathbb{S}^n \times (0, T))$.

The natural rescaling of the solution to (1) is

$$\tilde{X}(x,t) = \frac{1}{\sqrt{2(T-t)}}\left(X(x,t) - p\right),$$

where M_t contracts to the point $p \in \mathbb{R}^{n+1}$ at time T. The rescaled time parameter is

$$\tau = -\frac{1}{2}\ln\left(1 - \frac{t}{T}\right) \in [0,\infty).$$

The rescaled immersions $\tilde{M}_\tau$ evolve according to

$$\frac{\partial}{\partial\tau}\tilde{X}(x,\tau) = -F\left(\tilde{\mathcal{W}}(x,\tau)\right)\tilde{\nu}(x,\tau) + \tilde{X}(x,\tau), \qquad (6)$$

with initial condition

$$\tilde{X}(x,0) = \frac{1}{\sqrt{2T}}\left(X_0 - p\right).$$

By standard arguments, solutions to (6) have

- uniform positive lower and upper bounds on F;
- curvature pinching (homogeneous degree zero estimate).

Consequently, (6) is uniformly parabolic and $C^{2,\alpha}$ regularity of $\tilde{X}$ then follows by standard arguments for fully nonlinear equations (Krylov, Safanov).

For **asymptotic sphericity**, we need a geometric quantity whose extremum characterises a sphere, and whose monotonicity survives in the limit $\varepsilon \to 0$. Consider the family of flows

$$\frac{\partial}{\partial \tau} \tilde{X}^{\varepsilon}\left(x, \tau_{\varepsilon}\right) = -F^{\varepsilon}\left(\tilde{\mathcal{W}}^{\varepsilon}\left(x, \tau_{\varepsilon}\right)\right) \tilde{\nu}^{\varepsilon}\left(x, \tau_{\varepsilon}\right) + \tilde{X}^{\varepsilon}\left(x, \tau_{\varepsilon}\right), \qquad (7)$$

with initial condition

$$\tilde{X}^{\varepsilon}\left(x, 0\right) = \frac{1}{\sqrt{2T_{\varepsilon}}}\left(X_0 - p_{\varepsilon}\right).$$

For $\tilde{G}_0^\varepsilon := \tilde{G}_0\left(\tilde{\mathcal{W}}^\varepsilon\right)$ smooth, positive, increasing, concave, degree-one homogeneous and normalised, set

$$\tilde{G}^\varepsilon := \tilde{G}_0^\varepsilon + k\tilde{H}^\varepsilon \iff \tilde{g}^\varepsilon := \tilde{g}_0^\varepsilon + k\sum \tilde{\kappa}_i^\varepsilon$$

and $\tilde{Q}_\alpha^\varepsilon := \tilde{G}^\varepsilon - \alpha\tilde{F}^\varepsilon \iff \tilde{q}_\alpha^\varepsilon := \tilde{g}^\varepsilon - \alpha f^\varepsilon$, for numbers k and α.

Since $\tilde{G}$ is concave and $\tilde{F}$ is convex, the function $\frac{\tilde{G}}{\tilde{F}}$ has only one local maximum on $\Gamma^+ \cap \left\{\left|\tilde{A}^\varepsilon\right| = 1\right\}$, at $(1, \ldots, 1)$, implying the structural bound $\frac{\tilde{G}}{\tilde{F}} \leq 1$.

Lemma

For any $\overline{\alpha}$ there is an absolute constant $k_0 = k_0(\overline{\alpha})$ such that, under the rescaled flow (7), for any $\alpha \leq \overline{\alpha}$ and $k \geq k_0 > 0$,

$$\frac{\partial}{\partial \tau_\varepsilon}\tilde{Q}_\alpha^\varepsilon \geq \tilde{\mathcal{L}}^\varepsilon \tilde{Q}_\alpha^\varepsilon + \left(\dot{\tilde{F}}_\varepsilon^{pq}\tilde{h}^\varepsilon{}_p{}^m\tilde{h}_{mq}^\varepsilon - 1\right)\tilde{Q}_\alpha^\varepsilon. \quad (8)$$

We compute

$$\frac{\partial}{\partial \tau_\varepsilon} \tilde{Q}^\varepsilon_\alpha = \tilde{\mathcal{L}}^\varepsilon \tilde{Q}_\alpha + \left(\dot{\tilde{Q}}^\varepsilon_\alpha{}^{ij} \ddot{\tilde{F}}^{pq,mn}_\varepsilon - \dot{\tilde{F}}^{ij}_\varepsilon \ddot{\tilde{Q}}^\varepsilon_\alpha{}^{pq,mn} \right) \tilde{\nabla}_i{}^\varepsilon \tilde{h}^\varepsilon_{pq} \tilde{\nabla}_j{}^\varepsilon \tilde{h}^\varepsilon_{mn} + \left(\dot{\tilde{F}}^{pq}_\varepsilon \tilde{h}^\varepsilon_p{}^m \tilde{h}^\varepsilon_{mq} - 1 \right) \tilde{Q}^\varepsilon_\alpha . \quad (9)$$

We wish to choose $k > 0$ such that the whole $\tilde{\nabla} A^\varepsilon$ term in (9) is nonnegative. This requires the matrix inequality $\dot{\tilde{Q}}^\varepsilon_\alpha \geq 0$.

We have in coordinates that diagonalise the Weingarten map

$$\frac{\partial \tilde{q}^\varepsilon_\alpha}{\partial \tilde{\kappa}_i} = \frac{\partial \tilde{g}^\varepsilon_0}{\partial \tilde{\kappa}_i} + k - \alpha \frac{\partial \tilde{f}^\varepsilon}{\partial \tilde{\kappa}_i} > k_0 - \overline{\alpha} \overline{C}.$$

Taking $k_0 = \overline{\alpha} \overline{C}$ meets the requirement. □

Using the properties of $\tilde{G}^{\varepsilon}$ and the lower speed bound, we have

Lemma

There exists an absolute constant $\hat{C} > 0$ such that

$$\left(\hat{C} - \alpha\right) \tilde{F}^{\varepsilon} \leq \tilde{Q}^{\varepsilon}_{\alpha} \leq (1 - \alpha)\, \tilde{F}^{\varepsilon}. \tag{10}$$

Completion of proof of asymptotic sphericity

Consider $\tilde{Q}^{\varepsilon}_{\alpha_m}$ on the time interval $[m, m+1]$ for $m \geq 1$. Fix $\overline{\alpha} = 1$ and choose $\alpha = \alpha_0$ such that

$$\min_{M_0} \tilde{Q}^{\varepsilon}_{\alpha_0} = 0\,.$$

The lower bound in (10) implies $\alpha_0 > 0$, moreover, there is an upper bound on α_0 beyond which

$$\min_{M_t, t \in [m, m+1]} \tilde{Q}^{\varepsilon}_{\alpha_m} < 0.$$

The sequence $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is now generated by choosing on each interval $[m, m+1]$ the corresponding $\alpha = \alpha_m$ such that

$$\min_{M_{t_m}} \tilde{Q}^{\varepsilon}_{\alpha_m} = 0\,.$$

For all m we have $\alpha_m \leq 1$ since otherwise, by (10),

$$\max_{M_t, t\in[m,m+1]} \tilde{Q}^{\varepsilon}_{\alpha_m} < 0.$$

The evolution equation (8) implies that on $[m, m+1]$ the quantity $\tilde{Q}^{\varepsilon}_{\alpha_m}$ is non-negative.

We show that in fact $\min_{M_t} \tilde{Q}^{\varepsilon}_{\alpha_m}$ increases using the parabolic Harnack inequality.

First rewrite (8) in a local coordinate system around $B_\rho(x)$ for any $x \in M$,

$$\frac{\partial}{\partial \tau_\varepsilon} \tilde{Q}^\varepsilon_{\alpha m} \geq \dot{\tilde{F}}^{ij}_\varepsilon \left(\frac{\partial^2}{\partial x_i \partial x_j} \tilde{Q}^\varepsilon_{\alpha m} - \Gamma^k_{ij} \frac{\partial}{\partial x_k} \tilde{Q}^\varepsilon_{\alpha m} \right) + \left(\dot{\tilde{F}}^{pq}_\varepsilon \tilde{h}^\varepsilon{}_p{}^m \tilde{h}^\varepsilon_{mq} - 1 \right) \tilde{Q}^\varepsilon_{\alpha m} \,.$$

Using the ellipticity constants, we derive, for λ to be chosen,

$$\frac{\partial}{\partial \tau_\varepsilon} \left(e^{\lambda t} \sqrt{\tilde{Q}^\varepsilon_{\alpha m}} \right) \geq \dot{\tilde{F}}^{ij}_\varepsilon \frac{\partial^2}{\partial x_i \partial x_j} \left(e^{\lambda t} \sqrt{\tilde{Q}^\varepsilon_{\alpha m}} \right) + \frac{1}{2} \left(\dot{\tilde{F}}^{pq}_\varepsilon \tilde{h}^\varepsilon{}_p{}^m \tilde{h}^\varepsilon_{mq} - \frac{1}{2} \frac{\overline{C}^2}{\underline{C}} |\Gamma|^2 - 1 + \lambda \right) \left(e^{\lambda t} \sqrt{\tilde{Q}^\varepsilon_{\alpha m}} \right) \,.$$

Since the rescaled curvatures are bounded, there is a positive $\lambda = \lambda_0$ such that for

$$\tilde{Z}^{\varepsilon}_{\alpha_m} = \left(e^{\lambda_0 t} \sqrt{\tilde{Q}^{\varepsilon}_{\alpha_m}} \right),$$

$$\left(\frac{\partial}{\partial \tau_\varepsilon} - \dot{\tilde{F}}^{ij}_{\varepsilon} \frac{\partial^2}{\partial x_i \partial x_j} \right) \tilde{Z}^{\varepsilon}_{\alpha_m} \geq 0$$

in the local coordinate chart. The weak parabolic Harnack inequality implies, for each $x \in M$,

$$\min_{B_{\rho/2}(x) \times [m+1, m+2]} \tilde{Z}^{\varepsilon}_{\alpha_m} \geq c \int_{m-1}^{m} \left(\frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |\tilde{Z}^{\varepsilon}_{\alpha_m}|^\sigma dx \right)^{\frac{1}{\sigma}} d\tau \, .$$

for absolute positive σ and bounded c, independent of x.

Since $\tilde{X}_\varepsilon \in C^{2,\alpha}$, $\tilde{Z}^\varepsilon_{\alpha_m} \in C^{0,\alpha}$ and therefore

$$\min_{B_{\rho/2}(x)\times[m+1,m+2]} \tilde{Z}^\varepsilon_{\alpha_m} \geq c \max_{B_\rho(x)\times[m-1,m]} \tilde{Z}^\varepsilon_{\alpha_m} ,$$

where c depends on the absolute constants σ and α.

A parabolic chaining argument gives

$$\min_{M\times[m+1,m+2]} \tilde{Z}^\varepsilon_{\alpha_m} \geq c \max_{M\times[m-1,m]} \tilde{Z}^\varepsilon_{\alpha_m} .$$

Absorbing the exponential factor and squaring gives

$$\min_{M\times[m+1,m+2]} \tilde{Q}^\varepsilon_{\alpha_m} \geq c \max_{M\times[m-1,m]} \tilde{Q}^\varepsilon_{\alpha_m} , \tag{11}$$

where $c > 0$ is an absolute constant.

Estimating the maximum in (11) implies the recurrence relation

$$1-\alpha_{m+1} \leq -c(1-\alpha_m) + (1-\alpha_m) \leq (1-c)(1-\alpha_m)\,.$$

Since $\alpha_m < 1$ observe that $1-c > 0$ and so $1-c \in (0,1)$.

Iterating the recurrence relation, we have, for
$C = (1-\alpha_0) \in (0,1)$ and $\gamma = -\log(1-c) \in (0,\infty)$,

$$0 < 1-\alpha_{m+1} \leq (1-c)^m (1-\alpha_0) = (1-\alpha_0) e^{m\log(1-c)} \leq Ce^{-\gamma m}\,.$$

This holds for $t \in [m, m+1]$ and implies, for all $t \geq 0$,

$$0 < 1 - \min_{M_t} \frac{\tilde{G}^\varepsilon}{\tilde{F}^\varepsilon} \leq Ce^{-\gamma t}\,.$$

This implies, in turn, for all ε, uniform exponential convergence of $\frac{\tilde{G}^\varepsilon}{\tilde{F}^\varepsilon}$ to 1, a value of the ratio that is obtained only on a sphere. □

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