Automorphisms of Coxeter Groups

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Dedicated to

R. A. Franzsen

V. T. Franzsen

F. A. Wawryk

and the memory of

A. B. Franzsen (1930–1995)

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This thesis contains no material which has been accepted for the award of any other degree or diploma. All work in this thesis, except where duly attributed to another person, is believed to be original.

Contents

Introduction

There are surprisingly few results known about the automorphisms of infinite Coxeter groups. The only complete results are for finite rank *graph universal* Coxeter groups. A Coxeter group is graph universal if the labels on all edges in the Coxeter diagram are ∞. In the paper [Jam88], James found the automorphism groups of graph universal Coxeter groups whose diagrams have the form:

$$\bullet\overset{\infty}{-\!-}\bullet\overset{\infty}{-\!-}\bullet\quad\cdots\quad\bullet\overset{\infty}{-\!-}\bullet$$

This result was extended by Tits, [Tit88], to include all irreducible graph universal Coxeter groups whose diagrams do not contain triangles. Finally Mühlherr, [Mü98], found the automorphism group of any graph universal Coxeter group. In that paper Mühlherr gives a presentation for this automorphism group.

For the most part I will be looking at the automorphism group of a Coxeter group with finite labels. The only paper that I could find that deals with the finite label case is [HRT97]. In this paper Howlett, Rowley and Taylor show that the outer automorphism group of a Coxeter group of finite rank whose diagram has no infinite bonds is itself finite. The automorphisms of the finite irreducible Coxeter groups are well known.

In the following it will be shown that for a large class of finite rank Coxeter groups all automorphisms are inner by graph. That is $\mathrm{Aut}(W)$, the automorphism group of the Coxeter group W, is generated by the inner automorphisms and automorphisms arising from symmetries of the Coxeter diagram. Indeed, in most cases, $\mathrm{Aut}(W)$ is the semidirect product of $\mathrm{Inn}(W)$ and the group of graph automorphisms.

In the first chapter some basic properties of Coxeter groups are established. In particular it is shown that maximal finite standard parabolic subgroups are maximal finite subgroups. Finally it is shown that if the diagram of a given infinite Coxeter group, W, is a forest with labels in the set $\{2, 3, 4, 6\}$, then any automorphism of W that preserves the set of reflections is inner by graph.

The second chapter deals mainly with the automorphism groups of the irreducible finite Coxeter groups. In the finite case it is possible for a graph automorphism to be inner; for example, conjugation by the longest element in type A_n induces the obvious graph automorphism. The final section in Chapter 2 shows that this cannot happen if the Coxeter group has no finite irreducible components.

Nearly finite Coxeter groups are then defined; these are infinite Coxeter groups of rank n that have a finite standard parabolic subgroup of rank $n-1$. When W is a non-degenerate irreducible nearly finite Coxeter group then all reflection-preserving automorphisms are inner by graph. If this subgroup is of type A_{n-1} ($n \neq 6$), D_{2k+1}, E_6 or E_7 and W only has one conjugacy class of reflections, then all automorphisms are inner by graph.

Chapter 4 deals completely with the affine Weyl groups and the hyperbolic groups (in the sense of [Hum90]). Using the methods developed in previous chapters it is shown that in all cases the automorphism group is the semidirect product of $\mathrm{Inn}(W)$ and the group of graph automorphisms.

Finally the rank 3 Coxeter groups are considered in full. Having shown in Chapter 4 that if all the bonds are finite then the automorphisms are inner by graph the groups with infinite bonds are considered. The automorphisms of the rank 3 graph universal Coxeter groups are found. Finally the rank 3 groups with both finite and infinite bonds are considered. The case of rank 3 Coxeter groups with finite bonds is also covered in [FH].

Chapter 1

Properties of Coxeter Groups

We recall that a Coxeter group is a group with a presentation of the form

$$W = \mathrm{gp}\langle \{\, r_a \mid a \in \Pi \,\} \mid (r_a r_b)^{m_{ab}} = 1 \text{ for all } a, b \in \Pi \rangle$$

where Π is some indexing set, whose cardinality is called the *rank* of W, and the parameters m_{ab} satisfy the following conditions: $m_{ab} = m_{ba}$, $m_{ab} = 1$ if and only if $a = b$ and each m_{ab} lies in the set $\{\, m \in \mathbb{Z} \mid m \geq 1 \,\} \cup \{\infty\}$. The relation $(r_a r_b)^{m_{ab}} = 1$ is omitted if $m_{ab} = \infty$.

§1.1 Some Basic Facts

The *(Coxeter) diagram* of W is a graph with vertex set Π in which an edge (or *bond*) labelled m_{ab} joins $a, b \in \Pi$ whenever $m_{ab} \geq 3$. We say that the group is *irreducible* if this graph is connected.

Let V be the real vector space with basis Π and let B be a bilinear form on V with

$$B(a,b) = -\cos(\pi/m_{ab}) \qquad \text{if } m_{ab} \neq \infty,$$

and

$$B(a,b) \leq -1 \qquad \text{if } m_{ab} = \infty.$$

for all $a, b \in \Pi$. There is clearly a unique such form with $B(a,b) = -1$ whenever $m_{ab} = \infty$ and unless otherwise stated we assume that B is defined in this manner. For each $a \in V$ such that $B(a,a) = 1$ we define $\sigma_a : V \to V$ by $\sigma_a v = v - 2B(a,v)a$; it is well known (see, for example, Corollary 5.4 of [Hum90]) that W has a faithful representation on V given by $r_a \mapsto \sigma_a$ for all $a \in \Pi$. We shall identify elements of W with their images in this representation; thus $r_a = \sigma_a$ is the reflection in the hyperplane perpendicular to a. We also call this the *reflection along the root* a. The action of W on V preserves the form B. If the form B is degenerate, non-degenerate or positive definite then we will often just say that W is degenerate, non-degenerate or positive definite respectively.

The elements of the basis Π are called *simple roots*, and the reflections r_a for $a \in \Pi$ are called *simple reflections*. We call $\Phi = \{\, wa \mid w \in W,\ a \in \Pi \,\}$ the *root system* of W. Usually the reflection along the simple root a_i will be denoted by r_i to simplify subscripts.

The following lemma collects together some facts which will be useful later.

1.1 Lemma *Given the above representation of the Coxeter group W, the following are true.*

(1) *If $v \in \Phi$ and $v = \sum_{a\in\Pi} \lambda_a a$, then either $\lambda_a \geq 0$ for all $a \in \Pi$ or $\lambda_a \leq 0$ for all $a \in \Pi$. In the former case we call v a* positive root, *in the latter case a* negative root, *and we define Φ^+ and Φ^- to be the set of all positive roots and the set of all negative roots respectively.*

(2) *If $w \in W$ is a reflection, then $w = r_\alpha$ for some $\alpha \in \Phi$. Furthermore, $\alpha = xa$ for some $x \in W$ and $a \in \Pi$, whence $w = x r_a x^{-1}$ is conjugate to a simple reflection.*

(3) *W is a finite group if and only if the bilinear form B is positive definite.*

Proof Section 5.4 in [Hum90] contains a proof of 1, while 2 appears as Proposition 1.14. Theorem 4.1 in [Deo82] contains a proof of 3. ∎

1.2 Definition If w is an element of the Coxeter group W, then the *length* of w is the length of the shortest expression of w as a product of simple reflections. We denote this length by $l(w)$. Any expression of w as a product of $l(w)$ simple roots will be called a *reduced expression for w*.

Regarding w as a linear transformation of V we see that $\det(w) = (-1)^{l(w)}$, since all reflections have determinant -1.

It is easily seen that $l(w) = 0$ if and only if $w = 1$ and $l(w^{-1}) = l(w)$. If w can be expressed as a product of l simple reflections then it is obvious that for any $a \in \Pi$ the element $w' = wr_a$ can be expressed as a product of $l+1$ simple reflections. So $l(w') \leq l(w)+1$. Since also $w = w'r_a$ the same reasoning shows that $l(w) \leq l(w') + 1$. But since

$$(-1)^{l(w)} = \det w = (\det w')(\det r_a) = -\det w' = (-1)^{l(w')+1}$$

it follows that $l(w') \neq l(w)$. So we have proved.

1.3 Lemma *If $w \in W$ and $a \in \Pi$, then*

$$l(wr_a) = l(w) \pm 1.$$

1.4 Lemma *If $w, w' \in W$, then*

$$l(ww') \equiv l(w) + l(w') \pmod 2.$$

Proof We have $(-1)^{l(ww')} = \det(ww') = \det w \det w' = (-1)^{l(w)+l(w')}$. ∎

1.5 Definition If $w \in W$, then let

$$N(w) = \{\, a \in \Phi^+ \mid wa \in \Phi^- \,\}.$$

Define also $n(w) = |N(w)|$, the cardinality of the set $N(w)$. (It is shown below that the set $N(w)$ is always finite.)

1.6 Lemma *If $a \in \Pi$ is a simple root, then $N(a) = \{\, r_a \,\}$, in particular*

$$r_a(\Phi^+ \backslash \{\, a \,\}) = \Phi^+ \backslash \{\, a \,\}.$$

Proof As $r_a a = -a$ it remains to show that

$$r_a(\Phi^+ \backslash \{\, a \,\}) = \Phi^+ \backslash \{\, a \,\}.$$

Assuming that $c \in \Phi^+ \backslash \{\, a \,\}$, then $c = \sum_\Pi \lambda_b b$ where $\lambda_b \geq 0$ for all b, and there is a $b_0 \neq a$ such that $\lambda_{b_0} > 0$, as the only multiples of a in Φ are $\pm a$. Then

$$\begin{aligned} r_a c &= c - 2\,B(a,c)a \\ &= (\lambda_a - 2\,B(a,c))a + \lambda_{b_0} b_0 + \sum_{\Pi'} \lambda_b b. \end{aligned}$$

Now $\lambda_{b_0} > 0$ and so $r_a c \in \Phi^+$ and is clearly not a. ∎

1.7 Lemma *If $w \in W$ and $a \in \Pi$, then*

(a) *If $wa \in \Phi^+$, then $N(wr_a) = r_a N(w) \mathbin{\dot\cup} \{\, a \,\}$.*
(b) *If $wa \in \Phi^-$, then $N(wr_a) = r_a(N(w) \backslash \{\, a \,\})$.*
(c) *If $w^{-1}a \in \Phi^+$, then $N(r_a w) = N(w) \mathbin{\dot\cup} \{\, w^{-1}a \,\}$.*
(d) *If $w^{-1}a \in \Phi^-$, then $N(r_a w) = N(w) \backslash \{\, -w^{-1}a \,\}$.*

(Here the symbol $\dot\cup$ stands for the disjoint union.)

Proof Let $w \in W$ and $a \in \Pi$. If $wa \in \Phi^+$ then $a \notin N(w)$, while $wr_a a = w(-a) = -wa \in \Phi^-$, and so $a \in N(wr_a)$. By Lemma 1.6, $r_a N(w) \subset \Phi^+$ and $wr_a(r_a N(w)) = wN(w) \subset \Phi^-$ and so $r_a N(w) \subset N(wr_a)$. Now, if $b \in N(wr_a)$ and $b \neq a$, then $b \in \Phi^+$ but $w(r_a b) = wr_a b \in \Phi^-$ and so $r_a b \in N(w)$. Hence

$$N(wr_a) = r_a N(w) \,\dot{\cup}\, \{ a \} .$$

A similar argument proves the second claim.

Looking at the last claim, we have been given that $w^{-1}a \in \Phi^-$ and so $-w^{-1}a \in \Phi^+$, but $w(-w^{-1}a) = -a \in \Phi^-$. Thus $-w^{-1}a \in N(w)$. Notice however, that

$$r_a w(-w^{-1}a) = r_a(-a) = a \in \Phi^+$$

and so $-w^{-1}a \notin N(r_a w)$. Now if $b \in N(r_a w)$, then $r_a wb = -c \in \Phi^-$ where $c \in \Phi^+$. Observe that $c \neq a$, since $c = a$ would imply that $b = w^{-1}a$, contrary to the fact that $w^{-1}a \in \Phi^-$ and $b \in \Phi^+$. Thus

$$wb = -r_a c \in \Phi^- .$$

Therefore $b \in N(w)$. Now $b \neq -w^{-1}a$ and so we have

$$N(r_a w) \subseteq N(w) \backslash \left\{ -w^{-1}a \right\} .$$

The reverse inclusion is clear, since if $b \in N(w) \backslash \left\{ -w^{-1}a \right\}$, then

$$r_a wb \in r_a(\Phi^- \backslash \{ a \}) = \Phi^- \backslash \{ -a \}$$

and so $b \in N(r_a w)$. Thus:

$$N(r_a w) = N(w) \backslash \left\{ -w^{-1}a \right\} .$$

A similar argument proves the third claim. ∎

Suppose that $w = r_1 r_2 \dots r_t$, where $l(w) = t$. Using Lemma 1.7, if we build up $N(w)$ starting from $N(r_1)$, then

$$N(r_1 r_2) \subseteq r_2 N(r_1) \,\dot{\cup}\, \{ a_2 \} = \{ r_2 a_1, a_2 \}$$

where a_i is the simple root associated with the reflection r_i. An induction proof completes the proof of the following.

1.8 Corollary *If $w \in W$, then $n(w) \leq t = l(w)$. In particular $N(w)$ is always a finite set.*

The following is Theorem 1.7 of [Hum90], Part 3 being called the *Deletion Condition* there.

1.9 Theorem *Suppose that $w = r_1 r_2 \dots r_t$ and $n(w) < t$. Then we can find $1 \leq i < j \leq t$ such that*

(1) $a_i = r_{i+1} \dots r_{j-1} a_j$.

(2) $r_{i+1} r_{i+2} \dots r_j = r_i r_{i+1} \dots r_{j-1}$.

(3) $w = r_1 \dots \hat{r}_i \dots \hat{r}_j \dots r_t$.

As before r_i denotes the reflection along the simple root a_i.

Proof Following the steps in the proof of Corollary 1.8, it can be seen that if $n(w) < t$, then we can find a j such that $r_1 r_2 \dots r_{j-1} a_j \in \Phi^-$. Now $a_j \in \Phi^+$ and so we can find an index i such that

$$r_{i+1} \dots r_{j-1} a_j \in \Phi^+$$
$$r_i r_{i+1} \dots r_{j-1} a_j \in \Phi^- .$$

But $N(r_i) = \{ a_i \}$ and so

$$a_i = r_{i+1} \dots r_{j-1} a_j .$$

It is easily seen that $wr_vw^{-1} = r_{wv}$ and so

$$(r_{i+1}\dots r_{j-1})r_j(r_{j-1}\dots r_{i+1}) = r_{r_{i+1}\dots r_{j-1}a_j} = r_{a_i} = r_i.$$

Thus $r_{i+1}\dots r_j = r_i\dots r_{j-1}$ and therefore $r_i\dots r_j = r_{i+1}\dots r_{j-1}$. Using this equation in the expression for w:

$$\begin{aligned} w &= r_1\dots(r_i\dots r_j)\dots r_t \\ &= r_1\dots(r_{i+1}\dots r_{j-1})\dots r_t \\ &= r_1\dots\hat{r}_i\dots\hat{r}_j\dots r_t. \end{aligned}$$

∎

1.10 Corollary *If $w \in W$, then $n(w) = l(w)$.*

Proof If $l(w) = t$, then by the above we cannot have $n(w) < t$ or else we can shorten a reduced expression for w as a product of simple reflections. Thus $n(w) \geq l(w)$ and hence, by Corollary 1.8,

$$n(w) = l(w).$$

∎

1.11 Lemma *The Coxeter group W is finite if and only if the associated root system, Φ, is also finite.*

Proof If W is finite then the set of simple reflections is finite, and so Π is finite. As $\Phi = \{\, wa \mid w \in W, a \in \Pi \,\}$ it follows that Φ is also finite.

Conversely, suppose that Φ is finite. For $w \in W$ define $\rho_w \in \mathrm{Sym}(\Phi)$ by

$$\rho_w : a \mapsto wa.$$

If $\rho_w = \rho_{w'}$, then $w^{-1}w'$ fixes all roots $a \in \Phi$, in particular all positive roots. Thus

$$l(w^{-1}w') = n(w^{-1}w') = 0$$

and so $w^{-1}w' = 1$. Thus $\rho : W \to \mathrm{Sym}(\Phi)$ is injective. Hence, if Φ is finite,

$$|W| \leq |\,\mathrm{Sym}(\Phi)| = |\Phi|!.$$

∎

For each $I \subseteq \Pi$ we define $W_I = \mathrm{gp}\langle\{\, r_a \mid a \in I \,\}\rangle$; these subgroups are called the *standard parabolic subgroups* of W. Clearly W_I preserves the subspace V_I spanned by I; furthermore it acts on this subspace as a Coxeter group with root system $\Phi_I = \Phi \cap V_I$. A *parabolic subgroup* of W is any subgroup of the form wW_Iw^{-1} for some $w \in W$ and $I \subseteq \Pi$.

To save space in our later calculations we shall write $\mathrm{s}(\theta)$ for $\sin\theta$ and $\mathrm{c}(\theta)$ for $\cos\theta$. We shall also use π_k for π/k (for any positive integer k) and $u \cdot v$ for $B(u,v)$. It is readily checked that if $I = \{a, b\}$ is a two-element subset of Π, then Φ_I consists of all vectors v of the form

$$\frac{\mathrm{s}((h-1)\pi_m)}{\mathrm{s}(\pi_m)}a_1 + \frac{\mathrm{s}\left(h\pi_m\right)}{\mathrm{s}(\pi_m)}a_2, \tag{1.12}$$

where $h \in \mathbb{Z}$. Observe that $v \cdot a_1 = -\,\mathrm{c}\left(h\pi_m\right)$ and $v \cdot a_2 = \mathrm{c}((h-1)\pi_m)$. Replacing h by $m - h + 1$ gives the equivalent formula

$$\frac{\mathrm{s}\left(h\pi_m\right)}{\mathrm{s}(\pi_m)}a_1 + \frac{\mathrm{s}((h-1)\pi_m)}{\mathrm{s}(\pi_m)}a_2, \tag{1.13}$$

where now $v \cdot a = \mathrm{c}((h-1)\pi_m)$ and $v \cdot b = -\,\mathrm{c}\left(h\pi_m\right)$. The positive roots in Φ_I are the vectors of the form 1.12 or 1.13 with $1 \leq h \leq m$. We see that positive numbers appearing as coefficients of a or b in roots $v \in \Phi^+_{\{a,b\}}$ are never less than 1. This result in fact extends to the entire root system. First a technical lemma.

1.14 Lemma *Suppose that $v \in \Phi^+$ and $w = r_1 r_2 \dots r_l$ has minimal length such that $w^{-1}v = b \in \Pi$. Then $r_i \dots r_l b$ is positive for all i.*

Proof Suppose, to the contrary, that $r_j \dots r_l b \in \Phi^-$ for some j. Then we can find i such that $r_{i+1} \dots r_l b \in \Phi^-$ but $x = r_i \dots r_l b \in \Phi^+$. So, $x \in \Phi^+$ while $r_i x \in \Phi^-$, and therefore $x \in N(r_i) = \{ a_i \}$. Hence

$$a_i = (r_1 \dots r_{i-1})^{-1} x \in \Pi$$

which contradicts the minimality of l. ∎

1.15 Lemma *(Brink [Bri98]) Suppose that $v = \sum_{a \in \Pi} \lambda_a a \in \Phi^+$. For each $a \in \Pi$, if $\lambda_a > 0$, then $\lambda_a \geq 1$.*

Proof Choose w with minimal length such that $w^{-1}v = b \in \Pi$, say $l(w) = l$. We prove this lemma by induction on l.

If $l = 0$, then $v \in \Pi$ and the result is trivial. If $l = 1$, then

$$v = r_a b = b - 2(a \cdot b)a$$

where $a \cdot b = -1$ or $-\cos(\pi_m)$ for some $m \geq 2$. (Note that this formula corresponds to $h = 2$ in 1.12.) If $m = 2$, then $r_a b = b$. Otherwise the coefficient of a in v is either 2 or

$$2\cos(\pi_m) \geq 2\cos \pi_3 = 1.$$

Assuming that $l \geq 2$, let $r_1 r_2 \dots r_l$ be a reduced expression for w. Thus $v = r_1 r_2 \dots r_l b$. By Lemma 1.14, $r_2 \dots r_l b \in \Phi^+$, say $r_2 \dots r_l b = \sum_{a \in \Pi} \mu_a a$. By induction the μ's are all zero or at least 1. Then

$$\begin{aligned} v &= r_1 r_2 \dots r_l b \\ &= r_1 \sum_{a \in \Pi} \mu_a a \\ &= \sum_{a \in \Pi} \mu_a r_1 a. \end{aligned}$$

The only coefficient we need to check is the coefficient of a_1 which is

$$\mu_{a_1} + \sum_{a \in \Pi - \{a_1\}} \big(-2(a \cdot a_1) \big).$$

Now μ_{a_1} is zero or at least 1, as are all the terms of the form $-2(a \cdot a_1)$, and so the result follows. ∎

1.16 Definitions In a Coxeter group W let $\mathrm{Ref}(W)$ denote the set of reflections in W.

Clearly symmetries of the Coxeter diagram give rise to automorphisms which permute the simple reflections; we call these *graph automorphisms*. Let $\mathrm{Gr}(W)$ denote the group of graph automorphisms of W.

1.17 Lemma *Let W be a rank n Coxeter group with simple roots Π. If I and J are disjoint subsets of Π such that no edge of the Coxeter diagram joins a root in I with a root in J, then*

$$\mathrm{Ref}(W_{I \cup J}) = \mathrm{Ref}(W_I) \dot\cup \mathrm{Ref}(W_J).$$

Proof It is clear that

$$\mathrm{Ref}(W_I) \cup \mathrm{Ref}(W_J) \subseteq \mathrm{Ref}(W_{I\cup J}).$$

Suppose that r is a reflection in $W_{I\cup J}$. Then r is conjugate to a simple reflection by Lemma 1.1. Without loss we can find $w \in W_{I\cup J}$ and $a_i \in I$ such that

$$r = w r_i w^{-1}.$$

Now $W_{I\cup J} = W_I \times W_J$ and so $w = w_i w_j = w_j w_i$ for some $w_i \in W_I$ and $w_j \in W_J$. Thus

$$\begin{aligned} r &= w r_i w^{-1} \\ &= w_i w_j r_i w_j^{-1} w_i^{-1} \\ &= w_i r_i w_i^{-1} \in W_I. \end{aligned}$$

Hence r is a reflection in W_I.

Now if $r \in \mathrm{Ref}(W_I) \cap \mathrm{Ref}(W_J)$, then following the above proof we can find $r_i, w_i \in W_I$ and $r_j, w_j \in W_J$ such that

$$w_i r_i w_i^{-1} = r = w_j r_j w_j^{-1}$$

and hence $r_i = w_j^{-1} r_i w_j = w_i^{-1} r_j w_i = r_j$, which contradicts the fact that I and J are disjoint. ∎

1.18 Corollary *If W is a rank n Coxeter group, $I \subset \Pi$ and we can find a simple root $a \in \Pi \backslash I$ such that $a \cdot b = 0$ for all $b \in I$, then*

$$\mathrm{Ref}(W_{I\cup\{a\}}) = \mathrm{Ref}(W_I) \dot{\cup} \{r_a\}.$$

§1.2 Longest Elements

Let W be a finite Coxeter group and $w \in W$ have maximal length in W. If $N(w) \neq \Phi^+$, then we can find a simple root $a \notin N(w)$. (If $\Pi \subset N(w)$, all positive linear combinations of Π which are roots will be in $N(w)$, by Lemma 1.1, and this means $\Phi^+ \subset N(w)$.) Thus $wa \in \Phi^+$ and so, by Lemma 1.7, $l(wr_a) = n(wr_a) = n(w) + 1 = l(w) + 1$, which contradicts the maximality of $l(w)$. Hence if w has maximal length, then $l(w) = |\Phi^+|$.

Furthermore, if $u \in W$ is any element, then using an appropriate simple root at each step we may find an element $v \in W$ such that uv has maximal length and $l(uv) = l(u) + l(v)$. We denote by w_0 the element of maximal length in the finite Coxeter group, W.

1.19 Lemma *The element of maximal length in a finite Coxeter group is uniquely defined and is an involution.*

Proof Suppose that $u, v \in W$ have maximal length. Then $N(u) = N(v) = \Phi^+$. If $a \in \Pi$, then $va \in -\Phi^+$ and $uva \in \Phi^+$. Thus $l(uv) = 0$ and therefore $uv = 1$. If we note that $l(u^{-1}) = l(u)$ is also maximal we can see that $u^{-1}v = 1$ implying that $u = v$. Similarly, if w_0 is the (unique) element of maximal length, then $w_0 w_0 = 1$, and therefore w_0 is an involution. ∎

1.20 Lemma *If W is finite with longest element w_0, then for $w \in W$*

$$l(ww_0) = l(w_0) - l(w).$$

Proof We have seen that for any $w \in W$ we can find a $u \in W$ such that $uw = w_0 = w_0^{-1}$ and $l(w_0) = l(u) + l(w)$. But, then

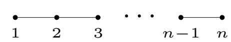
$$l(ww_0) = l(u^{-1}) = l(u) = l(w_0) - l(w).$$

∎

1.21 Definition If W is a possibly infinite Coxeter group and $I \subset \Pi$ such that the parabolic subgroup W_I is finite, then we denote the longest element of W_I by w_I.

Now consider the the irreducible finite Coxeter groups.

A_n: Let W be a Coxeter group of type A_n with diagram



It is well known that the number of positive roots is $n(n+1)/2$. Let $w_i = r_1 r_2 \cdots r_i$ where r_j is the reflection in the hyperplane perpendicular to the simple root a_j. Then

$$w_0 = w_n w_{n-1} \cdots w_2 w_1.$$

Observe that this expression for w_0 has length $n(n+1)/2$ which is equal to $|\Phi^+|$. Therefore this will be a reduced expression for w_0 if it does equal w_0. Note that if $j < i-1$, then $w_j a_i = a_i$, thus

$$w_n w_{n-1} \cdots w_2 w_1 a_i = w_n w_{n-1} \cdots w_{i-1} a_i$$

provided $i > 1$. Now

$$\begin{aligned} w_{i-1} a_i &= r_1 r_2 \cdots r_{i-1} a_i \\ &= r_1 r_2 \cdots r_{i-2} (a_{i-1} + a_i) \\ &= a_1 + a_2 + \cdots + a_i \end{aligned}$$

by induction, and therefore

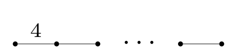
$$\begin{aligned} w_i w_{i-1} a_i &= w_i (a_1 + \cdots a_i) \\ &= r_1 r_2 \cdots r_i (a_1 + \cdots + a_i) \\ &= r_1 \cdots r_{i-1} (a_1 + \cdots a_{i-1}) \\ &= r_1 a_1 \qquad \text{(by induction)} \\ &= -a_1. \end{aligned}$$

Thus $w_n w_{n-1} \ldots w_1 a_i = w_n w_{n-1} \cdots w_{i+1} a_i$. Note that this is also valid for $i = 1$. Then

$$\begin{aligned} w_{i+1}(-a_1) &= r_1 r_2 (-a_1) \\ &= r_1 (-a_1 - a_2) \\ &= -a_2 \end{aligned}$$

and another induction will show that $w_n \cdots w_2 w_1 a_i = -a_{n-i+1}$. Therefore $w_n \cdots w_1$ does indeed send each positive root to a negative root and so must equal w_0. Unless $n = 1$ the longest element w_0 is not central, in fact conjugation by w_0 is the graph automorphism that interchanges r_i and r_{n-i+1}.

B_n: Let W be a Coxeter group of type B_n with diagram



In this case $|\Phi^+| = n^2$. This time we let $w_i = r_i r_{i-1} \cdots r_2 r_1 r_2 \cdots r_i$. Observe that this is the reflection along the root $b_i = r_i r_{i-1} \cdots r_2(a_1)$. A short calculation shows that

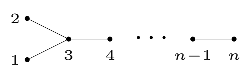
$$b_i = a_1 + \sqrt{2}a_2 + \cdots + \sqrt{2}a_i.$$

Now if $i > j > 1$ we find that

$$\begin{aligned} b_i \cdot b_j &= (b_j + \sqrt{2}a_{j+1} + \cdots + \sqrt{2}a_i) \cdot b_j \\ &= 1 + (\sqrt{2}a_{j+1} + \cdots + \sqrt{2}a_i) \cdot b_j \\ &= 1 + (\sqrt{2}a_{j+1}) \cdot (\sqrt{2}a_j) \\ &= 1 - 1 = 0, \end{aligned}$$

and a similar calculation also shows that $b_i \cdot b_1 = 0$ for $i > 1$. So the n roots $b_1, b_2, \ldots, b_n$ form an orthonormal basis for V, and it follows that $w_n w_{n-1} \cdots w_1$, the product of the reflections along the b_i, is the negative of the identity transformation on V. So $w_n w_{n-1} \cdots w_1(a) = -a$ for all positive roots a, and it follows that $w_n w_{n-1} \cdots w_1 = w_0$, the longest element. Furthermore, the given expression for w_i has length $2i - 1$, and so the resulting expression for w_0 has length $\sum_{i=1}^{n}(2i-1) = n^2 = |\Phi^+|$. So this expression is reduced. In these groups w_0 is central.

D_n: Let W be a Coxeter group of type D_n with diagram



In this case $|\Phi^+| = n(n-1)$. Let $w_i = r_i r_{i-1} \cdots r_2 r_1 r_3 r_4 \cdots r_i$ for $i > 2$, $w_2 = r_2$ and $w_1 = r_1$ let $w_0' = w_n w_{n-1} \cdots w_1$. Then the following lemma holds.

1.22 Lemma *Given the notation from above, w_0' acts as follows on the simple roots*

$$\begin{aligned} a_1 &\mapsto \begin{cases} -a_1 & \text{if } n \text{ is even} \\ -a_2 & \text{if } n \text{ is odd} \end{cases} \\ a_2 &\mapsto \begin{cases} -a_2 & \text{if } n \text{ is even} \\ -a_1 & \text{if } n \text{ is odd} \end{cases} \\ a_j &\mapsto -a_j \text{ if } j > 2. \end{aligned}$$

Proof We first prove that w_i interchanges a_1 and a_2 if $i > 2$ and fixes a_j for $i > j > 2$.

$$\begin{aligned} w_i a_1 &= r_i \cdots r_2 r_1 r_3 a_1 \\ &= r_i \cdots r_2 r_1 (a_1 + a_3) \\ &= r_i \cdots r_3 r_2 a_3 \\ &= r_i \cdots r_3 (a_2 + a_3) \\ &= r_i \cdots r_4 a_2 \\ &= a_2. \end{aligned}$$

As $w_i^2 = 1$ we can also see that $w_i a_2 = a_1$. If $i > j > 2$, then

$$\begin{aligned} w_i a_j &= r_i \cdots r_3 r_2 r_1 r_3 \cdots r_{j+1} a_j \\ &= r_i \cdots r_1 \cdots r_j (a_j + a_{j+1}) \\ &= r_i \cdots r_1 \cdots r_{j-1} a_{j+1} \\ &= r_i \cdots r_j a_{j+1} \\ &= r_i \cdots r_{j+1} (a_j + a_{j+1}) \\ &= r_i \cdots r_{j+2} a_j \\ &= a_j. \end{aligned}$$

Now we look at the cases $w_i a_i$ and $w_{i-1} a_i$, where $i > 3$.

$$\begin{aligned}
w_{i-1} a_i &= r_{i-1} \cdots r_1 \cdots r_{i-1} a_i \\
&= r_{i-1} \cdots r_1 \cdots r_{i-2}(a_{i-1} + a_i) \\
&= r_{i-1} a_i + r_{i-1} w_{i-2} a_{i-1} \\
&= a_i + a_{i-1} + (a_{i-1} + 2a_{i-2} + \cdots + 2a_3 + a_2 + a_1) \text{ by induction} \\
&= a_i + 2a_{i-1} + \cdots + 2a_3 + a_2 + a_1.
\end{aligned}$$

If we now observe that $a_i \cdot (a_i + 2a_{i-1} + \cdots + 2a_3 + a_2 + a_1) = 0$, then

$$\begin{aligned}
w_i a_i &= r_i w_{i-1} r_i a_i \\
&= -r_i w_{i-1} a_i \\
&= -w_{i-1} a_i
\end{aligned}$$

and hence $$w_i w_{i-1} a_i = -a_i.$$

If follows that if $i > 3$

$$\begin{aligned}
w_n w_{n-1} \cdots w_1 a_i &= w_n \cdots w_i w_{i-1} a_i \\
&= -w_n \cdots w_{i+1} a_i \\
&= -a_i.
\end{aligned}$$

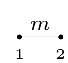
The same is true for $i = 3$, as is readily checked. Finally note that

$$\begin{aligned}
w_n w_{n-1} \cdots w_2 w_1 a_1 &= -w_n \cdots w_3 a_1 \\
&= -w_n \cdots w_4 a_2
\end{aligned}$$

and an induction proof finishes this argument. Similar calculations apply for a_2. ∎

Thus $w_0' = w_0$ is central if n is even, while if n is odd, then conjugation by w_0 is the graph automorphism that interchanges r_1 and r_2. We shall see in Chapter 2 that when n is even this graph automorphism is an outer automorphism. A simple calculation will confirm that the expression given for w_0 is reduced.

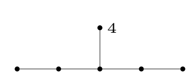
$I_2(m)$: Let W be a Coxeter group of type $I_2(m)$ with diagram



In all cases a reduced expression for the longest element is $r_1 r_2 \cdots$ where the product extends to m terms. If m is even, then this element is the half-turn η and so is central. If m is odd, then $r_1 r_2 \cdots$ is a reflection and so is not central, in fact conjugation by w_0 is the graph automorphism that interchanges r_1 and r_2.

The longest element and its effect upon the simple roots is merely stated for the remaining finite Coxeter groups.

E_6: Let W be a Coxeter group of type E_6 with diagram



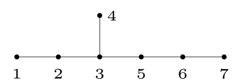
The longest element has length 36 and can be written, for example, as:

$$w_0 = r_6r_1r_2r_1r_3r_2r_1r_4r_3r_2r_1r_5r_3r_2r_1r_4r_3r_2r_5r_3r_4r_6r_5r_3r_2r_1r_4r_3r_2r_5r_3r_4r_6r_5r_3r_2,$$

$$\begin{aligned} w_0a_1 &= -a_6 \\ w_0a_2 &= -a_5 \\ w_0a_3 &= -a_3 \\ w_0a_4 &= -a_4 \\ w_0a_5 &= -a_2 \\ w_0a_6 &= -a_1. \end{aligned}$$

Thus conjugation by w_0 induces the obvious graph automorphism.

E_7: Let W be a Coxeter group of type E_7 with diagram

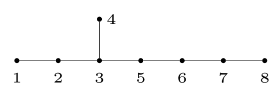


The longest element has length 63 and can be written as:

$$\begin{aligned} w_0 = {} & r_7r_1r_2r_1r_3r_2r_1r_4r_3r_2r_1r_5r_3r_2r_1r_4r_3r_2r_5r_3r_4r_6r_5r_3r_2r_1r_4r_3r_2r_5r_3r_4r_6r_5r_3r_2r_1 \\ & \times r_7r_6r_5r_3r_2r_1r_4r_3r_2r_5r_3r_4r_6r_5r_3r_2r_1r_7r_6r_5r_3r_2r_4r_3r_5r_6, \end{aligned}$$

w_0 is central.

E_8: Let W be a Coxeter group of type E_8 with diagram

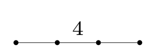


The longest element has length 120 and can be written as:

$$\begin{aligned} w_0 = {} & r_8r_1r_2r_1r_3r_2r_1r_4r_3r_2r_1r_5r_3r_2r_1r_4r_3r_2r_5r_3r_4r_6r_5r_3r_2r_1r_4r_3r_2r_5r_3r_4r_6r_5r_3r_2r_1 \\ & \times r_7r_6r_5r_3r_2r_1r_4r_3r_2r_5r_3r_4r_6r_5r_3r_2r_1r_7r_6r_5r_3r_2r_4r_3r_5r_6r_7r_8r_7r_6r_5r_3r_2r_1 \\ & \times r_4r_3r_2r_5r_3r_4r_6r_5r_3r_2r_1r_7r_6r_5r_3r_2r_4r_3r_5r_6r_7r_8r_7r_6r_5r_3r_2r_1r_4r_3r_2r_5r_3r_4 \\ & \times r_6r_5r_3r_2r_1r_7r_6r_5r_3r_2r_4r_3r_5r_6r_7, \end{aligned}$$

w_0 is central.

F_4: Let W be a Coxeter group of type F_4 with diagram

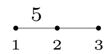


The longest element has length 24 and can be written as:

$$w_0 = r_4r_1r_2r_1r_3r_2r_1r_3r_2r_3r_4r_3r_2r_1r_3r_2r_3r_4r_3r_2r_1r_3r_2r_3,$$

w_0 is central.

H_3: Let W be a Coxeter group of type H_3 with diagram

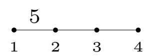


The longest element has length 15 and can be written as:

$$w_0 = r_3r_1r_2r_1r_2r_1r_3r_2r_1r_2r_1r_3r_2r_1r_2,$$

w_0 is central.

H_4: Let W be a Coxeter group of type H_4 with diagram



The longest element has length 60 and can be written as:

$$\begin{aligned} w_0 = {} & r_4r_1r_2r_1r_2r_1r_3r_2r_1r_2r_1r_3r_2r_1r_2r_3r_4r_3r_2r_1r_2r_1r_3r_2r_1r_2r_3r_4r_3r_2r_1r_2r_1r_3r_2r_1r_2 \\ & \times r_3r_4r_3r_2r_1r_2r_1r_3r_2r_1r_2r_3r_4r_3r_2r_1r_2r_1r_3r_2r_1r_2r_3, \end{aligned}$$

w_0 is central.

§1.3 Finite Subgroups of Infinite Coxeter Groups

If α is an automorphism of an infinite Coxeter group, W, then the image of a finite subgroup of W is again finite. In particular a maximal finite subgroup of W must be mapped to a maximal finite subgroup. In this section we characterise these subgroups.

Suppose that W is any Coxeter group with Π as the set of simple roots. We shall make use of the following result, which is due to Tits and appears in [Bou68], Exercise 2d, p. 130.

1.23 Lemma *If W is a Coxeter group and $H \le W$ is finite, then H is contained in a finite parabolic subgroup of W.*

Let V^* be the dual space of V and $\{\, \delta_a \mid a \in \Pi \,\}$ the basis of V^* such that for all $a, b \in \Pi$

$$\delta_a(b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \ne b. \end{cases}$$

For each $f \in V^*$ define

$$\zeta(f) = \left\{\, v \in \Phi^+ \mid f(v) < 0 \,\right\},$$

and let C be the set of all $f \in V^*$ for which $\zeta(f) = \emptyset$; equivalently,

$$C = \{\, f \in V^* \mid f(a) \ge 0 \text{ for all } a \in \Pi \,\}.$$

For each $w \in W$ and $f \in V^*$, define $fw : V \to \mathbb{R}$ by $v \mapsto f(wv)$; this gives a right action of W on V^*.

1.24 Lemma *Let $f \in V^*$ with $|\zeta(f)| < \infty$. Then $fw \in C$ for some $w \in W$.*

Proof Choose $w \in W$ with $|\zeta(fw)|$ minimal, and let $f' = fw$. Assume, for a contradiction, that $\zeta(f') \ne \emptyset$; that is, $f'(v) < 0$ for some $v \in \Phi^+$. Writing $v = \sum_{a\in\Pi} \lambda_a a$, we see that $f(v) = \sum a \in \Pi\lambda_a f(a)$, and since the coefficients λ_a are all non-negative we must have $f(a) < 0$ for at least one $a \in \Pi$. Now

$$(f'r_a)(a) = f'(r_a a) = f'(-a) = -f'(a) > 0,$$

and so $a \notin \zeta(f_a)$, writing f_a for $f'r_a$. It follows that $r_a c \in \Phi^+$ for all $c \in \zeta(f_a)$, and furthermore $r_a c \in \zeta(f')$ since

$$f'(r_a)(c) = (f_a)(c) < 0.$$

Therefore $v \mapsto r_a v$ is a one to one map from $\zeta(f_a)$ to $\zeta(f')$, and as a is not in the image of this map we deduce that $|\zeta(f_a)| < |\zeta(f')|$, contradicting the minimality of $|\zeta(fw)|$. ∎

1.25 Lemma *Let $f \in C$ and define $J = \{\, a \in \Pi \mid f(a) = 0 \,\}$. Then W_J is the stabilizer of f in W.*

Proof Let $S = \{\, w \in W \mid fw = f \,\}$, the stabilizer of f. If $a \in J$, then for all $v \in V$,

$$(fr_a)(v) = f(r_a v) = f(v - 2(v \cdot a)a) = f(v) - 2(v \cdot a)f(a) = f(v),$$

and so $fr_a = f$. Thus $W_J \subseteq S$.

To prove the reverse inclusion, we use induction on $l(w)$ to show that if $w \in S$ then $w \in W_J$. If $l(w) = 0$, then $w \in W_J$ trivially. If $w \neq 1$, let $w' \in W$ and $a \in \Pi$ be such that $w = w'r_a$ and $l(w') = l(w) - 1$. This tells us, by Lemma 1.7, that $w'a \in \Phi^+$. Then

$$0 \leq f(a) = (fw)(a) = f(wa) = f(w'r_a a) = -f(w'a) \leq 0$$

as $w'a \in \Phi^+$ and $f \in C$. Thus $f(a) = 0$; that is $a \in J$. Now by the first part of the proof we have $r_a \in S$, and so

$$fw' = (fw)r_a = fr_a = f.$$

Hence $w' \in W_J$ by induction, and it follows that $w = w'r_a \in W_J$. ∎

Proof (of 1.23) We use induction on $|\Pi|$. If $|W| < \infty$ we have nothing to prove; so suppose, without loss of generality, that W is infinite. Let $f = \sum_{a \in \Pi} \delta_a$, so that $f(a) = 1$ for all $a \in \Pi$ and thus $f(v) > 0$ for all $v \in \Phi^+$, and define $f' = \sum_{h \in H} fh$. Note that $f'h = f'$ for all $h \in H$. Define also $A = \bigcup_{h \in H} N(h)$, observing that A is finite, since H is finite and $N(h)$ is finite for all $h \in H$ (as $|N(w)| = l(w)$ for $w \in W$, by Corollary 1.10).

Let $v \in \Phi^+ \backslash A$. Then $hv \in \Phi^+$ for all $h \in H$ and so

$$f'(v) = \sum_{h \in H} (fh)(v) = \sum_{h \in H} f(hv) > 0.$$

Hence $\zeta(f') \cap (\Phi^+ \backslash A) = \emptyset$; that is $\zeta(f') \subseteq A$. In particular, $\zeta(f')$ is finite. By Lemma 1.24 there is a $w \in W$ such that $f' = f''w$ for some $f'' \in C$. Then for all $h \in H$,

$$f''whw^{-1} = (f''w)hw^{-1} = f'hw^{-1} = (f'h)w^{-1} = f'w^{-1} = f''.$$

Thus, by Lemma 1.25, $whw^{-1} \subseteq W_J$, where

$$J = \{\, a \in \Pi \mid f''(a) = 0 \,\}.$$

Now Φ^+ is infinite as W is infinite; so $\Phi^+ \backslash A \neq \emptyset$, and it follows in particular that $f' \neq 0$, since $f(v) > 0$ whenever $v \in \Phi^+ \backslash A$. Hence $f'' \neq 0$ also and thus $J \neq \Pi$. Now by induction there is $I \subset J$ and $u \in W$ such that W_I is finite and $u(whw^{-1})u^{-1} \subseteq W_I$. ∎

One immediate consequence of Lemma 1.23 is that every maximal finite subgroup of a Coxeter group is parabolic.

The following can be found as Theorem 2.7.4 of [Car85].

1.26 Lemma *(Kilmoyer) Let $I, J \subseteq \Pi$ and suppose that $d \in W$ is the minimal length element of $W_I d W_J$. Then $W_I \cap dW_J d^{-1} = W_K$, where $K = I \cap dJ$.*

1.27 Corollary *The intersection of a finite number of parabolic subgroups is a parabolic subgroup.*

Proof If H and K are parabolic subgroups of W, then we can find $w \in W$ such that $w^{-1}Hw$ is a standard parabolic subgroup of W. Using the above lemma we can see that $w^{-1}(H \cap K)w$ is a standard parabolic subgroup of W and hence $H \cap K$ is a parabolic subgroup. Induction completes the proof. ∎

1.28 Corollary *Let I, $J \subset \Pi$ and $t \in W$. Then for some $u \in W_I$,*

$$W_I \cap tW_Jt^{-1} = uW_Ku^{-1}$$

where $K = I \cap dJ \subseteq \Pi$, with d the minimal length element of W_ItW_J.

Proof We can write $t = udv$ where $u \in W_I$ and $v \in W_J$, and d is the minimal length element in W_ItW_J. By Lemma 1.26

$$W_I \cap tW_Jt^{-1} = u(W_I \cap dW_Jd^{-1})u^{-1} = uW_Ku^{-1}.$$

∎

Suppose that $I \subseteq \Pi$ is such that $|W_I|$ is finite while W_J is infinite for all J with $I \subsetneqq J \subseteq \Pi$. Then W_I is a *maximal finite standard parabolic subgroup* of W.

1.29 Lemma *Let W_I be a maximal finite standard parabolic subgroup. Then W_I is not conjugate to a subgroup of any other finite standard parabolic subgroup.*

Proof Suppose that $W_I \subseteq tW_Kt^{-1}$ for some $d \in W$ and some $K \subseteq \Pi$ such that W_K is finite. We may assume that t is of minimal length in W_ItW_K, and by Corollary 1.28 it follows that $I \subseteq tK$. Since W_I is a maximal finite standard parabolic subgroup, $t \neq 1$. So we may choose a simple root e such that $t^{-1}e = f$ is negative. As t has minimal length in tW_K it takes positive roots in the root system of W_K to positive roots. But $-f$ is a positive root while $t(-f) = -e$ is negative, and we conclude that f is not in the root system of W_K. Thus when $f = t^{-1}e$ is expressed as a linear combination of simple roots some $g \notin K$ appears with a negative coefficient. It follows that if h is any positive root in the root system of $W_{I\cup\{e\}}$ which is not in the root system of W_I, then $t^{-1}h$ involves g with a negative coefficient. But $W_{I\cup\{e\}}$ is infinite, while W_I is not. So t^{-1} takes an infinite number of positive roots to negative roots, and hence has infinite length, which is a contradiction. ∎

1.30 Corollary *If W is any infinite Coxeter group, then all maximal finite standard parabolic subgroups of W are maximal finite subgroups of W.*

Proof If W_I is a maximal finite standard parabolic subgroup but not a maximal finite subgroup then by Lemma 1.23 $W_I \leq tW_Jt^{-1}$ for some $t \in W$ and some $J \subseteq \Pi$ with $|W_I| < |W_J| < \infty$. But this contradicts Lemma 1.29. ∎

1.31 Corollary *If W is an infinite Coxeter group, H is a subgroup of W which can be written as the intersection of a finite collection of maximal finite subgroups and $\alpha \in \mathrm{Aut}(W)$, then $\alpha(H)$ is a parabolic subgroup of W.*

Proof From Corollary 1.30 if H is the intersection of a finite collection of maximal finite subgroups, then $\alpha(H)$ is the intersection of a finite collection of parabolic subgroups. The result follows from Corollary 1.27. ∎

1.32 Corollary *If W is an infinite Coxeter group, $\alpha \in \mathrm{Aut}(W)$ and r_i is a simple reflection such that $\langle r_i \rangle$ can be written as an intersection of maximal finite subgroups, then $\alpha(r_i)$ is a reflection.*

Proof By the above $\alpha\langle r_i \rangle$ is a parabolic subgroup of W of order 2. ∎

Thus, if every reflection is conjugate to a simple reflection, r_i, such that $\langle r_i \rangle$ can be written as an intersection of maximal finite subgroups, then every automorphism of W preserves reflections.

§1.4 Automorphisms That Preserve Reflections

Let W be a Coxeter group with set of simple roots Π, Coxeter diagram Γ and let α be an automorphism of W such that α preserves $\mathrm{Ref}(W)$, the set of reflections in W.

1.33 Lemma *Let W be a Coxeter group of finite rank and α an automorphism of W that preserves reflections, then*

$$\alpha\big(\mathrm{Ref}(W)\big) = \mathrm{Ref}(W),$$

and therefore α^{-1} also preserves reflections.

Proof As each reflection is conjugate to a simple reflection, by Lemma 1.1, if W has rank n, then there are at most n conjugacy classes of reflections. Say

$$\mathrm{Ref}(W) = \mathcal{C}_1 \dot\cup \mathcal{C}_2 \dot\cup \cdots \dot\cup \mathcal{C}_m.$$

If α is an automorphism of W with the property that $\alpha(r) \in \mathrm{Ref}(W)$ for all $r \in \mathrm{Ref}(W)$, then it is clear that for each $i \in \{1, 2, \ldots, m\}$ there is a $j \in \{1, 2, \ldots, m\}$ such that $\alpha(\mathcal{C}_i) = \mathcal{C}_j$. As there are only finitely many conjugacy classes of reflections α merely permutes them. ∎

Given a reflection-preserving automorphism α, define a function $\phi_\alpha : \Pi \to V$ as follows. If $a \in \Pi$, then $\alpha(r_a) = r_x$ for some $x \in \Phi \subset V$, let $\phi_\alpha(a) = x$ or $-x$ making the choice arbitrarily at present. If $a_i, a_j \in \Pi$ and the bond joining the corresponding vertices in Γ is labelled with an m, then $(r_i r_j)^m = 1$. If $\phi_\alpha(a_i) = \pm x_i$ and $\phi_\alpha(a_j) = \pm x_j$, then $r_{x_i} r_{x_j}$ also has order m and hence

$$\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) = (\pm x_i) \cdot (\pm x_j) = \mathrm{c}(l\pi_m)$$

for some l coprime to m.

1.34 Lemma *If Γ is a forest, then we can choose signs so that ϕ_α is a function such that*

$$\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) \leq 0$$

for all simple roots $a_i \neq a_j$.

Proof By induction on the rank of W. If W has rank 2, then we may choose the sign of $\phi_\alpha(a_1)$ at will, and then we can choose the sign of $\phi_\alpha(a_2)$ so that

$$\phi_\alpha(a_1) \cdot \phi_\alpha(a_2) \leq 0.$$

If W is any Coxeter group with Γ a forest, then we choose a vertex, a_i say, of degree one. Look at the (parabolic) subgroup $W_{\Pi \setminus \{a_i\}}$; by induction we may choose ϕ_α such that $\phi_\alpha(a_j) \cdot \phi_\alpha(a_k) \leq 0$ for all $a_j, a_k \neq a_i$. If a_i is joined to a_j, then we may choose the sign of $\phi_\alpha(a_i)$ so that

$$\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) \leq 0$$

without affecting any of the other inner products. ∎

Much of the time it will not be true that $\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) = a_i \cdot a_j$ even if they agree in sign. However, if $m = 2$, 3, 4 or 6, then the only numbers $l \in \{1, 2, \ldots, m-1\}$ coprime to m are $l = 1$ and $l = m - 1$. If we note that $\mathrm{c}\big((m-1)\pi_m\big) = -\mathrm{c}(\pi_m)$ then we see that $\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) = \pm\,\mathrm{c}(\pi_m)$. Hence we deduce the following:

1.35 Corollary *If W is a Coxeter group whose graph Γ is a forest with labels in the set $\{2, 3, 4, 6\}$, then we can choose ϕ_α so that*

$$\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) = a_i \cdot a_j$$

for all simple roots a_i and a_j.

We will see that if the labels are not all from the set $\{2, 3, 4, 6\}$, then B is not necessarily preserved. For Coxeter groups of types H_3 and H_4 in particular we will find automorphisms that preserve the signs of the inner products but

$$\phi_\alpha(a_1) \cdot \phi_\alpha(a_2) = -\,\mathrm{c}\,(2\pi_5) = \frac{1 - \sqrt{5}}{4}$$

instead of the original value of $-\,\mathrm{c}(\pi_5) = (-1 - \sqrt{5})/4$. In fact these are the only possibilities we need to consider if $m = 5$.

1.36 Definition From now on any label in a Coxeter diagram which does not come from the set $\{2,3,4,6\}$ will be called *unusual*. Furthermore the phrase *inner by graph* will mean that an automorphism lies in the subgroup of $\mathrm{Aut}(W)$ generated by the inner and graph automorphisms. The subgroup of inner automorphisms is a normal subgroup of $\mathrm{Aut}(W)$, therefore any inner by graph automorphism can be written as the product of an inner and a graph automorphism. Finally we will denote by $R(W)$ the subgroup of $\mathrm{Aut}(W)$ consisting of all automorphisms that preserve reflections.

Now Π is a basis for V and so ϕ_α can be extended to a linear map $V \to V$. If $\phi_\alpha(a_i)\cdot\phi_\alpha(a_j) = a_i \cdot a_j$ for all $a_i, a_j \in \Pi$, then ϕ_α is an orthogonal transformation.

1.37 Lemma *If α is an automorphism of W that preserves $\mathrm{Ref}(W)$ and ϕ_α is defined as above so that*

$$\phi_\alpha(a_i)\cdot\phi_\alpha(a_j) = a_i \cdot a_j$$

for all $a_i, a_j \in \Pi$, then $\alpha(w) = \phi_\alpha w \phi_\alpha^{-1}$ for all $w \in W$.

Proof It suffices to prove this when w is a simple reflection, since the general case then follows by a straightforward induction on $l(w)$. Now if $a \in \Pi$ and $v \in V$ we find that

$$\begin{aligned}
(\phi_\alpha r_a \phi_\alpha^{-1})(v) &= \phi_\alpha\big(r_a(\phi_\alpha^{-1}(v))\big) \\
&= \phi_\alpha\big(\phi_\alpha^{-1}(v) - 2\big(a\cdot\phi_\alpha^{-1}(v)\big)a\big) \\
&= v - 2\big(\phi_\alpha(a)\cdot v\big)\phi_\alpha(a) \\
&= r_{\phi_\alpha(a)}(v) \\
&= \alpha(r_a)(v)
\end{aligned}$$

by the definition of ϕ_α. Since this holds for all $v \in V$ we conclude that $\alpha(r_a) = \phi_\alpha r_a \phi_\alpha^{-1}$, as desired. ∎

Note If we have defined ϕ_α to satisfy the conditions of Lemma 1.37, then $-\phi_\alpha$ also satisfies those conditions.

The next result is Theorem 4.1 in [HRT97], the statement of which requires the following definition.

1.38 Definition Given a Coxeter group W with associated vector space V and bilinear form B a subset $\mathcal{P} \subset V$ is a *root basis* (relative to B) if

(*i*) for all $a, b \in \mathcal{P}$

$$\begin{aligned}
B(a,b) &= -\,\mathrm{c}\,\pi_{m_{ab}} && \text{if } m_{ab} \neq \infty \\
B(a,b) &\leq -1 && \text{if } m_{ab} = \infty
\end{aligned}$$

(*ii*) the zero vector is not contained in the set

$$\left\{ \sum_{a\in\mathcal{P}} \lambda_a a \;\middle|\; \lambda_a \geq 0 \text{ for all } a,\ \lambda_a \neq 0 \text{ for some } a \right\}.$$

In particular, notice that the set of simple roots of a Coxeter group forms a root basis which is in fact linearly independent.

1.39 Theorem *(Howlett, Rowley, Taylor [HRT97]) Let Φ_1 and Φ_2 be irreducible root systems spanning the spaces V_1 and V_2 with root bases Π_1 and Π_2 respectively. Suppose that $\varphi : V_1 \to V_2$ maps Φ_1 bijectively onto Φ_2 and takes the bilinear form on V_1 to that of V_2. Then $\Pi_2 = \pm\varphi w \Pi_1$ for some $w \in W_1$.*

To deal with the problems associated with reducible Coxeter groups we make the following definitions.

1.40 Definition If W is a Coxeter group with r_1 and r_2 any two simple reflections, then we say that there is a *chain joining* r_1 and r_2 if there is a path in the Coxeter diagram of W joining the node corresponding to r_1 to the node corresponding to r_2. Furthermore, if $r, r' \in \mathrm{Ref}(W)$, then we say that there is a *chain joining* r and r' if there are simple roots r_1 and r_2 such that r is conjugate to r_1, r' is conjugate to r_2 and there is a chain joining r_1 and r_2.

Let W be a Coxeter group of finite rank with diagram Γ, and suppose that the irreducible components of Γ are Γ_i for $1 \le i \le m$, say $\Pi = L_1 \dot\cup L_2 \dot\cup \cdots \dot\cup L_m$ is the corresponding decomposition of Π. Then, by Lemma 1.17

$$\mathrm{Ref}(W) = \mathrm{Ref}(W_{L_1}) \dot\cup \cdots \dot\cup \mathrm{Ref}(W_{L_m})$$

and the following lemma is clear.

1.41 Lemma *If $r, r' \in \mathrm{Ref}(W)$, then there is a chain joining r and r' if and only if $r, r' \in \mathrm{Ref}(W_{L_i})$ for some i.*

1.42 Lemma *Let α be an automorphism that preserves $\mathrm{Ref}(W)$ and $r, r' \in \mathrm{Ref}(W)$. Then there is a chain joining r and r' if and only if there is a chain joining $\alpha(r)$ and $\alpha(r')$.*

Proof Without loss we may assume that $r = r_1$ and $r' = r_2$ are simple reflections. Suppose that r_1 and r_2 correspond to adjacent nodes in the diagram of W; then r_1 and r_2 do not commute and hence $\alpha(r_1)$ and $\alpha(r_2)$ are reflections which do not commute. Therefore $\alpha(r_1)$ and $\alpha(r_2) \in \mathrm{Ref}(W_{L_i})$ for some i, as if they were in different components they would commute. Hence there is a chain joining $\alpha(r_1)$ and $\alpha(r_2)$. Induction on the length of the chain joining r_1 and r_2 finishes the proof of one of the implications. The reverse implication follows by applying the same argument to α^{-1}. ∎

1.43 Theorem *Let W be a finite rank Coxeter group and suppose that α is an automorphism that preserves reflections and we can define ϕ_α so that $\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) = a_i \cdot a_j$ for all i and j. Then α is inner by graph.*

Proof Using the notation developed above, $\Pi = L_1 \dot\cup \cdots \dot\cup L_m$ and

$$W = W_{L_1} \times W_{L_2} \times \cdots \times W_{L_m}.$$

Let $\alpha(r_1) \in \mathrm{Ref}(W_{L_t})$ where $a_1 \in L_1$. Then for all $a_i \in L_1$ there is a chain joining r_1 and r_i and so there is a chain joining $\alpha(r_1)$ and $\alpha(r_i)$ and hence

$$\alpha(W_{L_1}) \subseteq W_{L_t}.$$

If $r \in \mathrm{Ref}(W_{L_t}) \backslash \alpha(W_{L_1})$, then there is a chain joining r and $\alpha(r_1)$ as both lie in W_{L_t}. But $\alpha^{-1}(r) \notin W_{L_1}$ and so there is no chain joining $\alpha^{-1}(r)$ and r_1, a contradiction, by Lemma 1.42. Thus we have $\alpha(W_{L_1}) = W_{L_t}$.

Looking at $\phi_\alpha|_{\Phi_{L_1}}$ the conditions of Theorem 1.39 are satisfied and so there is a $w \in W_{L_1}$ such that

$$\Pi_{L_t} = \pm \phi_\alpha w \Pi_{L_1}.$$

Now we can change the sign of ϕ_α on L_1 without affecting the value of any inner products and can therefore ignore the possible negative. Furthermore if $L_j \neq L_1$, then $w\Pi_{L_j} = \Pi_{L_j}$ is fixed elementwise and so preceding α by conjugation by w^{-1} we may assume

$$\Pi_{L_t} = \phi_\alpha \Pi_{L_1}.$$

Repeating this for each component we find that, up to inner automorphisms

$$\phi_\alpha \Pi = \Pi.$$

Thus ϕ_α is a permutation of Π. Now if the nodes corresponding to a_i and a_j in Γ are joined by an edge labelled m, then $a_i \cdot a_j = -\,\mathrm{c}(\pi_m)$ and hence $\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) = -\,\mathrm{c}(\pi_m)$. Therefore the nodes corresponding to $\phi_\alpha(a_i)$ and $\phi_\alpha(a_j)$ must be joined by an edge labelled m. Thus ϕ_α induces an automorphism of Γ and hence a graph automorphism of W. It is easily seen that this automorphism is α which is therefore a graph automorphism. ∎

1.44 Proposition *If W is a Coxeter group whose Coxeter diagram is a forest with no unusual labels, then all automorphisms of W that preserve* $\mathrm{Ref}(W)$ *are the inner by graph automorphisms.*

Proof By Corollary 1.35 and Theorem 1.44. ∎

Chapter 2

Automorphisms of Finite Coxeter Groups

Let W be a finite irreducible Coxeter group, then W has type A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4 or $I_2(m)$. In the following α will be an automorphism of W.

2.1 Lemma *If w_0, the longest element in the finite Coxeter group W, is central and $l(w_0)$ is even, then*

$$\begin{aligned}\psi : W &\to W\\ w &\mapsto (w_0)^{l(w)}w\end{aligned}$$

defines an automorphism of W which is an outer automorphism except in the case $I_2(4l)$.

Proof If $w, w' \in W$, then by Lemma 1.4 $l(ww') = l(w) + l(w') + 2k$ for some integer k. By Lemma 1.19 w_0 is an involution and so $w_0^{2k} = 1$. So

$$\begin{aligned}\psi(ww') &= (w_0)^{l(ww')}ww'\\ &= (w_0)^{l(w)+l(w')+2k}ww'\\ &= (w_0)^{l(w)}(w_0)^{l(w')}ww'\\ &= (w_0)^{l(w)}w(w_0)^{l(w')}w'\\ &= \psi(w)\psi(w').\end{aligned}$$

If $\psi(w) = \psi(w')$, then $(w_0)^{l(w)}w = (w_0)^{l(w')}w'$, and if $l(w) + l(w')$ is even, it follows that $w = w'$. Now suppose that $l(w) + l(w')$ is odd in which case $w_0w = w'$. By Lemma 1.20

$$\begin{aligned}l(w') &= l(w_0w) = l(w_0) - l(w)\\ l(w_0) &= l(w) + l(w')\end{aligned}$$

and $l(w_0)$ is odd, contradicting the hypothesis that $l(w_0)$ is even. So ψ is injective and hence an automorphism. Note that the assumption that $l(w_0)$ is even is necessary, since otherwise we would find that

$$\alpha(w_0) = w_0^{l(w_0)}w_0 = 1,$$

whence α is not injective.

If $a \in \Phi$, then $w_0r_aw_0 = r_a$, as w_0 is central; so $w_0a = \pm a$. Since $w_0\big(\Phi^+\big) = \Phi^-$ we deduce that $w_0(a) = -a$ for all a, and hence w_0 acts as -1 on V. So $\psi(r_a) = w_0r_a = -r_a$ which has an $(n-1)$-dimensional -1-eigenspace and a 1-dimensional 1-eigenspace. Thus $\psi(r_a)$ is not a reflection unless $n = 2$. So we are finished unless W is of type $I_2(m)$.

Suppose that W is a Coxeter group of type $I_2(m)$; then $l(w_0)$ is even if and only if $m = 2k$ is even. In this case suppose the simple roots are a_1 and a_2 with corresponding simple reflections $r_1 = r_{a_1}$ and $r_2 = r_{a_2}$. The longest element is

$$w_0 = (r_1r_2)^{m/2} = (r_1r_2)^k.$$

Looking at the effect of ψ on r_1 and r_2

$$\begin{aligned}\psi(r_1) &= (w_0)^1r_1\\ &= (r_1r_2)^kr_1\\ &= (r_1r_2\cdots) \times r_i \times (r_1r_2\cdots)^{-1}\end{aligned}$$

where $r_1 r_2 \cdots$ has k terms and

$$r_i = \begin{cases} r_1 & \text{if } k \text{ is even and} \\ r_2 & \text{if } k \text{ is odd.} \end{cases}$$

Similarly $\psi(r_2) = (r_1 r_2 \cdots)' r_j (r_1 r_2 \cdots)'^{-1}$ where $(r_1 r_2 \cdots)'$ has $k-1$ terms and

$$r_j = \begin{cases} r_2 & \text{if } k \text{ is even and} \\ r_1 & \text{if } k \text{ is odd.} \end{cases}$$

Thus, if k is even,

$$\begin{aligned} \psi(r_1) &= (r_1 r_2)^{k/2} r_1 (r_1 r_2)^{-k/2} \\ \psi(r_2) &= (r_1 r_2 \cdots)' r_2 (r_1 r_2 \cdots)'^{-1} \\ &= (r_1 r_2 \cdots)' r_2 r_2 r_2 (r_1 r_2 \cdots)'^{-1} \\ &= (r_1 r_2)^{k/2} r_2 (r_1 r_2)^{-k/2} \end{aligned}$$

and ψ is conjugation by $(r_1 r_2)^{k/2}$. If k is odd a similar calculation shows that ψ is the graph automorphism that interchanges r_1 and r_2 followed by conjugation by $(r_1 r_2)^{k/2}$. Since r_1 and r_2 are not conjugate (since m is even by Lemma 3.19) this automorphism is outer. ∎

Thus ψ is an outer automorphism in the cases B_{2k}, D_{2k}, E_8, F_4, H_4 and $I_2\big(2(2l+1)\big)$.

2.2 Lemma *The automorphism ψ centralizes the inner automorphisms.*

Proof Recall that $l(w_0)$ is even. Let α be the inner automorphism that conjugates by w, and look at $\psi\alpha\psi$.

$$\begin{aligned} \psi\alpha\psi(r_i) &= \psi\alpha(w_0 r_i) \\ &= \psi\big(w w_0 r_i w^{-1}\big) \\ &= w_0^{2l(w)+1+l(w_0)} w_0 w r_i w^{-1} \\ &= w r_i w^{-1} = \alpha r_i. \end{aligned}$$

Hence $\psi\alpha\psi = \alpha$. ∎

Except for D_4 the group of graph automorphisms has order 1 or 2. When it has order 2 we denote the non-identity graph automorphism by γ.

2.3 Proposition *Let $w \in W$ be an involution, then there is an $I \subseteq \Pi$ such that w is conjugate to the longest element, w_I in the parabolic subgroup W_I. Furthermore w_I is central in W_I.*

Proof Let $L = \{\, a \in \Pi \mid wa = -a \,\}$. First observe that $\Phi_L^+ \subset N(w)$ is finite and so, by Lemma 1.11, W_L is finite. Let w_L be the longest element in W_L. Note also that if $a \in L$, then

$$w r_a w = r_{wa} = r_{-a} = r_a$$

and hence W_L centralizes w.

If $w = w_L$, then we are finished. So suppose $w_L w \neq 1$ and let $a \in N(w_L w) \cap \Pi$. If $wa \in \Phi^+$, then, as $w_L wa \in \Phi^-$ we have $wa \in N(w_L) = \Phi_L^+$, but then

$$a = w(wa) \in w\Phi_L^+ = \Phi_L^-$$

which is a contradiction. Hence $wa \in \Phi^-$. Thus $l(wr_a) = l(w) - 1$, by Lemmas 1.7 and 1.10. If $wa = -a$, then $a \in L$ and so

$$w_L wa = w_L(-a) \in w_L \Phi_L^- = \Phi_L^+$$

a contradiction, implying that $wa \neq -a$. Hence $wa = -b$ for some $b \in \Phi^+ \setminus \{\, a \,\}$, and

$$(wr_a)^{-1} a = r_a wa = r_a(-b) = -r_a b \in \Phi^-.$$

Thus $l\big(r_a(wr_a)\big) = l(wr_a) - 1$ and hence

$$l(r_a w r_a) = l(w) - 2$$

and we may proceed by induction on $l(w)$. ∎

There is, therefore, a one to one correspondence between the conjugacy classes of involutions in W and the classes of $I \subset \Pi$ such that w_I is central in W_I.

We now look at the various types of finite irreducible Coxeter groups individually.

§2.1 Type A

If W is a Coxeter group of type A_n it is well known that $W \cong \mathrm{Sym}_{n+1}$ the symmetric group on $n+1$ letters. Looking at W in this way, if w is an involution, then $w = w_k$ is a product of k disjoint 2-cycles for some $1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$. All such products are conjugate, and so

$$\begin{aligned} |\,\mathrm{cl}(w_k)| &= \#\text{products of } k \text{ disjoint 2-cycles} \\ &= \binom{n+1}{2}\binom{n-1}{2}\cdots\binom{n-2k+3}{2}\frac{1}{k!} \\ &= \binom{n+1}{2k} 1 \times 3 \times 5 \times \cdots \times (2k-1) \end{aligned}$$

where $\mathrm{cl}(w_k)$ is the conjugacy class of w_k. Under the isomorphism mentioned the simple reflections are (12), (23) and so on, up to $(n-1,n)$. These are all conjugate and so $\mathrm{cl}(w_1)$ is the only class of reflections and

$$|\,\mathrm{cl}(w_1)| = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

If $2 < 2k < n-1$, then $\binom{n+1}{2k} > \binom{n+1}{2}$ and hence $|\,\mathrm{cl}(w_k)| > |\,\mathrm{cl}(w_1)|$.

If $2k = n-1$, then $\binom{n+1}{2k} = \binom{n+1}{n-1} = \binom{n+1}{2}$, but for $k > 1$ we have $1.3.5\ldots > 1$ and again $|\,\mathrm{cl}(w_k)| > |\,\mathrm{cl}(w_1)|$.

If $2k = n$, n even, then

$$|\,\mathrm{cl}(w_k)| = \binom{n+1}{n} 1 \times 3 \times \cdots \times (n-1).$$

Thus $|\,\mathrm{cl}(w_k)| = |\,\mathrm{cl}(w_1)|$ only if $\frac{n}{2} = 1 \times 3 \times \cdots \times (n-1)$. Ignoring the case $n = 2$, where there is only one class of involutions, it is easily shown that $n/2 < 1 \times 3 \times \cdots \times (n-1)$.

If $2k = n+1$, n odd, then $|\,\mathrm{cl}(w_k)| = |\,\mathrm{cl}(w_1)|$ only if

$$\frac{n+1}{2} = 1 \times 3 \times \cdots \times (n-2).$$

For $n > 5$ it is easily shown that the right hand side is bigger, while for $n = 3$ it is smaller. If $n = 5$, then we have equality.

Hence, except possibly for $n = 5$, $k = 3$, no other class of involutions has the same size as the class of reflections. Hence, as the graph of A_n is a tree with no unusual labels all automorphisms preserve reflections and so are inner by graph, by Proposition 1.44. As we have seen earlier this graph automorphism is induced by conjugation by w_0 the longest element. Thus

$$\mathrm{Aut}(A_n) \cong W/Z(W) = W,$$

except possibly for $n = 5$.

Suppose that $W \cong \mathrm{Sym}_6$ is of type A_5. Define $\xi : W \to W$ by

$$\begin{aligned} \xi : (12) &\mapsto (13)(24)(56) = \sigma_1 \\ (23) &\mapsto (16)(25)(34) = \sigma_2 \\ (34) &\mapsto (14)(23)(56) = \sigma_3 \\ (45) &\mapsto (16)(24)(35) = \sigma_4 \\ (56) &\mapsto (12)(34)(56) = \sigma_5. \end{aligned}$$

Looking at the possible products:

$$\begin{array}{lll}
\sigma_1\sigma_2 = (145)(236) & \sigma_2\sigma_3 = (153)(264) & \sigma_3\sigma_4 = (125)(346) \\
\sigma_1\sigma_3 = (12)(34) & \sigma_2\sigma_4 = (23)(45) & \sigma_3\sigma_5 = (13)(24) \\
\sigma_1\sigma_4 = (15)(36) & \sigma_2\sigma_5 = (15)(26) & \sigma_3\sigma_6 = (154)(236) \\
\sigma_1\sigma_5 = (14)(23). & &
\end{array}$$

Hence ξ is a homomorphism. Looking now at ξ^2.

$$\begin{aligned}
\xi^2(12) &= \xi\big((13)(24)(56)\big) \\
&= \xi\big((23)(12)(23)\,(34)(23)(34)\,(56)\big) \\
&= (64)(53)(21)\,(45)(36)(21)\,(12)(34)(56) \\
&= (12).
\end{aligned}$$

Similar calculations for the other simple reflections show that $\xi^2 = 1$ and hence ξ is an outer automorphism of W. If α is any automorphism of W, then either the reflections are preserved, in which case α is inner, or $\xi\alpha$ preserves reflections and hence is inner. Hence $|\operatorname{Out}(W)| = 2$.

2.4 Proposition *If W is a Coxeter group of type A_n, then*

$$\operatorname{Aut}(W) \cong W$$

if $n \neq 5$, while

$$\operatorname{Aut}(\mathrm{Sym}_6) \cong \mathrm{Sym}_6 \rtimes \langle \xi \rangle.$$

So, for $n \neq 5$, any automorphism of a group of type A_n maps reflections to reflections, furthermore any automorphism of A_5 that does preserve reflections is inner.

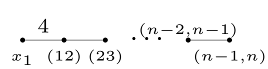
§2.2 Types B and D

In, for example, §2.10 of [Hum90] it is shown that groups of type B_n are isomorphic to $\mathcal{E}_n \rtimes \mathrm{Sym}_n$ where $\mathcal{E}_n$ is an elementary abelian 2-group, say

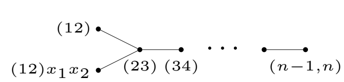
$$\mathcal{E}_n = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_n \rangle$$

where Sym_n acts to permute the x_i's. Similarly, groups of type D_n are isomorphic to $\mathcal{E}'_n \rtimes \mathrm{Sym}_n$ where $\mathcal{E}'_n$ is the subgroup of $\mathcal{E}_n$ generated by elements of the form $x_i x_j$. Following the treatment in [Hum90] the following diagrams show the association of the elements of the above groups to the simple reflections.

B_n :



D_n :



Thus elements of a group W of type B_n or D_n have the form σx for some $\sigma \in \mathrm{Sym}_n$ and $x \in \mathcal{E}^?_n$ where $\mathcal{E}^?_n$ is $\mathcal{E}_n$ or $\mathcal{E}'_n$ as appropriate. In much of the following B_n and D_n are treated simultaneously. We assume that $n > 2$, it being easy to check that for B_2 all automorphisms are inner by graph.

2.5 Lemma *Suppose W is a Coxeter group of type B_n or D_{2k} and let*

$$\theta(\sigma x) = \sigma x (w_0)^{m(\sigma)},$$

where $m(\sigma)$ is 0 if σ is even and 1 if σ is odd. Then θ is an outer automorphism of W.

Proof In all cases w_0, the longest element in W is a central involution and so θ is easily seen to be a homomorphism. Also $\theta^2 = 1$ and hence θ is an automorphism. Finally, θ does not map reflections to reflections and so is not an inner automorphism. ∎

It is worth noting $\psi = \theta$ if W is a group of type D_n (since elements of $\mathcal{E}_n'$ have even length).

2.6 Lemma *Suppose that α is an automorphism of a group of type B_n for $n \geq 3$ or D_n for $n > 4$. Then $\alpha(\mathcal{E}_n^?) = \mathcal{E}_n^?$.*

Proof Suppose that $\alpha \in \mathrm{Aut}(W)$ where W is a group of type B_n or D_n; then $\mathcal{E}_n^? \lhd W$ and hence $\alpha(\mathcal{E}_n^?) \lhd W$. Now if $x \in \mathcal{E}_n^?$, then $x^2 = 1$ and so, if $\alpha(x) = \sigma y$, then

$$1 = \alpha(x^2) = (\sigma y)^2 = \sigma^2 y^\sigma y.$$

Thus $\sigma^2 = 1$ and $y^\sigma = y$. As $\sigma^2 = 1$, σ is either 1 or a product of disjoint 2-cycles.

1. If $\sigma = (ab)$, then as $\alpha(\mathcal{E}_n^?) \lhd W$, for all $(cd) \in \mathrm{Sym}_n$ there is a $y' \in \mathcal{E}_n^?$ such that $(cd)y' \in \alpha(\mathcal{E}_n^?)$. In particular, $\alpha(\mathcal{E}_n^?)$ contains an element of the form $(bc)y'$, with $y' \in \mathcal{E}_n^?$ and $c \neq a$, and we find that

$$(ab)y(bc)y' = (ab)(bc)y^{(bc)}y' = (acb)y'' \in \alpha(\mathcal{E}_n^?).$$

But $(acb)y''$ has order a multiple of 3, which is a contradiction.

2. If $\sigma = (ab)(cd)(ef)\sigma'$ we can use the same argument with $(ac)(be)(df)\sigma'$, where σ' is 1 or a further product of disjoint 2-cycles.

3. If $\sigma = (ab)(cd)$ and $n > 4$ we can again use the same argument, with $(ae)(cd)$.

4. Finally, if W has type B_4, we may assume $\sigma = (12)(34)$. Now $y^{(12)(34)} = y$ and so we have

$$y = 1, \quad x_1x_2, \quad x_3x_4 \quad \text{or} \quad x_1x_2x_3x_4.$$

Given the normality of $\alpha(\mathcal{E}_4)$, and noting that

$$\begin{aligned}(12)(34) &= x_1\big((12)(34)x_1x_2\big)x_1\\ &= x_3\big((12)(34)x_3x_4\big)x_3\\ &= x_1x_3\big((12)(34)x_1x_2x_3x_4\big)x_1x_3.\end{aligned}$$

we deduce that $(12)(34)$ and $(12)(34)x_1x_2$ are both in $\alpha(\mathcal{E}_4)$. Hence $x_1x_2 \in \alpha(\mathcal{E}_n)$. By normality again we deduce that $\alpha(\mathcal{E}_4)$ contains all elements of the form $(ab)(cd)x_ix_j$. But since there are 18 such elements this contradicts the fact that $|\mathcal{E}_4| = 16$. ∎

For all $\sigma \in \mathrm{Sym}_n$ write

$$\alpha(\sigma) = \beta(\sigma)\delta(\sigma)$$

where $\beta(\sigma) \in \mathrm{Sym}_n$ and $\delta(\sigma) \in \mathcal{E}_n^?$. It is easy to see that β is an automorphism of Sym_n. Hence, up to inner automorphisms, except possibly when $n = 6$, we may assume that $\beta = 1$. So $\alpha(\sigma x) = \sigma\delta(\sigma)\alpha(x)$. Now

$$\begin{aligned}
\alpha(x\sigma) &= \alpha(\sigma x^\sigma) \\
\alpha(x)\alpha(\sigma) &= \sigma\delta(\sigma)\alpha(x^\sigma) \\
\alpha(x)\sigma\delta(\sigma) &= \sigma\delta(\sigma)\alpha(x^\sigma) \\
\alpha(x)^\sigma &= \alpha(x^\sigma)
\end{aligned}$$

for all $\sigma \in \mathrm{Sym}_n$ and $x \in \mathcal{E}_n^?$.

Considering type B_n, if $\alpha(x_1) = x_1$, then

$$\alpha(x_i) = \alpha(x_1^{(1i)}) = \alpha(x_1)^{(1i)} = x_i.$$

So suppose that for some $j \neq 1$:

$$\alpha(x_1) = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_j^1 \dots x_n^{\varepsilon_n}.$$

Then, for $i \neq 1$, j

$$\alpha(x_1) = \alpha(x_1^{(ij)}) = x_1^{\varepsilon_1} \dots x_j^{\varepsilon_i} \dots$$

and so $\varepsilon_i = 1$. Hence $\alpha(x_1) = x_1^{\varepsilon_1} x_2 x_3 \dots x_n$.

If $\alpha(x_1) = x_1 x_2 \dots x_n$, then $\alpha(x_2) = \alpha(x_1)^{(12)} = \alpha(x_1)$ which is a contradiction. Thus, if $\alpha \neq 1$, then $\alpha(x_1) = \hat{x_1} x_2 \dots x_n$ and hence

$$\alpha(x_i) = x_1 x_2 \dots \hat{x_i} \dots x_n,$$

where the $\hat{x_i}$ indicates that particular term is absent.

If n is odd and $\alpha|_{\mathcal{E}_n} = 1$, then

$$\begin{aligned}
\alpha(x_1 x_2 \dots x_n) &= (\hat{x_1} x_2 \dots x_n) \dots (x_1 x_2 \dots \hat{x_n}) \\
&= (x_1 x_2 \dots x_n)^{n-1} = 1
\end{aligned}$$

contradicting the injectivity of α. Thus if n is odd, $\alpha(x) = x$.

Now suppose that n is even. Then $\psi : w \mapsto w(w_0)^{l(w)}$ is an involutory automorphism of W. Now

$$x_i = (i-1, i) \dots (23)(12) x_1 (12)(23) \dots (i-1, i)$$

and from our previous discussion of the longest element w_0 we see that in fact

$$w_0 = x_1 x_2 \cdots x_n.$$

Now $l(x_i)$ is odd (since x_i is a reflection) and so

$$\psi(x_i) = x_i(x_1 \cdots x_n) = \alpha(x_i).$$

It follows that $(\psi\alpha)(x_i) = x_i$. Furthermore,

$$\begin{aligned}
\psi(\sigma) &= \psi\big(\sigma\delta(\sigma)\big) \\
&= \sigma\delta(\sigma) w_0^l
\end{aligned}$$

for some l. Thus, after replacing α by $\psi\alpha$ we still have $\beta = 1$, and we now also have $\alpha|_{\mathcal{E}_n} = 1$.

Thus, given $\alpha \in \mathrm{Aut}(W)$ where W is of type B_n, we may assume, up to inner and ψ that $\beta = 1$ and $\alpha = 1$.

Now look at the type D_n case. Using similar arguments either $\alpha(x_i x_j) = x_i x_j$ or $x_1 \dots \hat{x_i} \dots \hat{x_j} \dots x_n$. If $\alpha \neq 1$, then

$$\begin{aligned} \alpha(x_1x_2) &= x_3x_4\dots x_n \\ \alpha(x_2x_3) &= x_1x_4\dots x_n \\ \alpha(x_1x_3) &= x_2x_4\dots x_n. \end{aligned}$$

But then $\alpha(x_1x_3) \neq \alpha(x_1x_2)\alpha(x_2x_3)$ which is a contradiction. Hence $\alpha = 1$ in all cases for groups of type D_n.

Therefore, up to ψ in the B_{2k} case, we may assume that

$$\alpha(\sigma x) = \sigma\delta(\sigma)x.$$

2.7 Lemma *Suppose that $\alpha \in \mathrm{Aut}(W)$, where W is of type B_n or D_n, such that $\alpha(\sigma x) = \sigma\delta(\sigma)x$ as above. Then $\alpha = \theta$ or the action of δ on the simple reflections in Sym_n is either*

$$\delta(12) = x_1x_2 \qquad \textit{and} \qquad \delta(23) = \delta(34) = \dots = 1$$

or

$$\delta(12) = x_1x_2w_0 \qquad \textit{and} \qquad \delta(23) = \delta(34) = \dots = w_0.$$

Proof Suppose $y \in \mathcal{E}_n^?$, let $\alpha^y = i_y\alpha$, where i_y is the inner automorphism of W induced by y, and write $\alpha^y(\sigma x) = \beta'(\sigma)\delta'(\sigma)\alpha'(x)$. Then

$$\begin{aligned} \beta'(\sigma)\delta'(\sigma)\alpha'(x) &= y\alpha(\sigma x)y \\ &= \sigma\delta(\sigma)y^\sigma yx. \end{aligned}$$

Thus $\beta' = 1$, $\alpha' = 1$ and $\delta' = \delta y^\sigma y$. Following automorphisms with conjugation by elements of $\mathcal{E}_n^?$ only affects δ. Also note that as $\alpha(1) = 1$ we have $\delta(1) = 1$. Now

$$\begin{aligned} \alpha(\sigma\tau) &= \alpha(\sigma)\alpha(\tau) \\ \sigma\tau\delta(\sigma\tau) &= \sigma\delta(\sigma)\tau\delta(\tau) \\ \delta(\sigma\tau) &= \delta(\sigma)^\tau\delta(\tau). \end{aligned}$$

In particular if σ is an involution.

$$\begin{aligned} 1 = \delta(\sigma^2) &= \delta(\sigma)^\sigma\delta(\sigma) \\ \delta(\sigma)^\sigma &= \delta(\sigma). \end{aligned}$$

Similarly for any 3-cycle (abc), $1 = \delta\big((abc)^3\big) = \delta(abc)^{(abc)^2}\delta(abc)^{(abc)}\delta(abc)$. Thus if

$$\delta(abc) = \prod x_i^{\varepsilon_i},$$

then

$$1 = (x_ax_bx_c)^{\varepsilon_a+\varepsilon_b+\varepsilon_c} \prod_{i \notin \{a,b,c\}} x_i^{3\varepsilon_i}$$

and so $\varepsilon_a + \varepsilon_b + \varepsilon_c \equiv 0 \pmod 2$ and $\varepsilon_i = 0$ otherwise. Hence

$$\delta(abc) = 1, \qquad x_ax_b, \qquad x_ax_c \qquad \text{or} \qquad x_bx_c.$$

Suppose that $\delta \neq 1$. We first look at the case where $\delta(abc) \neq 1$ for some 3-cycle (abc), without loss we may assume that $\delta(123) \neq 1$, and conjugating by x_2 or x_3 as necessary (x_2x_4 or x_3x_4 in D_n) we may assume that

$$\delta(123) = x_1x_2.$$

If there is an $i > 3$ such that $\delta(i23) \neq 1$, then conjugating by x_i (or x_ix_4, or x_ix_5 for D_n, $n \geq 5$) we may assume

$$\begin{aligned}
\delta(123) &= x_1x_2 \\
\delta(i23) &= x_2x_3 \\
\text{Now} \qquad (23)(12) &= (123) \\
\delta(23)^{(12)}\delta(12) &= x_1x_2 \\
\text{Conjugating by } (12) \qquad \delta(23)\delta(12) &= x_1x_2. \\
\text{Similarly} \qquad \delta(23)\delta(i2) &= x_3x_i \\
\text{multiplying the last two} \qquad \delta(12)\delta(i2) &= x_1x_2x_3x_i \\
\text{conjugating by } (i2) \qquad \delta(12)^{(i2)}\delta(i2) &= x_1x_2x_3x_i \\
\delta(1i2) &= x_1x_2x_3x_i.
\end{aligned}$$

This is a contradiction as $\delta(1i2) = 1$, x_1x_2, x_1x_i or x_2x_i. Thus $\delta(i23) = 1$ for all $i > 3$.

Now $(23)(i2) = (i23)$ and so

$$\delta(23)^{(i2)}\delta(i2) = \delta(i23) = 1.$$

Conjugating by $(i2)$ we find that $\delta(23) = \delta(2i)$ for all $i > 3$. For $i \geq 3$ we have

$$\delta(23) = \delta(2i) = \delta(2i)^{(2i)} = \delta(23)^{(2i)}$$

and hence $\delta(23) = 1$, x_1, $x_2 \ldots x_n$ or $x_1x_2 \ldots x_n$. From earlier $\delta(12)\delta(23) = x_1x_2$ and so

$$\delta(12) = x_1x_2, \qquad x_2, \qquad x_1\hat{x}_2 \ldots x_n \qquad \text{or} \qquad x_3x_4 \ldots x_n.$$

We know that $\delta(12)^{(12)} = \delta(12)$ and so we can only have the first or last alternatives. If $\delta(12) = x_1x_2$ then $\delta(23) = 1$, while if $\delta(12) = x_3 \cdots x_n$ then $\delta(23) = x_1x_2 \cdots x_n$. Hence

$$\begin{aligned}
\delta(12) = x_1x_2 \quad &\text{and} \quad \delta(23) = \delta(24) = \cdots = 1, \quad \text{or} \\
\delta(12) = x_3 \cdots x_n \quad &\text{and} \quad \delta(23) = \delta(24) = \cdots = x_1x_2 \cdots x_n
\end{aligned}$$

(Note that the second case cannot occur in type D_{2k+1}.)

If i, $j \geq 3$, then $\delta(ij) = \delta\big((2i)(2j)(2i)\big) = \delta(2i)^{(2j)(2i)}\delta(2j)^{(2i)}\delta(2i)$ and so

$$\delta(ij) = 1 \qquad \text{or} \qquad x_1 \ldots x_n$$

respectively. Similarly $\delta(1i) = \delta\big((2i)(21)(2i)\big)$ and so

$$\delta(1i) = x_1x_i \qquad \text{or} \qquad \hat{x_1}x_2 \ldots \hat{x_i} \ldots x_n.$$

Thus there are two possibilities for δ.

$$\delta(12) = x_1x_2 \qquad \text{and} \qquad \delta(23) = \delta(34) = \cdots = 1$$

or

$$\delta(12) = x_1x_2w_0 \qquad \text{and} \qquad \delta(23) = \delta(34) = \cdots = w_0$$

Now suppose that $\delta \neq 1$ but $\delta(abc) = 1$ for all 3-cycles (abc). Then without loss we may assume that $\delta(12) \neq 1$. Suppose $\delta(12) = (12)y$ for some $1 \neq y \in E_n^?$. If $\delta(2i) = (2i)y'$, then

$$\alpha\big((12)(2i)\big) = \alpha(12i) = (12i) = (12)(2i).$$

But then

$$\begin{aligned}(12)(2i) &= \alpha\big((12)(2i)\big)\\ &= \alpha(12)\alpha(2i)\\ &= (12)y(2i)y'\\ &= (12i)y^{(2i)}y'\end{aligned}$$

whence $y^{(2i)}y' = 1$. From earlier we know that $\delta(2i)^{(2i)} = \delta(2i)$ and hence $y'^{(2i)} = y'$. Therefore $(yy')^{(2i)} = 1$ and we deduce $y = y'$. Repeating this we may show that $\delta(ij) = y$ for all i and j. But this implies that $y^{(ij)} = y$ for all i and j, together with the fact that $y \neq 1$ we have shown that $y = w_0$. Noting that this is not possible in the D_{2k+1} case. In the other cases is it easily shown that

$$\alpha(\sigma x) = \sigma x w_0^{m(\sigma)} = \theta(\sigma x).$$

∎

In view of this lemma if α, or α followed by θ, is not inner we may assume that $\delta(12) = x_1x_2$ while $\delta(i, i+1) = 1$ for $i \geq 2$. It can be seen that the automorphism we are left with is conjugation by the element x_1. In type B_n this is clearly an inner automorphism.

In type D_{2k} we find

$$\begin{aligned}\alpha(12) &= (12)x_1x_2\\ \alpha\big((12)x_1x_2\big) &= (12)x_1x_2x_1x_2 = (12)\\ \alpha(i, i+1) &= (i, i+1)\end{aligned}$$

and so α is the graph automorphism.

If this is an inner automorphism, then n is even and we can find σy such that

$$(i, i+1)^{\sigma y} = (i, i+1)$$

for all i. Hence $\sigma = 1$. Now $(i, i+1)^y = (i, i+1)$ tells us that

$$y = x_1^{\varepsilon_1}(x_2 \dots x_n)^{\varepsilon_2}.$$

But $y^{(12)}y = x_1x_2$ implies that $\varepsilon_1 + \varepsilon_2 = 1$ and hence

$$y = x_1 \qquad \text{or} \qquad x_2 \dots x_n$$

neither of which lie in D_{2k}. Thus θ is not an inner automorphism.

Hence, up to inner automorphisms we have:

If W has type B_{2k+1}, then $\alpha \in \langle\theta\rangle$.
If W has type B_{2k}, then $\alpha \in \langle\psi\rangle \times \langle\theta\rangle$.
If W has type D_{2k+1}, then $\alpha = 1$.
If W has type D_{2k}, then $\alpha \in \langle\psi\rangle \times \langle\delta\rangle$.

(It is easily established that ψ commutes with θ and the graph automorphisms as appropriate.) It only remains to deal with the cases of $n = 6$ and $\beta \neq 1$ and D_4.

Suppose that $n = 6$ and β is an outer automorphism of Sym_6. Without loss we may assume that

$$\begin{aligned} \beta : (12) &\mapsto (13)(24)(56) \\ (23) &\mapsto (16)(25)(34) \\ (34) &\mapsto (14)(23)(56) \\ (45) &\mapsto (16)(24)(35) \\ (56) &\mapsto (12)(34)(56). \end{aligned}$$

Now x_5x_6 is not central in either B_6 or D_6 and so $\alpha(x_5x_6)$ cannot be central. Suppose $\alpha(x_5x_6) = \prod x_i^{\varepsilon_i}$; then

$$\begin{aligned} \alpha(x_5x_6(12)) &= \alpha((12)x_5x_6) \\ \beta(12)\delta(12)\alpha(x_5x_6)^{\beta(12)} &= \beta(12)\delta(12)\alpha(x_5x_6) \\ \left(\prod x_i^{\varepsilon_i}\right)^{(13)(24)(56)} &= \prod x_i^{\varepsilon_i}. \end{aligned}$$

Thus $\varepsilon_1 = \varepsilon_3$, $\varepsilon_2 = \varepsilon_4$ and $\varepsilon_5 = \varepsilon_6$. A similar argument with $(23)x_5x_6$ shows that $\varepsilon_1 = \varepsilon_6$, $\varepsilon_2 = \varepsilon_5$ and $\varepsilon_3 = \varepsilon_4$. Hence

$$\varepsilon_1 = \varepsilon_6 = \varepsilon_5 = \varepsilon_2 = \varepsilon_4 = \varepsilon_3$$

and $\alpha(x_5x_6) = w_0^{\varepsilon_1}$ is central, a contradiction. Thus we have no new automorphisms.

Finally suppose that W is of type D_4. Looking for parabolic subgroups with longest element central, to find the classes of involutions, we discover 6 possibilities:

$$\begin{aligned} &\langle r_1\rangle && \text{longest element: } r_1 \\ &\langle r_1, r_2\rangle && \text{longest element: } r_1r_2 \\ &\langle r_1, r_4\rangle && \text{longest element: } r_1r_4 \\ &\langle r_2, r_4\rangle && \text{longest element: } r_2r_4 \\ &\langle r_1, r_2, r_4\rangle && \text{longest element: } r_1r_2r_4 \\ &W && \text{longest element: } w_0. \end{aligned}$$

To see that r_1r_2, r_1r_4 and r_2r_4 are not conjugate observe that in the notation used above

$$\begin{aligned} r_1r_2 &= x_1x_2 \\ r_1r_4 &= (12)(34) \end{aligned}$$

are clearly not conjugate and, by the symmetry of the diagram, r_2r_4 must also belong to a separate class. There are many places to find the sizes of these conjugacy classes, for example [Car72]. We find:

$$\begin{aligned} |\operatorname{cl}(r_1)| &= 12 \\ |\operatorname{cl}(r_1r_2)| &= 6 \\ |\operatorname{cl}(r_1r_4)| &= 6 \\ |\operatorname{cl}(r_2r_4)| &= 6 \\ |\operatorname{cl}(r_1r_2r_3)| &= 12 \\ |\operatorname{cl}(w_0)| &= 1 \end{aligned}$$

The automorphism ψ does not preserve reflections and so $\operatorname{cl}(r_1w_0) \neq \operatorname{cl}(r_1)$. This implies $\operatorname{cl}(r_1w_0) = \operatorname{cl}(r_1r_2r_3)$. If α is any automorphism of W, then we may assume, up to ψ, that α preserves reflections. Thus, by Proposition 1.44, α is inner by graph. We have seen above that the graph automorphisms of order 2 are outer. Suppose that $\alpha : r_1 \mapsto r_2$, $r_2 \mapsto r_4$, $r_3 \mapsto r_3$ and $r_4 \mapsto r_1$ is inner. Then $(12)^{\sigma y} = (34)$ and $\big((12)x_1x_2\big)^{\sigma y} = (12)$, and therefore

$$(12) = (12)^{\sigma} = (34)$$

is a contradiction and all the graph automorphisms are outer automorphisms. Hence

$$\operatorname{Aut}(W) \cong (\operatorname{Inn}(W) \rtimes \mathrm{Sym}_3) \times \langle\psi\rangle = (W/\langle w_0\rangle \rtimes \mathrm{Sym}_3) \times \langle\psi\rangle.$$

The following proposition summarizes what we have shown.

2.8 Proposition

(a) *If W is a group of type B_n for n odd, then*

$$\mathrm{Aut}(W) \cong (W/\langle w_0 \rangle) \rtimes \langle \theta \rangle.$$

(b) *If W is a group of type B_n for n even, then*

$$\mathrm{Aut}(W) = \big((W/\langle w_0 \rangle) \rtimes \langle \theta \rangle\big) \times \langle \psi \rangle.$$

(c) *If W is a group of type D_n for n odd, then*

$$\mathrm{Aut}(W) \cong W.$$

(d) *If W is a group of type D_n for n even $n > 4$, then*

$$\mathrm{Aut}(W) \cong \big((W/\langle w_0 \rangle) \rtimes \langle \gamma \rangle\big) \times \langle \psi \rangle,$$

where γ is the graph automorphism.

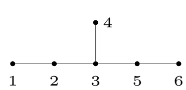
(e) *If W is a group of type D_4, then*

$$\mathrm{Aut}(W) \cong (W/\langle w_0 \rangle \rtimes \mathrm{Sym}_3) \times \langle \psi \rangle.$$

In particular all automorphisms of D_{2k+1} map reflections to reflections. All automorphisms of B_{2k+1} map r_1 to a reflection while any automorphism of B_n that does preserve reflections must be inner.

§2.3 Type E

Suppose that W is of type E_6 with diagram



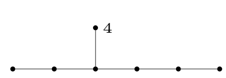
The following table lists representatives of the conjugacy classes of parabolic subgroups with longest element central, and the sizes of the corresponding classes of involutions (see [Car72]).

W_I	Type	$\lvert \mathrm{cl}(w_I) \rvert$
$\langle r_1 \rangle$	A_1	36
$\langle r_1, r_3 \rangle$	$A_1 \times A_1$	270
$\langle r_1, r_3, r_6 \rangle$	$A_1 \times A_1 \times A_1$	540
$\langle r_2, r_3, r_4, r_5 \rangle$	D_4	45.

Thus any automorphism must preserve reflections and so is inner by graph. We have seen that the graph automorphism is inner, being conjugation by w_0, and so

$$\mathrm{Aut}(W) \cong W.$$

Suppose that W is of type E_7 with diagram



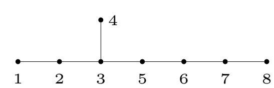
The following table lists representatives of the conjugacy classes of parabolic subgroups with longest element central, and the sizes of the corresponding classes of involutions.

W_I	Type	$\lvert \mathrm{cl}(w_I) \rvert$
$\langle r_1 \rangle$	A_1	63
$\langle r_1, r_3 \rangle$	$A_1 \times A_1$	945
$\langle r_2, r_5, r_7 \rangle$	$A_1 \times A_1 \times A_1$	3780
$\langle r_4, r_5, r_7 \rangle$	$A_1 \times A_1 \times A_1$	315
$\langle r_2, r_4, r_5, r_7 \rangle$	$A_1 \times A_1 \times A_1 \times A_1$	3780
$\langle r_2, r_3, r_4, r_5 \rangle$	D_4	315
$\langle r_2, r_3, r_4, r_5, r_7 \rangle$	$A_1 \times D_4$	945
$\langle r_2, r_3, r_4, r_5, r_6, r_7 \rangle$	D_6	63
W	E_7	1

If r is a reflection, then rw_0 is an involution and so the other class of size 63 must be $w_0 \,\mathrm{cl}(r_1) = \mathrm{cl}(r_1 w_0)$. Now $l(w_0) = 63$ and so $l(r_1 w_0) = 62$ is even. Therefore $\langle \mathrm{cl}(r_1 w_0) \rangle \neq W$ as all the elements on the left have even length. Thus all automorphisms of W preserve reflections and hence are inner. W has no graph automorphisms.

$$\mathrm{Aut}(W) \cong W/\langle w_0 \rangle.$$

Finally suppose that W is of type E_8 with diagram



The classes of involutions are as follows:

W_I	Type	$\lvert \mathrm{cl}(w_I) \rvert$
$\langle r_1 \rangle$	A_1	120
$\langle r_1, r_3 \rangle$	$A_1 \times A_1$	3780
$\langle r_1, r_3, r_6 \rangle$	$A_1 \times A_1 \times A_1$	37800
$\langle r_1, r_3, r_6, r_8 \rangle$	$A_1 \times A_1 \times A_1 \times A_1$	113400
$\langle r_2, r_3, r_4, r_5 \rangle$	D_4	3150
$\langle r_2, r_3, r_4, r_5, r_7 \rangle$	$D_4 \times A_1$	37800
$\langle r_2, r_3, r_4, r_5, r_6, r_7 \rangle$	D_6	3780
$\langle r_1, r_2, r_3, r_4, r_5, r_6, r_7 \rangle$	E_7	120
W	E_8	1

In this case ψ is an automorphism that must interchange the classes of size 120. Thus, up to ψ, automorphisms preserve reflections and hence are inner, by Proposition 1.44.

$$\mathrm{Aut}(W) \cong (W/\langle w_0 \rangle) \rtimes \langle \psi \rangle.$$

The following proposition summarizes what we have shown.

2.9 Proposition

(a) *If W is a group of type E_6, then*

$$\mathrm{Aut}(W) \cong W.$$

(b) *If W is a group of type E_7, then*

$$\mathrm{Aut}(W) \cong W/\langle w_0 \rangle.$$

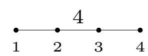
(c) *If W is a group of type E_8, then*

$$\mathrm{Aut}(W) \cong (W/\langle w_0 \rangle) \rtimes \langle \psi \rangle.$$

Note that all automorphisms of groups of type E_6 or E_7 are inner.

§2.4 Type F

There is of course only one finite group of this type. Suppose that W is of type F_4 with diagram



By Lemmas 3.18 and 3.19 it can be seen that there are two classes of reflections with representatives r_1 and r_4. In particular this proves that the graph automorphism is outer. The classes of involutions are as follows:

W_I	Type	$\lvert \mathrm{cl}(w_I) \rvert$
$I_1 = \langle r_1 \rangle$	A_1	12
$I_2 = \langle r_4 \rangle$	A_1	12
$I_3 = \langle r_2, r_3 \rangle$	$I_2(4)$	18
$I_4 = \langle r_1, r_4 \rangle$	$A_1 \times A_1$	72
$I_5 = \langle r_1, r_2, r_3 \rangle$	B_3	12
$I_6 = \langle r_2, r_3, r_4 \rangle$	B_3	12
$I_7 = W$	F_4	1

Define by $\psi_l : r_1 \mapsto r_1 w_0$, $r_2 \mapsto r_2 w_0$, $r_3 \mapsto r_3$ and $r_4 \mapsto r_4$. Similarly define ψ_r by $r_1 \mapsto r_1$, $r_2 \mapsto r_2$, $r_3 \mapsto r_3 w_0$ and $r_4 \mapsto r_4 w_0$. It is clear that ψ_l and ψ_r are automorphisms of W that are outer as they do not map reflections to reflections. It can be seen that:

(*i*) γ interchanges I_1 and I_2 and interchanges I_5 and I_6.

(*ii*) ψ_l fixes I_2 and I_5 while interchanging I_1 and I_6.

(*iii*) ψ_r fixes I_1 and I_6 while interchanging I_2 and I_5.

Let $\mathcal{C}_j$ be the conjugacy class containing the central element of W_{I_j}. Then $\mathcal{C}_1 \cup \mathcal{C}_2$ generates W, but it can be checked that $\mathcal{C}_1 \cup \mathcal{C}_6$ does not. Indeed both $\mathcal{C}_1$ and $\mathcal{C}_6$ are contained in the kernel of the homomorphism $W \to \{\pm 1\}$ given by $r_1, r_2 \mapsto 1$ and $r_3, r_4 \mapsto -1$.

So no automorphism maps $\{\mathcal{C}_1, \mathcal{C}_2\} \mapsto \{\mathcal{C}_1, \mathcal{C}_6\}$. Similarly, there is no automorphism that maps this set to $\{\mathcal{C}_2, \mathcal{C}_5\}$. So the possible targets for $(\mathcal{C}_1, \mathcal{C}_2)$ are:

$$
\begin{aligned}
(\mathcal{C}_1, \mathcal{C}_2) \mapsto & (\mathcal{C}_1, \mathcal{C}_2) && i \\
& (\mathcal{C}_1, \mathcal{C}_5) && \psi_r = \gamma\psi_l\gamma \\
& (\mathcal{C}_2, \mathcal{C}_1) && \gamma \\
& (\mathcal{C}_2, \mathcal{C}_6) && \psi_l\gamma \\
& (\mathcal{C}_5, \mathcal{C}_1) && \gamma\psi_l \\
& (\mathcal{C}_5, \mathcal{C}_6) && \psi_l\psi_r\gamma = \psi_l\gamma\psi_l \\
& (\mathcal{C}_6, \mathcal{C}_2) && \psi_l \\
& (\mathcal{C}_6, \mathcal{C}_5) && \psi_l\psi_r = \psi = \psi_l\gamma\psi_l\gamma.
\end{aligned}
$$

If automorphisms α_1 and α_2 have the property that $\alpha_1\big(\{\mathcal{C}_1, \mathcal{C}_2\}\big) = \alpha_2\big(\{\mathcal{C}_1, \mathcal{C}_2\}\big)$ then $\alpha_2^{-1}\alpha_1$ preserves reflections, and so is inner by graph (by Proposition 1.44). So, modulo inner automorphisms the above 8 possibilities are the only ones. Thus $\mathrm{Out}(W) = \langle \gamma, \psi_l \rangle$ is dihedral of order 8. The following proposition summarizes what we have shown.

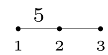
2.10 Proposition

$$\mathrm{Aut}(W) \cong W/\langle w_0 \rangle \rtimes \langle \gamma, \psi_l \rangle$$

where $\langle \gamma, \psi_l \rangle$ is a dihedral group of order 8.

§2.5 Type H

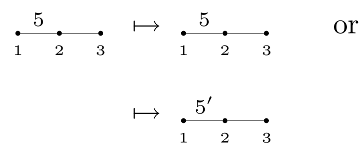
Suppose W is of type H_3 with diagram



The classes of involutions are as follows:

W_I	Type	$\lvert \mathrm{cl}(w_I) \rvert$
$\langle r_1 \rangle$	A_1	15
$\langle r_1, r_3 \rangle$	$A_1 \times A_1$	15
W	H_3	1

Now the elements in $\mathrm{cl}(r_1 r_3)$ all have even length and so all automorphisms map reflections to reflections. Unfortunately the graph of W contains an edge labelled with a 5 and so, up to inner automorphisms, there are two possibilities:



where in the second case $a_1' \cdot a_2' = -\,\mathrm{c}(2\pi_5) = -m/2$ where $m = (\sqrt{5}-1)/2$, recalling that $a_1 \cdot a_2 = -l/2$ where $l = (\sqrt{5}+1)/2$.

Define $\xi : W \to W$ by

$$\begin{aligned} \xi : r_1 &\mapsto r_{a_1'} = r_2 r_1 r_3 r_2 r_3 r_1 r_2 r_1 r_3 r_2 r_3 r_1 r_2 \\ r_2 &\mapsto r_2 \\ r_3 &\mapsto r_3 \end{aligned}$$

where $a_1' = r_2 r_1 r_3 r_2 r_3 r_1 a_2 = -(l+1)a_1 - 2la_2 - la_3$. Then $a_1' \cdot a_2 = -m/2 = -\,\mathrm{c}(2\pi_5)$ and so $r_{a_1'} r_2$ has order 5. Furthermore, $a_1' \cdot a_3 = 0$, and so $r_{a_1'} r_3$ has order 2. Thus ξ is a homomorphism. It is easily checked that $\xi^2 = 1$ and therefore ξ is an outer automorphism of W.

Now suppose that α is an outer automorphism of W. Then up to inner automorphisms we may assume

$$\begin{aligned} \alpha : r_1 &\mapsto r_\beta \\ r_2 &\mapsto r_\gamma \\ r_3 &\mapsto r_\delta \end{aligned}$$

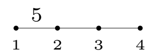
where $\beta \cdot \gamma = -m/2$, $\beta \cdot \delta = 0$ and $\gamma \cdot \delta = -1/2$. Then

$$\begin{aligned} \alpha\xi : r_1 &\mapsto r_{r_\gamma r_\beta r_\delta r_\gamma r_\delta r_\beta \gamma} = r_{\beta'} \\ r_2 &\mapsto r_\gamma \\ r_3 &\mapsto r_\delta \end{aligned}$$

where $\beta' = (m-1)\beta - 2m\gamma - m\delta$. Now $\beta' \cdot \gamma = -l/2$, $\beta' \cdot \delta = 0$ and $\gamma \cdot \delta = -1/2$ and hence $\alpha\xi$ is inner. Hence

$$\mathrm{Aut}(W) \cong W/\langle w_0 \rangle \rtimes \langle \xi \rangle.$$

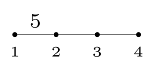
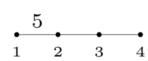
Now suppose that W is of type H_4 with diagram

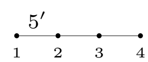


The classes of involutions are as follows:

W_I	Type	$\lvert \mathrm{cl}(w_I) \rvert$
$\langle r_1 \rangle$	A_1	60
$\langle r_1, r_3 \rangle$	$A_1 \times A_1$	450
$\langle r_1, r_2, r_3 \rangle$	H_3	60
W	H_4	1

In this case $\psi : x \mapsto x(w_0)^{l(x)}$ is an automorphism that interchanges the two classes of size 60 and so, up to ψ, we may assume that an automorphism α maps reflections to reflections. Thus, up to inner automorphisms we have

 $\mapsto$  or

$\mapsto$ 

As in the type H_3 case there is an automorphism ξ such that

$$\begin{aligned} \xi : r_1 &\mapsto r_{a_1'} \\ r_2 &\mapsto r_2 \\ r_3 &\mapsto r_3 \\ r_4 &\mapsto r_4 \end{aligned}$$

where

$$\begin{aligned} a_1' &= r_1 r_2 r_3 r_4 r_1 r_2 r_3 r_1 r_2 r_3 r_4 r_1 r_2 r_3 r_4 r_1 r_2 r_3 r_1 a_2 \\ &= -(3l+2)a_1 - (3l+3)a_2 - 2(l+1)a_3 - (l+1)a_4. \end{aligned}$$

A similar, but much longer, calculation shows that $\alpha\xi$ is inner and hence

$$\mathrm{Aut}(W) \cong W/\langle w_0 \rangle \rtimes \langle \xi, \psi \rangle = \big(W/\langle w_0 \rangle \rtimes \langle \xi \rangle\big) \times \langle \psi \rangle.$$

The following proposition summarizes what we have shown.

2.11 Proposition

(a) *If W is a group of type H_3, then*

$$\mathrm{Aut}(W) \cong W/\langle w_0 \rangle \rtimes \langle \xi \rangle.$$

(b) *If W is a group of type H_4, then*

$$\mathrm{Aut}(W) \cong \big(W/\langle w_0 \rangle \rtimes \langle \xi \rangle\big) \times \langle \psi \rangle.$$

Any automorphism of a group of type H_3 maps reflections to reflections.

§2.6 **The Dihedral Groups**

Suppose that W is of type $I_2(m)$ with diagram

$$\underset{1}{\bullet}\overset{m}{\text{———}}\underset{2}{\bullet}$$

It is easily checked that the roots in Φ have the form 1.12, namely:

$$v_h = \frac{\mathrm{s}\big((h-1)\pi_m\big)}{\mathrm{s}(\pi_m)}a_1 + \frac{\mathrm{s}(h\pi_m)}{\mathrm{s}(\pi_m)}a_2$$

where $\pi_m = \pi/m$ and $0 \le h < 2m$. Observe that $v_0 = -a_1$, $v_1 = a_2$ and $v_{h+m} = -v_h$ for all h. It is readily checked that $v_h \cdot v_0 = \mathrm{c}(h\pi_m)$, and the permutation of the root system defined by $v_h \mapsto v_{h+1}$ corresponds to a rotation through π_m.

Since the reflections are precisely the non-central involutions in W it is trivial that all automorphisms map reflections to reflections. Of course, an automorphism is completely determined by its effect on r_1 and r_2. Clearly there is an automorphism that takes r_{v_h} to $r_{v_{h+1}}$ for all h; we denote this automorphism by α. Since v_h and v_{h+m} correspond to the same reflection, α has order m. Note also that the group generated by α acts transitively on the set of reflections in W.

2.12 Notation Let $ab\overset{n}{\cdots}$ denote the product of the first n terms of the alternating sequence $a, b, a, b, \ldots$.

Every reflection in W can be expressed in the form $r_2r_1\overset{n}{\cdots}$ for some odd integer n: indeed, $n = 2h-1$ gives the reflection along v_h. It follows that $r_1r_{v_h} = (r_1r_2)^h$, which has order m if and only if $\gcd(h,m) = 1$. It follows that there is an automorphism α_h such that $r_1 \mapsto r_1$ and $r_2 \mapsto r_{v_h}$ whenever $\gcd(h,m) = 1$, and these are the only automorphisms that fix r_1. So if we define $\mathrm{Aut}_1(W) = \{\, \alpha_h \mid \gcd(h,m) = 1 \,\}$, then it follows that $\mathrm{Aut}(W) = \mathrm{Aut}_1(W)\langle\alpha\rangle$.

Observe that if $\gcd(h,m) = 1$ and $\gcd(k,m) = 1$ then

$$(\alpha_h\alpha_k)(r_2) = \alpha_h(r_2r_1\overset{2k-1}{\cdots\cdots}) = \alpha_h(r_2)r_1\overset{2k-1}{\cdots\cdots} = r_2r_1\overset{2hk-1}{\cdots\cdots} = \alpha_{hk}(r_2).$$

Obviously also $(\alpha_h\alpha_k)(r_1) = r_1$, and so it follows that $\alpha_h\alpha_k = \alpha_{hk}$. So $\mathrm{Aut}_1(W)$ is isomorphic to the group of units of the ring of integers modulo m. Furthermore, it is easily verified that $\alpha_h\alpha = \alpha^h\alpha_h$, this being the automorphism that takes the reflection along v_k to the reflection along v_{hk+h} for each k. So conjugation by α_h acts on the cyclic group of order m generated by α by raising elements to the power h. Thus $\mathrm{Aut}(W)$ is the holomorph of the cyclic group of order m.

Note that α_{m-1} fixes r_1 and takes r_2 to $r_2r_1\overset{2m-3}{\cdots\cdots} = r_1r_2r_1$; hence α_{m-1} is conjugation by r_1. If γ is the nontrivial graph automorphism then $\gamma\alpha_{m-1}$ takes $r_{v_0} = r_1$ to $r_2 = r_{v_1}$ and r_2 to $r_2r_1r_2 = r_{v_3}$. Thus $\gamma\alpha_{m-1} = \alpha$. It follows that $\langle\alpha, \alpha_{m-1}\rangle = \langle\gamma, \alpha_{m-1}\rangle$ is the group of all automorphisms that are inner by graph.

If m is even then the graph automorphism γ is outer, since it interchanges the two classes of reflections. If m is odd then γ is inner, being conjugation by w_0. The following proposition summarizes what we have shown.

2.13 Proposition *Suppose that W is a group of type $I_2(m)$ and use the notation from above. Then all automorphisms of W preserve reflections,* $\mathrm{Aut}(W) = \mathrm{Aut}_1(W)\langle\alpha\rangle$ *is the holomorph of the cyclic group of order m and* $\mathrm{Aut}_1(W)$ *is isomorphic to the group of units of the ring of integers modulo m.*

§2.7 Automorphisms of Infinite Coxeter Groups

In the finite case the subgroups of inner automorphisms and graph automorphisms are not always disjoint. We have seen that in the cases A_n, D_{2k+1}, $I_2(2k+1)$ and E_6 the non-trivial graph automorphism is the same as conjugation by the longest element of W. This cannot happen if W is an infinite irreducible Coxeter group; in fact we can say slightly more.

2.14 Lemma *If W is any infinite Coxeter group with no finite irreducible components, then the only graph automorphism that is inner is the identity.*

Proof Suppose that conjugation by $w \in W$ is a graph automorphism. If W is not irreducible then

$$W = W_{L_1} \times \cdots \times W_{L_m}$$

where $\Pi = L_1 \dot{\cup} L_2 \dot{\cup} \cdots \dot{\cup} L_M$. As the automorphism is inner it is clear that each component is fixed and, by looking at the restriction to W_{L_i}, we may assume that Π is connected. That is, we now assume that W is irreducible. Now let

$$J = \{\, a \in \Pi \mid wa \in \Phi^- \,\}.$$

If $J = \emptyset$ then $N(w) = \emptyset$, whence $l(w) = 0$ by Corollary 1.10, giving $w = 1$. If $J = \Pi$ then $w\Phi^+ = \Phi^-$ and $l(w) = |\Phi^+|$. But this is impossible since Φ is infinite, by Lemma 1.11. Thus $\emptyset \subsetneqq J \subsetneqq \Pi$ and hence both J and $\Pi \backslash J$ are non-empty.

As conjugation by w is a graph automorphism, for each $a \in \Pi$ there is an $a' \in \Pi$ such that $wr_a w^{-1} = r_{a'}$. This gives $wa = \pm a'$. It follows that $wJ = -K$ for some $K \subset \Pi$ and $w(\Pi \backslash J) = \Pi \backslash K$. Now let $a \in J$ and $b \in \Pi \backslash J$. Then $-wa = a' \in K$ and $wb = b' \in \Pi \backslash K$, and by the definition of the bilinear form it follows that $a \cdot b \leq 0$ and $a' \cdot b' \leq 0$. But $a \cdot b = wa \cdot wb = -a' \cdot b'$, and so we conclude that $a \cdot b = 0$. This result holds for all $a \in J$ and $b \in \Pi \backslash J$, and the two sets are non-empty, contradicting the irreducibility of W. ∎

Combined with Corollary 1.44 the following result has been proved.

2.15 Corollary *If W is an infinite Coxeter group, as above, whose diagram is a forest with no unusual labels and $R(W)$ is the group of automorphisms of W that preserve* $\mathrm{Ref}(W)$ *then*

$$R(W) = \mathrm{Inn}(W) \rtimes \mathrm{Gr}(W).$$

The fact that $\mathrm{Inn}(W) \cong W/Z(W)$, where $Z(W)$ is the centre of W, is well-known. When W is irreducible, apart from the finite groups, the centre is trivial and so $\mathrm{Inn}(W) \cong W$.

2.16 Lemma *If W is an infinite irreducible Coxeter group then $Z(W) = \{\,1\,\}$.*

Proof Let $w \in Z(W)$. Then for all $a \in \Phi$ we have $r_{wa} = wr_a w^{-1} = r_a$ and so $wa = \pm a$. Hence $\Pi = L \dot{\cup} K$ where

$$L = \{\, a \in \Pi \mid wa = -a \,\}$$
$$K = \{\, a \in \Pi \mid wa = a \,\}.$$

Now for all $a \in L$ and $b \in K$ we find that $a \cdot b = wa \cdot wb = -a \cdot b$ and hence $a \cdot b = 0$. This contradicts the irreducibility of W unless one of L or K is empty. If $K = \emptyset$ then $wa = -a$ for all $a \in \Phi^+$, and so $N(w) = \Phi^+$, which is infinite by Lemma 1.11. However this contradicts Corollary 1.8. So $L = \phi$, and hence $w = 1$. So we conclude that $Z(W) = \{\,1\,\}$ as required. ∎

2.17 Corollary *If W is an infinite irreducible Coxeter group whose diagram is a tree with no unusual labels and $R(W)$ is the set of automorphisms of W that preserve* $\mathrm{Ref}(W)$ *then*

$$R(W) \cong \mathrm{Inn}(W) \rtimes \mathrm{Gr}(W) \cong W \rtimes \mathrm{Gr}(W).$$

Chapter 3

Nearly Finite Coxeter Groups

3.1 Definition Given the Coxeter group W with set of simple reflections Π indexed by positive integers, if $a_i \in \Pi$, then we let W_i denote the standard parabolic subgroup $W_{\Pi \setminus \{a_i\}}$. If W_i is finite for some a_i, then we denote by w_i the longest element in W_i.

3.2 Definition We shall call the infinite Coxeter group W *nearly finite* if there is a simple root a_i such that W_i is finite.

§3.1 Automorphisms That Preserve Reflections

Before considering the nearly finite groups we need some linear algebra.

3.3 Definition The $n \times n$ matrix M is *reducible* if there are non-empty sets I and J such that $I \dot\cup J = \{1, \ldots n\}$ and $m_{ij} = 0$ for all $i \in I$ and $j \in J$. Otherwise M is said to be *irreducible.*

3.4 Lemma *Let M be the Gram matrix of the finite rank Coxeter group, W, then M is irreducible if and only if W is irreducible.*

Proof This is clear since $m_{ij} = a_i \cdot a_j$ is zero if and only if vertices i and j are not connected in the Coxeter diagram. ∎

3.5 Lemma *Suppose that M is a positive definite matrix such that $m_{ii} = 1$ for all i and $m_{ij} \leq 0$ for all $i \neq j$. Let $C = M^{-1}$, then $c_{ij} \geq 0$ for all i and j. Further, if M is irreducible, then $c_{ij} > 0$ for all i and j.*

Proof Let e_i be the i^{th} standard basis vector, written as a column vector. The ii^{th} entry of C is equal to the ii^{th} cofactor of M divided by $\det(M)$. As M is positive definite the principal minors are all positive and hence the ii^{th} entry of C is non-zero. Also $v^t M v \geq 0$ for all vectors v with equality if and only if $v = 0$.

Let $C_i = \sum_{j=1}^{n} \lambda_j e_j$ be the i^{th} column of C^t, $J = \{ j \mid \lambda_j \geq 0 \}$ and $K = \{ k \mid \lambda_k < 0 \}$. Setting $C' = \sum_J \lambda_j e_j$ and $C'' = \sum_K \lambda_k e_k$ we have $C_i = C' + C''$. Now $C_i^t M = e_i^t$ and so

$$C_i^t M e_j = e_i^t e_j \geq 0$$

for all i and j. Since $\lambda_k < 0$ for $k \in K$,

$$\begin{aligned} 0 &\geq \sum_{k \in K} \lambda_k (C_i^t M e_k) \\ &= C_i^t M C'' \\ &= {C'}^t M C'' + {C''}^t M C'' \\ &= \sum_{\substack{j \in J \\ k \in K}} \lambda_j \lambda_k e_j^t M e_k + {C''}^t M C'' \\ &= \sum_{\substack{j \in J \\ k \in K}} \lambda_j \lambda_k m_{jk} + {C''}^t M C''. \end{aligned}$$

Since $\lambda_j \geq 0$, $\lambda_k <$ and $m_{jk} \leq 0$ each term $\lambda_j \lambda_k m_{jk}$ is non-negative. Hence

$$0 \geq \sum_{J,K} \lambda_j \lambda_k m_{jk} + {C''}^t M C'' \geq {C''}^t M C'' \geq 0.$$

Thus $C'' = 0$ and $C_i = C'$ only has non-negative entries. Hence C has non-negative entries.

Suppose that C has at least one entry equal to 0, say $c_{rk} = 0$. Let $I = \{\, j \mid c_{rj} > 0 \,\}$ and $J = \{\, j \mid c_{rj} = 0 \,\}$; then $I \dot{\cup} J = \{\, 1, \ldots, n \,\}$, remembering that $c_{ij} \geq 0$ for all i and j. Now $c_{rr} \neq 0$ and so $r \in I$, while $k \in J$. Hence both I and J are nonempty. If $j \in J$, then $j \neq r$; thus

$$\begin{aligned} 0 = e_r^t e_j &= C_r^t M e_j \\ &= \sum_{i \in I} c_{ri} e_i^t M e_j \\ &= \sum_{i \in I} c_{ri} m_{ij}. \end{aligned}$$

We have $c_{ri} > 0$ for $i \in I$, while $m_{ij} \leq 0$ as $i \neq j$. Therefore $m_{ij} = 0$ for all $i \in I$. Thus $m_{ij} = 0$ for all $i \in I$ and $j \in J$ and hence M is reducible. ∎

3.6 Proposition *Suppose that W is a finite Coxeter group with simple roots $\{a_1, \ldots, a_n\}$, and let $d_1, \ldots, d_n$ be a set of non-negative real numbers. Let $x = \sum \delta_i a_i$ be the solution of the system of equations $x \cdot a_i = d_i$ for all i. Then $\delta_i \geq 0$ for all i, and if some d_j is nonzero then $\delta_i \geq 0$ for all i and $\delta_k > 0$ for all a_k in the same connected component as a_j.*

In particular if W is irreducible and d_j is non-zero for some j then $\delta_i > 0$ for all i.

Proof If M is the Gram matrix of the finite Coxeter group W, then M is positive definite, by Lemma 1.1. For all i we have $m_{ii} = a_i \cdot a_i = 1$, whereas if $i \neq j$, then

$$m_{ij} = a_i \cdot a_j = -\operatorname{c}(\pi_{m_{ij}}) \leq 0.$$

By Lemma 3.4 M is irreducible if and only if W is irreducible. Thus we may use Lemma 3.5.

Observe that the values of δ_i are given by

$$(\delta_i) = M^{-1}(d_i).$$

The entries of M^{-1} are all non-negative, by Lemma 3.5, and hence the δ_i are non-negative given that the d_i are non-negative.

If W is reducible, then the vector space spanned by the a_i is the orthogonal sum of subspaces associated with the irreducible components and the Gram matrix M is the diagonal sum of the Gram matrices for the components. Thus we may consider each component separately and so suppose that W is irreducible. In that case the entries of M^{-1} are all positive. Given $(\delta_i) = M^{-1}(d_i)$ if any one d_i is positive while the rest are non-negative, then each δ_i is the sum of non-negative terms including at least one positive term. Hence each δ_i is positive. ∎

3.7 Lemma *Suppose W is a Coxeter group, r_i and r_j are simple reflections and α is an automorphism of W such that $\alpha(r_i) = r_k$ and $\alpha(r_j) = r_l$ are also simple reflections. Then $a_i \cdot a_j = a_k \cdot a_l$.*

Proof Let $a_i \cdot a_j = -\operatorname{c}(\pi_s)$ and $a_k \cdot a_l = -\operatorname{c}(\pi_t)$. The order of $r_i r_j$ is s which equals t the order of $r_k r_l$ and hence $a_i \cdot a_j = a_k \cdot a_l$. ∎

3.8 Definition We say that a Coxeter diagram is of *finite type* if the corresponding Coxeter group is finite.

A set of simple reflections in a Coxeter group is of *finite type* if the parabolic subgroup they generate is finite.

3.9 Theorem *Suppose that W is irreducible, non-degenerate and nearly finite. Suppose that the diagram of W has no infinite bonds. Let Δ be the set of simple reflections and suppose that there exist $r_1, r_y \in \Delta$ (possibly equal) and an automorphism $\alpha : W \to W$ such that the following properties hold:*

(1) $\Delta_1 = \Delta \setminus \{\, r_1 \,\}$ *and* $\Delta_y = \Delta \setminus \{\, r_y \,\}$ *are both of finite type, and* $\alpha(\Delta_1) = \Delta_y$.

(2) $\alpha \in R(W)$.

Then α is inner by graph.

Proof Let V_1 and V_y be the subspaces of V spanned by $\Pi\setminus\{\,a_1\,\}$ and $\Pi\setminus\{\,a_y\,\}$ respectively, and let x_i be the projection of $-a_i$ onto V_i for $i = 1$ and y. We shall show that $x_y \cdot x_y \le x_1 \cdot x_1$. Since the same argument with r_1 and r_y interchanged, α replaced by α^{-1} and using Lemma 1.33 will show that $x_1 \cdot x_1 \le x_y \cdot x_y$, it follows that $x_1 \cdot x_1 = x_y \cdot x_y$.

Write $\{\,2, 3, \ldots, n\,\} = J^{(1)} \cup J^{(2)} \cup \cdots \cup J^{(k)}$ where the $J^{(j)}$ correspond to the irreducible components of $\Pi\setminus\{\,a_1\,\}$. For each $i \in \{\,2, 3, \ldots, n\,\}$ let

$$a_i \cdot a_1 = -\operatorname{c}(\pi_{m_{i1}}) = -c_i$$

so that by the definition of x_1,

$$a_i \cdot x_1 = c_i$$

for all $i \in \{\,2, 3, \ldots, n\,\}$. Write

$$u_1 = x_1 + a_1$$

so that u_1 spans the orthogonal complement of V_1 in V. (As W is non-degenerate V_1 has dimension $n-1$, $V_1^{\perp}$ is one dimensional and $a_1 \notin V_1$.)

Let $x_1 = \sum_{i=2}^{n} \mu_i a_i$. Note that

$$x_1 = x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(k)}$$

where $x_1^{(j)}$ is the projection of $-a_1$ onto the subspace spanned by $\left\{\, a_i \mid i \in J^{(j)} \,\right\}$. Now for all $j \in \{\,1, 2, \ldots, k\,\}$ and $i \in J^{(j)}$,

$$x_1^{(j)} \cdot a_i = -a_1 \cdot a_i = c_i \ge 0;$$

moreover, since W is irreducible there is at least one $i \in J^{(j)}$ such that $c_i > 0$. Since

$$x_1^{(j)} = \sum_{i \in J^{(j)}} \mu_i a_i$$

it follows from Proposition 3.6 that $\mu_i > 0$ for all i.

Let $x \in \Phi^+$ be such that $\alpha(r_1) = r_x$, and let σ be the permutation of $\{\,1, 2, \ldots, n\,\}$ such that $\alpha(r_i) = r_{\sigma i}$ for $i \in \{\,2, 3, \ldots, n\,\}$. (Thus $\sigma 1 = y$.) As $r_{\sigma i} r_x$ has the same order as $r_i r_1$, namely m_{i1}, we have

$$x \cdot a_{\sigma i} = \operatorname{c}(j_i \pi_{m_{i1}}) = d_i$$

for some j_i coprime to m_{i1}. Note that $c_i \ge |d_i|$ for all $i \in \{\,2, 3, \ldots, n\,\}$, and $d_i = 0$ if and only if $c_i = 0$. Let $x_0 = \sum_{i=2}^{n} \lambda_{\sigma i} a_{\sigma i}$ be the projection of x onto V_y, so that $x_0 \cdot a_{\sigma i} = d_i$ for each i, and

$$x = x_0 + \omega u_y$$

for some scalar ω, where $u_y = x_y + a_y$ spans the orthogonal complement of V_y in V. Examining the coefficient of a_y in the above equation for x, and using Lemma 1.15, we see that $\omega \ge 1$. Note also that

$$x_0 = x_0^{(1)} + x_0^{(2)} + \cdots + x_0^{(k)}$$

where $x_0^{(j)}$ is the projection of x onto the space spanned by $\left\{\, a_{\sigma i} \mid i \in J^{(j)} \,\right\}$.

Since $c_i \ge |d_i|$ for all $i \in J^{(j)}$ (where $j \in \{\,1, 2, \ldots, k\,\}$ is arbitrary), we have for all $i \in J^{(j)}$,

$$\begin{aligned}
0 \le c_i - d_i &= x_1 \cdot a_i - x_0 \cdot a_{\sigma i} \\
&= \left(\sum_{l \in J^{(j)}} \mu_l a_l \cdot a_i \right) - \left(\sum_{l \in J^{(j)}} \lambda_{\sigma l} a_{\sigma l} \cdot a_{\sigma i} \right) \\
&= \left(\sum_{l \in J^{(j)}} \mu_l a_l \cdot a_i \right) - \left(\sum_{l \in J^{(j)}} \lambda_{\sigma l} a_l \cdot a_i \right) && \text{by Lemma 3.7} \\
&= \sum_{l \in J^{(j)}} (\mu_l - \lambda_{\sigma l})\, a_l \cdot a_i.
\end{aligned}$$

By Proposition 3.6 it follows that $\mu_l \geq \lambda_{\sigma l}$ for all $l \in J^{(j)}$; moreover, if $\mu_l - \lambda_{\sigma l} = 0$ for some $l \in J^{(j)}$ then we must have $c_i - d_i = 0$ for all $i \in J^{(j)}$. Similarly,

$$0 \leq c_i + d_i = \sum_{l \in J^{(j)}} (\mu_l + \lambda_{\sigma l})\, a_l \cdot a_i$$

for all $i \in J^{(j)}$; so $\mu_l + \lambda_{\sigma l} \geq 0$ for all $l \in J^{(j)}$, equality occurring for some l only if $c_i + d_i = 0$ for all $i \in J^{(j)}$. Note in particular that $\mu_l \geq |\lambda_{\sigma l}|$ for all $l \in \{1, 2, \ldots, n-1\}$, and if $\mu_l = |\lambda_{\sigma l}|$ then $c_l = |d_l|$.

Each $z \in V$ can be written in the form $z = z_0 + \nu u_1$ with $z_0 \in V_1$ and $\nu \in \mathbb{R}$, and if $u_1 \cdot u_1 \geq 0$ this gives

$$z \cdot z = z_0 \cdot z_0 + \nu^2 u_1 \cdot u_1 \geq 0,$$

contrary to the fact that W is not of positive type. So $u_1 \cdot u_1 < 0$, and, by the same reasoning $u_y \cdot u_y < 0$.

Since $x \in \Phi$

$$1 = x \cdot x = (x_0 + \omega u_y) \cdot (x_0 + \omega u_y) = x_0 \cdot x_0 + \omega^2 u_y \cdot u_y$$

and we also have that

$$x_0 \cdot x_0 = \sum_{i=2}^{n} \lambda_{\sigma i} a_{\sigma i} \cdot x_0 = \sum_{i=2}^{n} \lambda_{\sigma i} d_i.$$

Similarly,

$$1 = a_1 \cdot a_1 = (-x_1 + u_1) \cdot (-x_1 + u_1) = x_1 \cdot x_1 + u_1 \cdot u_1$$

and also

$$x_1 \cdot x_1 = \sum_{i=2}^{n} \mu_i a_i \cdot x_1 = \sum_{i=2}^{n} \mu_i c_i.$$

Thus

$$u_1 \cdot u_1 + \sum_{i=2}^{n} \mu_i c_i = \omega^2 u_y \cdot u_y + \sum_{i=2}^{n} \lambda_{\sigma i} d_i,$$

and so

$$\sum_{i=2}^{n} (\mu_i c_i - \lambda_{\sigma i} d_i) = \omega^2 u_y \cdot u_y - u_1 \cdot u_1.$$

Since $\mu_i \geq |\lambda_{\sigma i}|$ and $c_i \geq |d_i|$ for all i we see that $\sum_{i=2}^{n} (\mu_i c_i - \lambda_{\sigma i} d_i) \geq 0$, and so $\omega^2 u_y \cdot u_y \geq u_1 \cdot u_1$. But $\omega^2 \geq 1$, and since $u_y \cdot u_y < 0$ it follows that $u_y \cdot u_y \geq \omega^2 u_y \cdot u_y$, and hence $u_y \cdot u_y \geq u_1 \cdot u_1$. Since $1 = x_1 \cdot x_1 + u_1 \cdot u_1$ (shown above) and $1 = x_y \cdot x_y + u_y \cdot u_y$ similarly it follows that $x_y \cdot x_y \leq x_1 \cdot x_1$, as desired.

In view of our earlier remarks, we must have $u_y \cdot u_y = u_1 \cdot u_1$, and

$$0 \leq \sum_{i=2}^{n} (\mu_i c_i - \lambda_{\sigma i} d_i) = (\omega^2 - 1) u_1 \cdot u_1 \leq 0$$

since $\omega \geq 1$ and $u_1 \cdot u_1 < 0$. Thus $(\omega^2 - 1) u_1 \cdot u_1 = 0$, giving $\omega = 1$, and $\sum_{i=2}^{n} (\mu_i c_i - \lambda_{\sigma i} d_i) = 0$, giving $\mu_i c_i = \lambda_{\sigma i} d_i = |\lambda_{\sigma i} d_i|$ for all $i \in \{2, 3, \ldots, n\}$.

Since $0 \leq (\mu_l - |\lambda_{\sigma l}|) c_l \leq \mu_l c_l - |\lambda_{\sigma l}|\, |d_l| = 0$ it follows that, for all $l \in \{2, 3, \ldots, n\}$, either $c_l = 0$ or $|\lambda_{\sigma l}| = \mu_l$. For each $j \in \{1, 2, \ldots, k\}$ we may choose $l \in J^{(j)}$ with $c_l > 0$; then either $\lambda_{\sigma l} = \mu_l$ and $d_i = c_i$ for all $i \in J^{(j)}$, or else $\lambda_{\sigma l} = -\mu_l$, and $d_i = -c_i$ for all $i \in J^{(j)}$. In the former case we have

$$x_0^{(j)} \cdot a_{\sigma i} = d_i = c_i = x_1^{(j)} \cdot a_i$$

for all $i \in J^{(j)}$, and it follows that $x_0^{(j)} = \tilde{\sigma}(x_1^{(j)})$, where $\tilde{\sigma}$ is the isomorphism $V_1 \to V_y$ given by $a_i \mapsto a_{\sigma i}$. In the latter case,

$$x_0^{(j)} \cdot a_{\sigma i} = d_i = -c_i = -x_1^{(j)} \cdot a_i$$

for all $i \in J^{(j)}$, giving $x_0^{(j)} = -\tilde{\sigma}(x_1^{(j)})$.

Let w be the product of the longest elements of the parabolic subgroups corresponding to those sets $\sigma(J^{(j)})$ for which $x_0^{(j)} = \tilde{\sigma}(x_1^{(j)})$, let β be the inner automorphism of W given by conjugation by w, and let $\alpha' = \beta\alpha$. Since β induces a graph automorphism on $\Delta \setminus \{ r_y \}$, we see that α' satisfies the same hypotheses as α. Now $\alpha'(r_1) = w r_x w^{-1} = r_{wx}$, and

$$\begin{aligned} wx &= wx_0 + u_y \\ &= wx_0^{(1)} + wx_0^{(2)} + \cdots + wx_0^{(k)} + u_y. \end{aligned}$$

where $wx_0^{(j)}$ is the projection of wx onto the span of $\left\{ a_{\sigma i} \,\middle|\, i \in J^{(j)} \right\}$. Applying to α' the arguments used for α enables us to deduce that for each j

$$wx_0^{(j)} = \pm\tilde{\sigma}(x_1^{(j)}) = \pm \sum_{i \in J^{(j)}} \mu_i a_{\sigma i}.$$

But w was chosen so that $wx_0^{(j)}$ is a negative linear combination of $\left\{ a_{\sigma i} \,\middle|\, i \in J^{(j)} \right\}$ for each j, and so we conclude that $wx_0^{(j)} = -\tilde{\sigma}(x_1^{(j)})$. Thus

$$\begin{aligned} wx &= -\tilde{\sigma}(x_1^{(1)}) - \tilde{\sigma}(x_1^{(2)}) - \cdots - \tilde{\sigma}(x_1^{(k)}) + u_y \\ &= -\tilde{\sigma}(x_1) + u_y \end{aligned}$$

showing that $wx \cdot a_{\sigma i} = -c_i = a_1 \cdot a_i$ for all $i \in \{ 2, 3, \ldots, n \}$. Lemma 3.7 together with Theorem 1.44 shows that α' is inner by graph. (Indeed we have shown that the positive roots wx, $a_{\sigma 2}$, $a_{\sigma 3}$, $\ldots$, $a_{\sigma n}$ form a base for the root system that is isomorphic to Π. From this we can conclude that $wx = a_y$ and that α' is in fact a graph automorphism.) ∎

§3.2 Reflections and Components

3.10 Definitions If W is a Coxeter group and $r \in W$, then we denote the conjugacy class of r in W by $\mathcal{C}(r)$.

(1) If r and r' are reflections in W then we say that $\mathcal{C}(r)$ and $\mathcal{C}(r')$ are *linked* if we can find $r_1 \in \mathcal{C}(r)$ and $r_2 \in \mathcal{C}(r')$ such that r_1 and r_2 do not commute.

(2) If r and r' are reflections in W then we say that there is *a chain joining* $\mathcal{C}(r)$ *and* $\mathcal{C}(r')$ if we can find reflections r_0, $\ldots$, r_n such that $r_0 = r$, $r_n = r'$ and $\mathcal{C}(r_i)$ and $\mathcal{C}(r_{i+1})$ are linked for $i = 0$, $\ldots$, $n-1$.

Recall from Lemma 1.17 that if r is any reflection in W then r lies in an irreducible component of W. Clearly, conjugate reflections lie in the same irreducible component.

3.11 Lemma *Suppose that W is a Coxeter group with r and r' any two reflections, then r and r' are in the same irreducible component of W if and only if there is a chain joining $\mathcal{C}(r)$ and $\mathcal{C}(r')$.*

Proof By Lemma 1.1 each conjugacy class of reflections contains at least one simple reflection. If r and r' are in the same irreducible component then there are simple roots a_0 and a_0' such that $r_0 \in \mathcal{C}(r)$ and $r_0' \in \mathcal{C}(r')$. Furthermore there is a path in the Coxeter diagram joining a_0 and a_0'. If $a_0, a_1, \ldots, a_n = a_0'$ is such a path then the $\mathcal{C}(r_i)$ form a chain joining $\mathcal{C}(r)$ and $\mathcal{C}(r')$.

Conversely, suppose that r and r' are in different components of W. If $\mathcal{C}(r_0), \mathcal{C}(r_1), \ldots$ is a chain joining r and r', then we can find a j such that $\mathcal{C}(r_j)$ and $\mathcal{C}(r_{j+1})$ are in different components. But this implies that each reflection in $\mathcal{C}(r_j)$ commutes with every reflection in $\mathcal{C}(r_{i+1})$. This contradicts the fact that $\mathcal{C}(r_j)$ is linked to $\mathcal{C}(r_{j+1})$. Thus if r and r' are in different components then there is no chain joining them. ∎

Note that a simple corollary of this is that there is a chain joining $\mathcal{C}(r)$ and $\mathcal{C}(r')$ if and only if there is a chain joining r and r'.

3.12 Lemma *Suppose that W and W' are finite rank Coxeter groups. If α is an isomorphism such that $\alpha\big(\operatorname{Ref}(W)\big) = \operatorname{Ref}(W')$, then α maps each irreducible component of W onto an irreducible component of W'.*

Proof Suppose the irreducible components of W are $W_{J(i)}$. If r and r' are any two reflections in $\operatorname{Ref}(W_{J(i)})$ then by Lemma 3.11 there is a chain joining $\mathcal{C}(r)$ and $\mathcal{C}(r')$. As α preserves reflections and conjugacy classes there is a chain joining $\mathcal{C}\big(\alpha(r)\big)$ and $\mathcal{C}\big(\alpha(r')\big)$. Hence $\alpha(r)$ and $\alpha(r')$ are in the same irreducible component of W'. Thus $\alpha(W_{J(i)})$ is contained in an irreducible component of W'. Applying this to α^{-1} finishes the proof. ∎

3.13 Proposition *Suppose that $\alpha : W \to W'$ is an isomorphism of finite Coxeter groups that maps reflections to reflections. Then W and W' have the same type.*

Proof By Lemma 3.12 we may concentrate upon irreducible Coxeter groups. If two finite irreducible groups are isomorphic then they have the same order.

All finite Coxeter groups have even order and so if W is a finite irreducible Coxeter group then we can find a group W' of type $I_2(m)$ such that $|W| = |W'|$. However exactly half the elements of a group of type $I_2(m)$ are reflections, and this is a property not possessed by any of the other types. So there is no reflection preserving isomorphism between a group of type $I_2(m)$ and an irreducible group of any other type. The only other coincidences of order for finite irreducible Coxeter groups occurs for types A_4 and H_3, which both have order 120. They are not isomorphic since, for example, A_4 has trivial centre while H_3 does not. ∎

Suppose that W is a nearly finite Coxeter group of rank $n \geq 4$ such that W_i is a finite irreducible Coxeter group and let α be an automorphism of W. From Corollary 1.30 we know that $\alpha(W_i)$ is a maximal finite parabolic subgroup. Up to inner automorphisms we may assume that $\alpha(W_i)$ is a standard parabolic subgroup and therefore has rank at most $n-1$. If $\alpha(W_i)$ is reducible then it follows that W_i is isomorphic to a product of smaller Coxeter groups, for which the sum of the ranks does not exceed the rank of W_i. We proceed to show that this cannot occur by examining all cases.

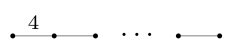
In the paper [Max98], Maxwell finds all the normal subgroups of the finite irreducible Coxeter groups. In all cases W^+ will denote the subgroup of elements of even length; other normal subgroups will be given as the kernels of surjective homomorphisms to other Coxeter groups. Maxwell's list is as follows.

A_n: If W is a group of type A_n then the normal subgroups of W are:

$$\{1\}, \qquad W \quad \text{and} \quad W^+.$$

Clearly no group of type A_n is reducible as W does not have two normal subgroups H and K such that $|H| \times |K| = |W|$.

B_n: If W is a group of type B_n with diagram



then the normal subgroups of W are:

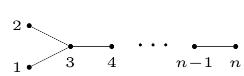
$$
\begin{aligned}
N_0 &= \{1\} \\
N_1 &= \langle w_0 \rangle = Z(W) \\
N_2 &= W^+ \\
N_3 &= W \\
N_4 &= \ker(\phi_4) \qquad \text{where } \phi_4 : W \to \langle s_1 \rangle \text{ type } A_1 \\
&\qquad \phi_4(r_1) = s_1,\ \phi_4(r_i) = 1 \\
N_5 &= \ker(\phi_5) \qquad \text{where } \phi_5 : W \to \langle s_1 \rangle \text{ type } A_1 \\
&\qquad \phi_5(r_1) = 1,\ \phi_5(r_i) = s_1 \\
N_6 &= \ker(\phi_6) \qquad \text{where } \phi_6 : W \to \langle s_1 \rangle \times \langle \sigma_1 \rangle \text{ type } A_1 \times A_1 \\
&\qquad \phi_6(r_1) = s_1,\ \phi_6(r_i) = \sigma_1 \\
N_7 &= \ker(\phi_7) \qquad \text{where } \phi_7 : W \to \langle s_i \rangle \text{ type } A_{n-1} \\
&\qquad \phi_7(r_1) = 1,\ \phi_7(r_i) = s_i \\
N_8 &= \ker(\phi_8) \qquad \text{where } \phi_8 : W \to \langle s_i \rangle \times \langle \sigma_1 \rangle \text{ type } A_{n-1} \times A_1 \\
&\qquad \phi_8(r_1) = \sigma_1,\ \phi_8(r_i) = s_i \\
N_9 &= \ker(\phi_9) \qquad \text{where } \phi_9 : W \to \langle s_i \rangle \text{ type } A_2 \\
&\qquad \phi_9(r_1) = 1,\ \phi_9(r_2) = s_1,\ \phi_9(r_3) = s_2,\ \phi_9(r_4) = s_1 \\
N_{10} &= \ker(\phi_{10}) \qquad \text{where } \phi_{10} : W \to \langle s_1 \rangle \times \langle \sigma_1 \rangle \text{ type } A_2 \times A_1 \\
&\qquad \phi_9(r_1) = \sigma_1,\ \phi_9(r_2) = s_1,\ \phi_9(r_3) = s_2,\ \phi_9(r_4) = s_1
\end{aligned}
$$

where the last two cases only occur when $n = 4$. A consideration of orders shows that the only possible decompositions of W are as $N_1 \times N_2$, $N_1 \times N_4$ or $N_1 \times N_5$. Now w_0 is a product of elements conjugate to r_1; thus $\phi_5(w_0) = 1$ and hence $w_0 \in N_5$. Thus $N_1 \cap N_5 \neq \{1\}$. If n is even then $\phi_4(w_0) = r_1^n = 1$, whence $w_0 \in N_4$, and $l(w_0) = n^2$ is even, whence $w_0 \in N_2$. Hence W has no direct product decomposition if n is even. If n is odd then

$$W = N_1 \times N_2 = N_1 \times N_4.$$

In fact N_4 is a Coxeter group of type D_{2k+1} and therefore a group of type B_{2k+1} is isomorphic to a group of type $A_1 \times D_{2k+1}$. The Coxeter group of type D_{2k+1} is not abstractly isomorphic to any other indecomposable Coxeter groups, since the only one with the correct order is $I_2(2^{2k-1}(2k+1)!)$, and this has non-trivial centre whereas D_{2k+1} has trivial centre. Thus $A_1 \times D_n$ is the only decomposition as a product of groups abstractly isomorphic to Coxeter groups. However, $A_1 \times D_n$ has greater rank than B_n, and so cannot occur as a parabolic subgroup of any Coxeter group that has a maximal parabolic subgroup of type B_n.

D_n: If W is a group of type D_n with diagram



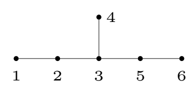
then the normal subgroups of W are:

$$
\begin{aligned}
N_0 &= \{1\} \\
N_1 &= \langle w_0 \rangle \text{ (only for } n \text{ even)} \\
N_2 &= W^+ \\
N_3 &= W \\
N_4 &= \ker(\phi_4) \qquad \text{where } \phi_4 : W \to \langle s_i \rangle \text{ type } A_{n-1}
\end{aligned}
$$

$$\phi_4(r_1) = s_1,\ \phi_4(r_2) = s_1,\ \phi_4(r_i) = s_{i-1}$$
$$N_5 = \ker(\phi_5) \qquad \text{where } \phi_5 : W \to \langle s_1, s_2 \rangle \text{ type } A_2$$
$$\phi_5(r_1) = \phi_5(r_2) = \phi_5(r_4) = s_1,\ \phi_5(r_3) = s_2$$
$$N_6 = \ker(\phi_6) \qquad \text{where } \phi_6 : W \to \langle s_1, s_2 \rangle \text{ type } A_2$$
$$\phi_6(r_1) = \phi_6(r_2) = \phi_6(r_3) = s_1,\ \phi_6(r_4) = s_2$$
$$N_7 = \ker(\phi_7) \qquad \text{where } \phi_7 : W \to \langle s_1, s_2 \rangle \text{ type } A_2$$
$$\phi_7(r_2) = \phi_7(r_3) = \phi_7(r_4) = s_1,\ \phi_7(r_1) = s_2$$

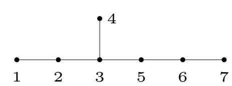
where the last three cases only occur if $n = 4$. Looking at orders the only possible decomposition is $W = N_1 \times N_2$. But $l(w_0)$ is even and so $N_1 \subset N_2$, whence W is indecomposable.

E_6: If W is a group of type E_6 with diagram



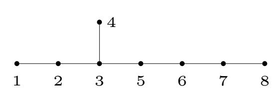
then the normal subgroups of W are $\{1\}$, W^+ and W. Hence W is indecomposable.

E_7: If W is a group of type E_7 with diagram



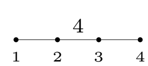
then the normal subgroups of W are $\{1\}$, $\langle w_0 \rangle$, W^+ and W. Since $l(w_0)$ is odd we see that $W = \langle w_0 \rangle \times W^+$. The only finite Coxeter group of order $|W|/2$ is of type $I_2(|W|/4)$, but a group of type $A_1 \times I_2(|W|/4)$ has further normal subgroups and so cannot be isomorphic to a group of type E_7. Thus W does not decompose as a product of Coxeter groups.

E_8: If W is a group of type E_8 with diagram



then the normal subgroups of W are $\{1\}$, W^+ and W. Hence W is indecomposable.

F_4: If W is a group of type F_4 with diagram



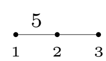
then the normal subgroups are:

$$N_0 = \{1\}$$
$$N_1 = \langle w_0 \rangle$$
$$N_2 = W^+$$
$$N_3 = W$$
$$N_4 = \ker(\phi_4) \qquad \text{where } \phi_4 : W \to \langle s_1 \rangle \text{ type } A_1$$
$$\phi_4(r_1) = 1,\ \phi_4(r_2) = 1,\ \phi_4(r_3) = s_1,\ \phi_4(r_4) = s_1$$
$$N_5 = \ker(\phi_5) \qquad \text{where } \phi_5 : W \to \langle s_1 \rangle \text{ type } A_1$$
$$\phi_5(r_1) = s_1,\ \phi_5(r_2) = s_1,\ \phi_5(r_3) = 1,\ \phi_5(r_4) = 1$$
$$N_6 = \ker(\phi_6) \qquad \text{where } \phi_6 : W \to \langle s_1 \rangle \times \langle \sigma_1 \rangle \text{ type } A_1 \times A_1$$
$$\phi_6(r_1) = s_1,\ \phi_6(r_2) = s_1,\ \phi_6(r_3) = \sigma_1,\ \phi_6(r_4) = \sigma_1$$

$$\begin{aligned}
N_7 = \ker(\phi_7) &\qquad \text{where } \phi_7 : W \to \langle s_1, s_2\rangle \text{ type } A_2\\
&\phi_7(r_1) = 1,\ \phi_7(r_2) = 1,\ \phi_7(r_3) = s_1,\ \phi_7(r_4) = s_2\\
N_8 = \ker(\phi_8) &\qquad \text{where } \phi_8 : W \to \langle s_1, s_2\rangle \text{ type } A_2\\
&\phi_8(r_1) = s_1,\ \phi_8(r_2) = s_2,\ \phi_8(r_3) = 1,\ \phi_8(r_4) = 1\\
N_9 = \ker(\phi_9) &\qquad \text{where } \phi_9 : W \to \langle s_1, s_2\rangle \times \langle \sigma_1\rangle \text{ type } A_2 \times A_1\\
&\phi_9(r_1) = s_1,\ \phi_9(r_2) = s_2,\ \phi_9(r_3) = \sigma_1,\ \phi_9(r_4) = \sigma_1\\
N_{10} = \ker(\phi_{10}) &\qquad \text{where } \phi_{10} : W \to \langle s_1, s_2\rangle \times \langle \sigma_1\rangle \text{ type } A_1 \times A_1\\
&\phi_{10}(r_1) = \sigma_1,\ \phi_{10}(r_2) = \sigma_1,\ \phi_{10}(r_3) = s_1,\ \phi_{10}(r_4) = s_2\\
N_{11} = \ker(\phi_{11}) &\qquad \text{where } \phi_{11} : W \to \langle s_1, s_2\rangle \times \langle \sigma_1, \sigma_2\rangle \text{ type } A_2 \times A_2\\
&\phi_{11}(r_1) = s_1,\ \phi_{11}(r_2) = s_2,\ \phi_{11}(r_3) = \sigma_1,\ \phi_{11}(r_4) = \sigma_2
\end{aligned}$$

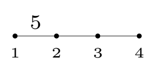
As before a consideration of orders suggests that the only possible decompositions are $N_1 \times N_2$, $N_1 \times N_4$ or $N_1 \times N_5$. However, w_0 is an element of each of N_2, N_4 and N_5 and so W is indecomposable.

H_3: If W is a group of type H_3 with diagram



then the normal subgroups of W are $\{1\}$, $\langle w_0\rangle$, W^+ and W. As with type E_7, $W = \langle w_0\rangle \times W^+$ but W^+ is not a Coxeter group.

H_4: If W is a group of type H_4 with diagram



then the normal subgroups of W are $\{1\}$, W^+ and W. Hence W is indecomposable.

Note that W_i cannot have type $I_2(m)$ since we have assumed that W has rank at least 4, and hence W_i has rank at least 3. Automorphisms of Coxeter groups of rank 3 are dealt with in a separate chapter.

Therefore in all cases if W_i is a finite irreducible Coxeter group, then $\alpha(W_i)$ is a parabolic subgroup of the same type. Hence we have proved the following theorem.

3.14 Theorem *If W is a nearly finite Coxeter group of rank $n \geq 4$ such that W_i is a finite irreducible Coxeter group, then any automorphism of W will map W_i to a conjugate of a maximal standard parabolic subgroup W_j of the same type as W_i.*

§3.3 Graph Automorphisms and Unusual Labels

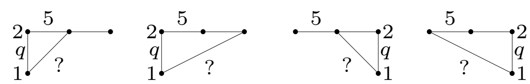
Suppose that W is an irreducible nearly finite Coxeter group. Renumbering if necessary we may assume that W_1 is finite. We shall consider the situation in which W_1 has a component of type H_3, H_4 or $I_2(m)$ for $m > 3$ and there is an $a_i \neq a_1$ such that W_i has the same type as W_1. Renumbering again if necessary suppose that W_1 and W_2 have the same type. We shall classify all such Coxeter groups. In particular, it turns out that in almost all cases there is a graph automorphism that interchanges a_1 and a_2.

For convenience in the following identify a_i with the vertex it corresponds to in the Coxeter diagram, Γ, of W. Let Γ_k denote the graph obtained by deleting vertex a_k from Γ, and write $\deg_k$ for $\deg(a_k)$. Since W_1 and W_2 have the same type, $\sum_{k\neq 1} \deg_k = \sum_{k\neq 2} \deg_k$, and so it follows that $\deg_1 = \deg_2$.

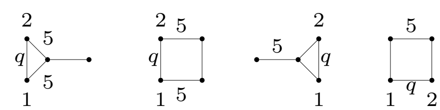
First suppose that W_1 and W_2 are irreducible. If we have a graph where a_1a_2 is not an edge, then we can obtain a new graph by including this edge. This new graph will still have the property that W_1 and W_2 have the same type. Therefore we will look for graphs where a_1a_2 is an edge and obtain the remaining possibilities by deleting this edge. We will deal with type H_3 in detail and then state the possibilities for H_4 and $I_2(m)$.

Suppose that W_1 and W_2 have type H_3. Then $\deg_2$ in Γ_1 is either 1 or 2. Thus $2 \leq \deg_2 = \deg_1 \leq 3$ in Γ.

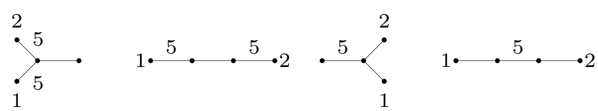
Case (a): $\deg_1 = \deg_2 = 2$. As a_1a_2 is an edge in Γ we can see that a_2 has degree 1 in Γ_1. There are four possibilities:



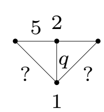
As W_2 also has type H_3 the remaining label can be determined, and we have the following possibilities.



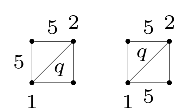
Deleting the edge a_1a_2 gives four further possibilities.



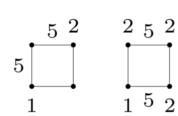
Case (b): $\deg_1 = \deg_2 = 3$. This time a_2 has degree 2 in Γ_1 and there is only one possibility. As $\deg_1 = 3$ there is an edge from a_1 to each vertex in Γ_1.



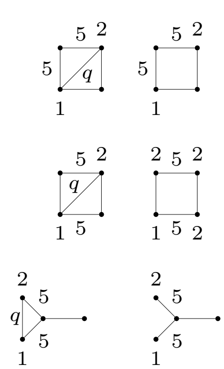
Again W_2 has type H_3 and the labels can be determined. There are two possibilities.

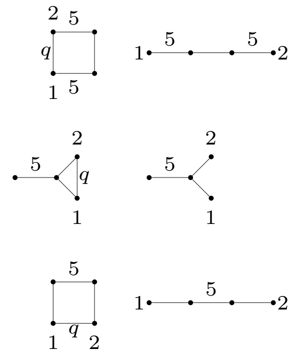


Deleting the edge a_1a_2 gives the following possibilities.



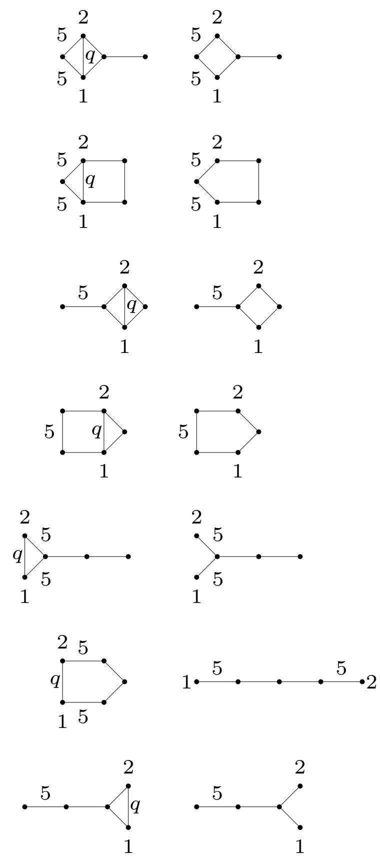
Thus there are the following 12 possibilities for Γ.



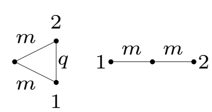


Note that in each case there is a graph automorphism that interchanges a_1 and a_2.

Similar arguments to the above, lead to the following 14 possibilities if W_1 and W_2 have type H_4.



There are only 2 possibilities if W_1 and W_2 have type $I_2(m)$ for $m > 3$.

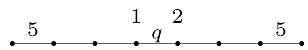


In every case there is a graph automorphism that interchanges a_1 and a_2.

Secondly, suppose that W_1 is reducible. Then Γ_1 contains at least two components. Thus $\deg_1 \geq 2$ as a_1 is joined to each component of Γ.

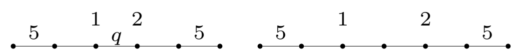
Case (a): a_2 is not contained in the given component of Γ_1 of type H_3, H_4 or $I_2(m)$. Then Γ_2 contains a component of type H_3, H_4 or $I_2(m)$ with at least one edge added (the edge joining a_1 to this component). This does not give a diagram of finite type if the original component had type H_4 or $I_2(m)$ for $m > 5$. So the original component is H_3, $I_2(4)$ or $I_2(5)$.

If Γ_1 has a component of type H_3 and a_2 is not in this component, then $\deg_1 \leq 3$ in Γ, for otherwise Γ_2 would contain a component with an edge labelled with 5 and a vertex of degree at least 3, contradicting the fact that Γ_2 is of finite type. If $\deg_1 = 3$ or if a_2 and a_1 are not adjacent, then Γ_2 contains a component with at least two edges added to a graph of type H_3 and again such a component is not of finite type. Thus $\deg_1 = 2$, Γ_1 has two components and a_2 is adjacent to a_1. Thus Γ_2 has a component of type H_4 and so W_1 and W_2 are of type $H_3 \times H_4$. Thus Γ is the following.



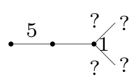
Again there is a graph automorphism that interchanges a_1 and a_2.

If Γ_1 has a component of type $I_2(5)$ and a_2 is not in this component, then again $\deg_1 \leq 3$. In the case $\deg_1 = 2$, Γ_1 has 2 components and hence Γ_2 has two components. The component containing a_1 is of type H_3 or H_4, and so W_1 and W_2 are of type $I_2(5) \times H_3$ or $I_2(5) \times H_4$. There are two possibilities.

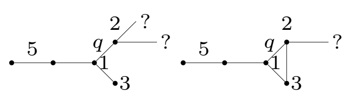


In each case there is a graph automorphism that interchanges a_1 and a_2.

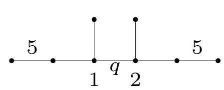
If $\deg_1 = \deg_2 = 3$ then we have the following partial diagram:



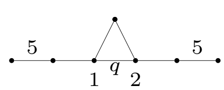
Thus in Γ_2 two edges have been added to $I_2(5)$ and so W_2 contains a component of type H_4 and we have the following.



In Γ_2 the vertex a_3 has degree 1 and so $\deg_3 = 1$ or 2 in Γ. If $\deg_3 = 1$ then Γ_1 has components A_1, $I_2(5)$ and H_4 and Γ is the following.



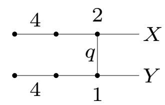
There is a graph automorphism swapping a_1 and a_2. If $\deg_3 = 2$ in Γ, then a_2a_3 is an edge in Γ_1 and W_2 has two components. Thus W_1 and W_2 have type $I_2(5) \times H_4$ and Γ is the following.



Again there is a graph automorphism swapping a_1 and a_2.

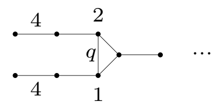
If Γ_1 has a component of type $I_2(4)$ and a_2 is not in this component, then yet again $\deg_1 \leq 3$. If $\deg_1 = 2$ then Γ_1 has two components, of types $I_2(4)$ and B_k for some k. Thus Γ is the following

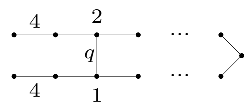
and there is a graph automorphism swapping a_1 and a_2. If $\deg_1 = 3$, then a_1 and a_2 must be adjacent as in the $I_2(5)$ case. Then Γ_2 contains a component of type B_l, as well as one of type $I_2(4)$. Thus we have



where there may or may not be edges between the parts of the graph labelled X and Y.

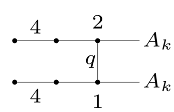
If X and Y are joined then Γ_1 and Γ_2 have 2 components and must have type $I_2(4) \times B_l$. As B_l does not have a vertex of degree more than 2, any vertex other than a_1 and a_2 having degree 3 in Γ must be adjacent to both a_1 and a_2. There is at most one such vertex, and so there are two possibilities.





There is a graph automorphism swapping a_1 and a_2 in each case.

Finally suppose that X and Y are not joined. Then Γ_1 has 3 components, and since the component of Γ_1 corresponding to Y is of type A_k for some k, we see that Γ_1 and Γ_2 have type $I_2(4) \times A_k \times B_l$. Hence Γ is the following,

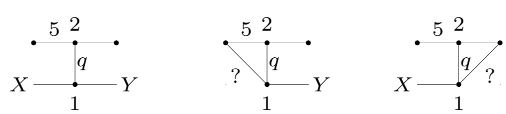


and there is a graph automorphism swapping a_1 and a_2. This completes the case where a_2 is not in the given component of type H_3, H_4 or $I_2(m)$.

Case (b): a_2 is in the given component of type H_3, H_4 or $I_2(m)$ for $m > 3$.

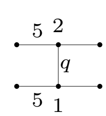
Suppose that Γ_1 contains a component of type H_3 and a_2 is in this component. Thus $\deg_2 \leq 2$ in Γ_1 and so $2 \leq \deg_1 = \deg_2 \leq 3$ in Γ.

Suppose $\deg_1 = \deg_2 = 3$. The vertices a_1 and a_2 are joined, and there may or may not be edges joining a_1 to the other vertices of the H_3 component of Γ_1. Since a_1 has degree 3 the possibilities for Γ are as follows.

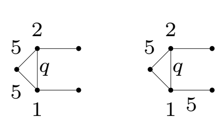


In the first case Γ_2 has 3 components, two of which are of type A_1, while in the second and third cases Γ_2 has two components, one of which is of type A_1. In all three cases the remaining component of Γ_2 must be of type H_3.

In the first case the only possibility is as follows,

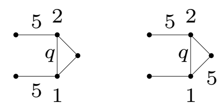


and there is a graph automorphism swapping a_1 and a_2. In the second case we have two possibilities, as follows.



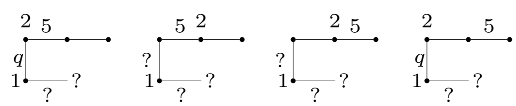
In the first of these cases there is a graph automorphism swapping a_1 and a_2, while the other is the first of our exceptions.

In the third case we also have two possibilities, as follows.

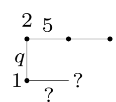


Again there is a graph automorphism swapping a_1 and a_2 in the first case. The other case is isomorphic to the exceptional case above.

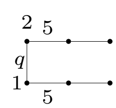
Now suppose that $\deg_1 = \deg_2 = 2$, still in the case where Γ_1 has a component of type H_3 that contains a_2. Only one of the two edges from a_1 connects to the given H_3 component of Γ_1, otherwise Γ_1 is irreducible, a case that has already been dealt with. There are four possibilities.



If Γ is

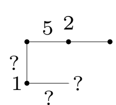


then Γ_2 has a component of type A_2, and so Γ_1 and Γ_2 have type $A_2 \times H_3$. So we have the following diagram,

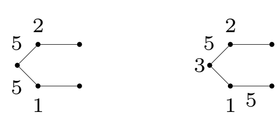


and there is a graph automorphism swapping a_1 and a_2.

If Γ is

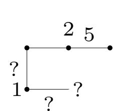


then Γ_1 has a component of type A_1, and so Γ_1 and Γ_2 have type $A_1 \times H_3$. So we have the following possible diagrams.

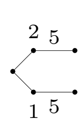


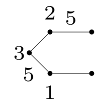
In the first case there is a graph automorphism swapping a_1 and a_2, while the other is the final exceptional case. In fact it is the same as the previous exceptional case but with $q = 2$. Furthermore, in this case Γ_3 is also of finite type and is the unique subgraph of type $A_2 \times I_2(5)$.

If Γ is

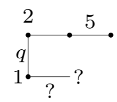


then again we find that Γ_1 and Γ_2 are of type $A_1 \times H_3$. We have the following possible diagrams.

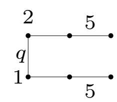




In the first case there is a graph automorphism swapping a_1 and a_2, while the other is isomorphic to the exception just considered.
If Γ is



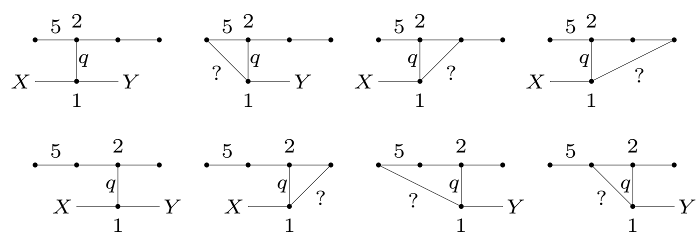
then Γ_1 and Γ_2 are of type $I_2(5) \times H_3$. We have the following diagram.



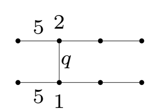
There is a graph automorphism swapping a_1 and a_2. This completes the case where Γ has a component of type H_3.

Now suppose that Γ_1 contains a component of type H_4 and a_2 is a vertex in this component. As in the equivalent H_3 case this means that $2 \leq \deg_1 = \deg_2 \leq 3$.

If $\deg_1 = \deg_2 = 3$, again a_1 and a_2 must be adjacent. So we have the following eight possibilities.

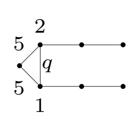


(i) In the first diagram of the above cases it is clear that Γ_1 and Γ_2 must be of type $A_1 \times A_2 \times H_4$. We obtain the diagram.



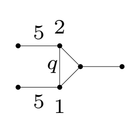
and there is a graph automorphism swapping a_1 and a_2.

(ii) In the second diagram it is clear that Γ_1 and Γ_2 must be of type $A_2 \times H_4$. We obtain the diagram.



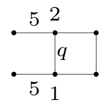
and there is a graph automorphism swapping a_1 and a_2.

(iii) In the third diagram it is clear that Γ_1 and Γ_2 must be of type $A_1 \times H_4$. We obtain the diagram.



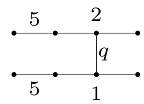
and there is a graph automorphism swapping a_1 and a_2.

(iv) In the fourth diagram it is clear that Γ_1 and Γ_2 must be of type $A_1 \times A_2 \times H_4$. We obtain the diagram.



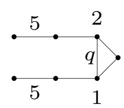
and there is a graph automorphism swapping a_1 and a_2.

(v) In the fifth diagram it is clear that Γ_1 and Γ_2 must be of type $A_1 \times I_2(5) \times H_4$. We obtain the diagram.



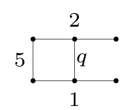
and there is a graph automorphism swapping a_1 and a_2.

(vi) In the sixth diagram it is clear that Γ_1 and Γ_2 must be of type $I_2(5) \times H_4$. We obtain the diagram.



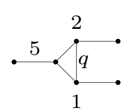
and there is a graph automorphism swapping a_1 and a_2.

(vii) In the seventh diagram it is clear that Γ_1 and Γ_2 must be of type $A_1 \times H_4$. We obtain the diagram.



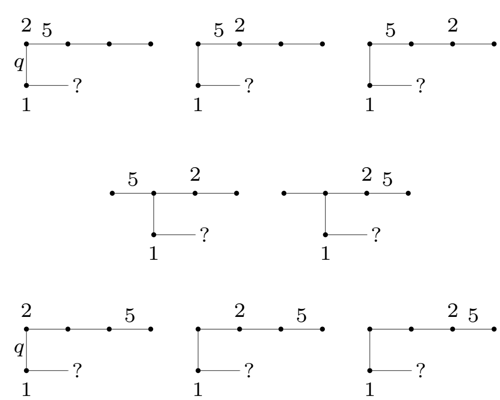
and there is a graph automorphism swapping a_1 and a_2.

(viii) In the final diagram it is clear that Γ_1 and Γ_2 must be of type $A_1 \times H_4$. We obtain the diagram.

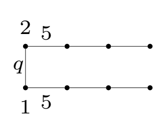


and there is a graph automorphism swapping a_1 and a_2.

Suppose finally that $\deg_1 = \deg_2 = 2$. As we are assuming that Γ_1 is reducible, a_1 is joined to the given H_4 component by exactly one edge. We have the following possibilities.

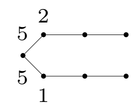


In the first diagram Γ_1 and Γ_2 are of type $A_3 \times H_4$ and we have

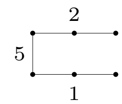


and there is a graph automorphism swapping a_1 and a_2. This can be repeated for each diagram; the results are summarized below.

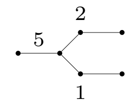
Γ_1 and Γ_2 have type $A_2 \times H_4$



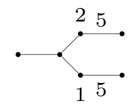
Γ_1 and Γ_2 have type $A_1 \times H_4$



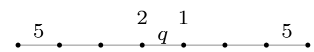
Γ_1 and Γ_2 have type $A_1 \times H_4$



Γ_1 and Γ_2 have type $A_1 \times H_4$



Γ_1 and Γ_2 have type $H_3 \times H_4$

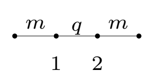


Γ_1 and Γ_2 have type $I_2(5) \times H_4$

Γ_1 and Γ_2 have type $A_1 \times H_4$

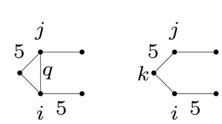
In each case there is a graph automorphism swapping a_1 and a_2.

It remains to consider the case that Γ_1 contains a component of type $I_2(m)$ for $m > 3$ and a_2 is a vertex in this component. Since a_2 has degree 1 in Γ_1 it has degree 2 in Γ. Thus $\deg_1 = \deg_2 = 2$, and since Γ_1 is reducible if follows that one of the edges from a_1 connects to the given $I_2(m)$ and the other does not. Therefore Γ_1 and Γ_2 both have type $A_1 \times I_2(m)$. The only possibility is as follows



and there is a graph automorphism swapping a_1 and a_2. This completes the proof of the following result.

3.15 Proposition *Suppose that W is a Coxeter group and there are two simple roots $a_i \neq a_j$ such that W_i and W_j are finite Coxeter groups of the same type. If W_i and W_j contain at least one component of type H_3, H_4 or $I_2(m)$ for $m > 3$, then there is a graph automorphism of W that interchanges W_i and W_j, or else W corresponds to one of the following diagrams.*



In the second of these cases there is simple root a_k such that W_k is the unique maximal standard parabolic subgroup of type $A_2 \times I_2(5)$.

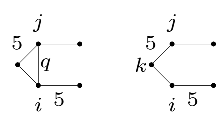
§3.4 Nearly Finite Coxeter Groups

3.16 Theorem *If W is an irreducible non-degenerate nearly finite Coxeter group with finite labels, then any automorphism of W that preserves reflections is inner by graph.*

Proof Renumbering if necessary we may suppose that W_1 is finite. Let $\alpha \in R(W)$ be any automorphism that preserves reflections. Up to inner automorphisms we may assume that $\alpha(W_1)$ is a standard parabolic subgroup. By Proposition 3.13 and the fact that $\alpha|_{W_1}$ preserves reflections W_1 and $\alpha(W_1)$ have the same type and hence the same rank. Thus $\alpha(W_1) = W_i$ for some i possibly equal to 1.

Suppose that W_1 does not contain any component of type H_3, H_4 or $I_2(m)$ for $m > 3$. Then Γ_1 is a forest with no unusual labels. As W_1 and W_i have the same type there is a permutation σ of the simple roots such that $\tilde{\sigma} : W_i \to W_1$ given by $\tilde{\sigma}(r_i) = r_{\sigma i}$ is an isomorphism. Then $\alpha\tilde{\sigma}$ is a reflection preserving automorphism of a Coxeter group whose diagram is a forest with no unusual labels. By Proposition 1.44 $\alpha\tilde{\sigma}$ is inner by graph. Thus, up to inner automorphisms of W_i, we may assume that $\alpha\tilde{\sigma}$ is a graph automorphism of W_i and hence that α maps r_j to a simple root for each $j \in \{2, 3, \ldots, n\}$. The result follows from Theorem 3.9.

So suppose that W_1 contains at least one component of type H_3, H_4 or $I_2(m)$ with $m > 3$. By Proposition 3.15 there are two possibilities: either W has a graph automorphism interchanging W_1 and W_2, so that up to automorphisms that are inner by graph we may assume that $\alpha(W_1) = W_1$, or else W has one or other of the following two diagrams.



The first of these requires special treatment, which we defer. The second may be dealt with by relabelling a_k to be a_1, since then W_1 becomes the unique maximal standard parabolic subgroup of type $A_2 \times I_2(5)$. Modifying α by an inner automorphism then permits us to assume that $\alpha(W_1) = W_1$.

Proceeding under the assumption that $\alpha(W_1) = W_1$, by Lemma 1.34 we may define ϕ_α so that

$$\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) \leq 0$$

for all i, $j \geq 2$ with $i \neq j$. As in Corollary 1.35 it follows that

$$\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) = a_i \cdot a_j \tag{3.17}$$

unless $m_{ij} = 5$ or $m_{ij} \geq 7$. Furthermore, these values for m_{ij} can only occur if a_i and a_j lie in an irreducible component of W_1 of type H_3, H_4 or $I_2(m)$, and then only for one pair of simple roots in the component.

If Eq. 3.17 does hold for all i, $j \geq 2$ then by Theorem 1.39 there exists $w \in W_1$ such that $\pm\phi_\alpha(wa_i) \in \Pi \setminus \{a_1\}$ for all $i \geq 2$, and hence conjugation by w followed by α permutes the simple reflections of W_1. Theorem 3.9 then applies, and it follows that α is inner by graph. It remains to deal with those cases in which W_1 has at least one irreducible component of type H_3, H_4 or $I_2(m)$ (where $m = 5$ or $m \geq 7$) for which Eq. 3.17 does not hold. We do this by adapting the argument used in the proof of Theorem 3.9 to the present situation.

Suppose that a_i and a_j are simple roots of W_1 for which Eq. 3.17 fails, and let $m = m_{ij}$. Thus $a_i \cdot a_j = -\,\mathrm{c}(\pi_m)$, and $\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) = -\,\mathrm{c}(j\pi_m)$ for some j coprime to m. Since $\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) \leq 0$ we have that $1 < j < m/2$. Note that for types H_3 and H_4 we have $m = 5$, and $j = 2$ is the only possibility.

We assume now that Eq. 3.17 fails for some i and j. We shall show that this leads to a contradiction.

Let V_1 be the subspace of V spanned by $\Pi \setminus \{a_1\}$, and let M be the Gram matrix of the restriction of our bilinear form to V_1 computed relative to the above basis. Let M' be the Gram matrix for the same form computed relative to the basis $\phi_\alpha(\Pi \setminus \{a_1\})$. We assume that the simple roots are ordered so that M is a diagonal sum of matrices corresponding to the various irreducible components of W_1. Since $\phi_\alpha(a_i) \cdot \phi_\alpha(a_j) = 0$ whenever $a_i \cdot a_j = 0$, we see that M' is also a diagonal sum, with blocks of the same sizes as those of M.

The blocks of M corresponding to components of types H_3, H_4 and $I_2(m)$ are as follows (assuming the ordering is chosen appropriately).

$$M_3 = \begin{bmatrix} 1 & -\operatorname{c}(\pi_5) & 0 \\ -\operatorname{c}(\pi_5) & 1 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 1 & -\operatorname{c}(\pi_5) & 0 & 0 \\ -\operatorname{c}(\pi_5) & 1 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{bmatrix}$$

$$M_I = \begin{bmatrix} 1 & -\operatorname{c}(\pi_m) \\ -\operatorname{c}(\pi_m) & 1 \end{bmatrix}$$

Letting $T_* = M_*^{-1}$ be the corresponding inverses we find:

$$T_3 = \begin{bmatrix} \frac{9+3\sqrt{5}}{2} & 4+2\sqrt{5} & 2+\sqrt{5} \\ 4+2\sqrt{5} & 6+2\sqrt{5} & 3+\sqrt{5} \\ 2+\sqrt{5} & 3+\sqrt{5} & \frac{5+\sqrt{5}}{2} \end{bmatrix}$$

$$T_4 = \begin{bmatrix} 28+12\sqrt{5} & 33+15\sqrt{5} & 22+10\sqrt{5} & 11+5\sqrt{5} \\ 33+15\sqrt{5} & 42+18\sqrt{5} & 28+12\sqrt{5} & 14+6\sqrt{5} \\ 22+10\sqrt{5} & 28+12\sqrt{5} & 20+8\sqrt{5} & 10+4\sqrt{5} \\ 11+5\sqrt{5} & 14+6\sqrt{5} & 10+4\sqrt{5} & 6+2\sqrt{5} \end{bmatrix}$$

$$T_I = \begin{bmatrix} 1/\operatorname{s}^2(\pi_m) & \operatorname{c}(\pi_m)/\operatorname{s}^2(\pi_m) \\ \operatorname{c}(\pi_m)/\operatorname{s}^2(\pi_m) & 1/\operatorname{s}^2(\pi_m) \end{bmatrix}$$

If ϕ_α does not preserve the form on such a component then the corresponding blocks of M' are as follows.

$$M'_3 = \begin{bmatrix} 1 & -\operatorname{c}(2\pi_5) & 0 \\ -\operatorname{c}(2\pi_5) & 1 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix}$$

$$M'_4 = \begin{bmatrix} 1 & -\operatorname{c}(2\pi_5) & 0 & 0 \\ -\operatorname{c}(2\pi_5) & 1 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{bmatrix}$$

$$M'_I = \begin{bmatrix} 1 & -\operatorname{c}(j\pi_m) \\ -\operatorname{c}(j\pi_m) & 1 \end{bmatrix}$$

Letting $T'_* = {M'}_*^{-1}$ be the corresponding inverses we find:

$$T'_3 = \begin{bmatrix} \frac{9-3\sqrt{5}}{2} & -4+2\sqrt{5} & -2+\sqrt{5} \\ -4+2\sqrt{5} & 6-2\sqrt{5} & 3-\sqrt{5} \\ -2+\sqrt{5} & 3-\sqrt{5} & \frac{5-\sqrt{5}}{2} \end{bmatrix}$$

$$T'_4 = \begin{bmatrix} 28-12\sqrt{5} & -33+15\sqrt{5} & -22+10\sqrt{5} & -11+5\sqrt{5} \\ -33+15\sqrt{5} & 42-18\sqrt{5} & 28-12\sqrt{5} & 14-6\sqrt{5} \\ -22+10\sqrt{5} & 28-12\sqrt{5} & 20-8\sqrt{5} & 10-4\sqrt{5} \\ -11+5\sqrt{5} & 14-6\sqrt{5} & 10-4\sqrt{5} & 6-2\sqrt{5} \end{bmatrix}$$

$$T'_I = \begin{bmatrix} 1/\operatorname{s}^2(j\pi_m) & \operatorname{c}(j\pi_m)/\operatorname{s}^2(j\pi_m) \\ \operatorname{c}(j\pi_m)/\operatorname{s}^2(j\pi_m) & 1/\operatorname{s}^2(j\pi_m) \end{bmatrix}$$

Note that each entry t'_{ij} of T'_* is positive but strictly less that the corresponding entry t_{ij} of T_*. (Recall that $\frac{m}{2} > j > 1$ so that, $\pi/2 > j\pi_m > \pi_m$ and hence $\mathrm{s}(j\pi_m) > \mathrm{s}(\pi_m)$ and $\mathrm{c}(j\pi_m) < \mathrm{c}(\pi_m)$.)

Hence if $T = (t_{ij}) = M^{-1}$ and $T' = (t'_{ij}) = M'^{-1}$, then we have $t_{ij} \geq t'_{ij}$ for all $i, j \in \{2, 3, \ldots, n\}$. Since there is a component for which the form is not preserved, there is a block on which $t_{ij} > t'_{ij}$ for all i and j corresponding to roots in that block.

As in the proof of Theorem 3.9 we suppose that $\alpha(r_1) = r_x$ where $x \in \Phi^+$, and let x_0 be the projection of x onto V_1. Let x_1 be the projection of $-a_1$ onto V_1. Then $u_1 = x_1 + a_1$ spans the orthogonal complement of V_1 in V, and $u_1 \cdot u_1 < 0$ since W is non-degenerate and infinite. Finally $x = x_0 + \omega u_1$ for some scalar ω, and by Lemma 1.15 we have $\omega \geq 1$.

For each $i \in \{2, 3, \ldots, n\}$ let

$$a_i \cdot a_1 = -\mathrm{c}(\pi_{m_{i1}}) = -c_i.$$

Write $x_1 = \sum_{i=2}^{n} \mu_i a_i$. Now $x_1 \cdot a_i = -a_1 \cdot a_i = c_i$, for $i \geq 2$ and so

$$\mu_i = \sum_{j=2}^{n} t_{ij} c_j.$$

For each $i \geq 2$ there is an integer j_i such that

$$x_0 \cdot \phi_\alpha(a_i) = x \cdot \phi_\alpha(a_i) = \mathrm{c}(j_i \pi_{m_{i1}}) = d_i,$$

where $|d_i| \leq c_i$ and $d_i = 0$ if and only if $c_i = 0$. Writing $x_0 = \sum_{i=2}^{n} \lambda_i \phi_\alpha(a_i)$, we have

$$\lambda_i = \sum_{j=2}^{n} t'_{ij} d_j$$

for $i \in \{2, 3, \ldots, n\}$. Now observe the following.

$$x_0 \cdot x_0 = \sum_{i=2}^{n} \lambda_i x_0 \cdot \phi_\alpha(a_i) = \sum_{i=2}^{n} \lambda_i d_i = \sum_{i=2}^{n} \sum_{j=2}^{n} t'_{ij} d_i d_j$$

and

$$x_1 \cdot a_1 = \sum_{i=2}^{n} \mu_i a_i \cdot a_1 = -\sum_{i=2}^{n} \mu_i c_i = -\sum_{i=2}^{n} \sum_{j=2}^{n} t_{ij} c_i c_j.$$

Since $c_i c_j \geq |d_i d_j| \geq d_i d_j$ and $t_{ij} \geq t'_{ij} \geq 0$ for all $i, j \in \{2, 3, \ldots, n\}$

$$\sum_i \sum_j t_{ij} c_i c_j \geq \sum_i \sum_j t'_{ij} d_i d_j.$$

But there is an irreducible component of W_1 for which $t_{ij} > t'_{ij}$. As W is irreducible there is an edge joining a_1 to this component, and hence there is an a_k in this component for which $c_k > 0$. Then $t_{kk} c_k^2 \geq t_{kk} d_k^2 > t'_{kk} d_k^2$,and so

$$-x_1 \cdot a_1 = \sum_i \sum_j t_{ij} c_i c_j > \sum_i \sum_j t'_{ij} d_i d_j = x_0 \cdot x_0.$$

Therefore $1 + x_1 \cdot a_1 < 1 - x_0 \cdot x_0$. Now

$$\begin{aligned} u_1 \cdot u_1 &= (x_1 + a_1) \cdot u_1 \\ &= a_1 \cdot u_1 \\ &= a_1 \cdot a_1 + a_1 \cdot x_1 \\ &= 1 + a_1 \cdot x_1. \end{aligned}$$

Thus $u_1 \cdot u_1 < 1 - x_0 \cdot x_0$ and since $u_1 \cdot u_1 < 0$,

$$1 > \frac{1 - x_0 \cdot x_0}{u_1 \cdot u_1}.$$

Since $x \in \Phi$

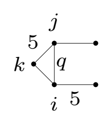
$$\begin{aligned} 1 &= x \cdot x \\ &= (x_0 + \omega u_1) \cdot (x_0 + \omega u_1) \\ &= x_0 \cdot x_0 + \omega^2 u_1 \cdot u_1. \end{aligned}$$

Hence

$$\omega^2 = \frac{1 - x_0 \cdot x_0}{u_1 \cdot u_1} < 1.$$

But $\omega \geq 1$ and we have obtained the desired contradiction.

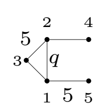
It only remains to deal with groups that have Coxeter diagram



(for which we cannot assume that $\alpha(W_1) = W_1$). These are considered in detail in the next section. ∎

§3.5 Some Examples

To complete the proof of Theorem 3.16 and to foreshadow the arguments used in the next chapter we look at two examples. We find $\mathrm{Aut}(W)$ for the nearly finite Coxeter group with diagram



where $q > 2$. After this example we consider nearly finite groups with W_1 irreducible of type A_{n-1} for $n \neq 6$, D_{2k+1}, E_6 or E_7. First some preliminary results.

3.18 Lemma *Let W be a Coxeter group with a_i, a_j two simple roots. If the edge joining the nodes corresponding to a_i and a_j is labelled with an odd number, then r_i and r_j are conjugate.*

Proof Suppose the edge joining the nodes is labelled $2k+1$. Then $(r_i r_j)^{2k+1} = 1$ and so

$$r_i = (r_j r_i)^k r_j (r_i r_j)^k = (r_j r_i)^k r_j (r_i r_j)^{-k}$$

whence r_i and r_j are conjugate. ∎

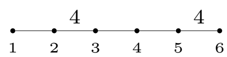
3.19 Lemma *Let W be a Coxeter group with Π the set of simple roots. If we can find subsets I and J of Π such that $\Pi = I \dot{\cup} J$ and any edge joining a vertex in I to a vertex in J has an even label (or is ∞), then no reflection in W_I is conjugate to a reflection in W_J.*

Proof For each simple reflection r_i define $f(r_i) \in \{1, -1\}$ as follows: if $a_i \in I$, then $f(r_i) = 1$ while if $a_j \in J$, then $f(r_j) = -1$. If $a_i \in I$ and $a_j \in J$ then $m_{ij} = 2k$ for some k, and hence

$$\big(f(r_i)f(r_j)\big)^{m_{ij}} = (-1)^{2k} = 1.$$

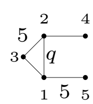
(If $m_{ij} = \infty$ there is no relation.) For other values of i and j the equation holds trivially. Thus f defines a homomorphism from W to an abelian group. If $r_i \in W_I$ and $r_j \in W_J$, then $f(r_i) \neq f(r_j)$ and therefore r_i and r_j cannot be conjugate. ∎

Using these last two results it is possible to write down representatives of the conjugacy classes of reflections. Choosing a simple reflection r_i, any other simple reflection connected to r_i by a path whose labels are all odd is in the conjugacy class of r_i. No other simple reflection is in this conjugacy class. Consider, for example, the following diagram:



There are three conjugacy classes of reflections with representatives r_1, r_3 and r_6.

Suppose now that W is a Coxeter group with diagram



Then every simple reflection is connected to r_1 by a path on which every edge has an odd label. By Lemma 3.18 this implies that all simple reflections are conjugate to r_1, and hence that W has a single conjugacy class of reflections.

As we know, W_1 and W_2 are maximal finite standard parabolic subgroups of type $A_1 \times H_3$. If $q > 5$, then $W_{\{a_1,a_2\}}$ is also a maximal finite subgroup. Thus $\langle r_2 \rangle = W_1 \cap W_{\{a_1,a_2\}}$ and by Lemma 1.32 every automorphism of W maps r_2 to a reflection. There is only one class of reflections and hence all automorphisms preserve reflections. Similarly, if $q \leq 5$ then $W_{\{a_1,a_2,a_4\}}$ is a maximal finite subgroup, and since

$$\langle r_4 \rangle = W_1 \cap W_2 \cap W_{\{a_1,a_2,a_4\}}$$

we see that again automorphisms of W preserve reflections. Thus $\mathrm{Aut}(W) = R(W)$.

Let $\alpha \in \mathrm{Aut}(W)$. Since α must map the maximal finite subgroup W_1 to another maximal finite subgroup, up to inner automorphisms we may assume that $\alpha(W_1)$ is W_1 or W_2. If $\alpha(W_1) = W_1$ then the argument used in Theorem 3.16 shows that α is inner by graph. So suppose that $\alpha(W_1) = W_2$.

The simple reflection r_5 is central in W_1; therefore

$$\alpha(r_5) \in Z(W_2) = \langle r_4, w_2 \rangle,$$

where w_2 is the longest element in W_2. As $\alpha(r_5)$ is a reflection and r_4 is the only reflection in $Z(W_2)$ we can see that $\alpha(r_5) = r_4$. By Lemma 1.17 $\mathrm{Ref}(W_1) = \mathrm{Ref}(W_{\{a_2,a_3,a_4\}}) \dot\cup \{r_5\}$ while $\mathrm{Ref}(W_2) = \mathrm{Ref}(W_{\{a_1,a_3,a_5\}}) \dot\cup \{r_4\}$. As $\alpha(r_5) = r_4$ we can see that the reflections in $W_{\{a_2,a_3,a_4\}}$ are mapped to reflections in $W_{\{a_1,a_3,a_5\}}$ by α, and thus

$$\alpha(W_{\{a_2,a_3,a_4\}}) = W_{\{a_1,a_3,a_5\}}.$$

Let σ be the permutation $(12)(345)$. Then $\tilde\sigma : W_{\{a_1,a_3,a_5\}} \to W_{\{a_2,a_4,a_3\}}$ given by $\tilde\sigma(r_i) = r_{\sigma i}$ is an isomorphism. Thus $\alpha\tilde\sigma$ is an automorphism of $W_{\{a_1,a_3,a_5\}}$, a Coxeter group of type H_3. Using Proposition 2.11 $\alpha\tilde\sigma$ is either inner or we may follow $\alpha\tilde\sigma$ by an inner automorphism and obtain the automorphism ξ. Hence, following α by conjugation by an element of

$W_{\{a_1,a_3,a_5\}}$ we may assume that $\alpha(W_1) = W_2$, $\alpha(r_5) = r_4$ and either $\alpha|_{W_{\{a_1,a_3,a_5\}}} = \xi$ or else $\alpha(\{r_2, r_3, r_4, r_5\}) = \{r_1, r_3, r_4, r_5\}$. In the second case α is inner by graph by Theorem 3.9.

Suppose, for a contradiction, that $\alpha|_{W_{\{a_1,a_3,a_5\}}} = \xi$. Then we have

$$\begin{aligned}
\alpha(r_1) &= r_x \\
\alpha(r_2) &= r_1 \\
\alpha(r_3) &= r_{a_5'} \\
\alpha(r_4) &= r_3 \\
\alpha(r_5) &= r_4
\end{aligned}$$

where $a_5' = -2la_1 - la_3 - (l+1)a_5$, $l = (1+\sqrt{5})/2$. Note that $a_5' \cdot a_1 = (1-\sqrt{5})/4$, $a_5' \cdot a_3 = 0$ and $a_5' \cdot a_4 = 0$. As $r_x r_{a_5'}$ and $r_1 r_3$ both have order 3, we can see that $x \cdot a_5' = \pm 1/2$. Given that $r_x = r_{-x}$ we may replace x with $-x$, if necessary to ensure that $x \cdot a_5' = -1/2$. Similar arguments show that $x \cdot a_1 = \mathrm{c}(j\pi_q)$ for some j coprime to q, $x \cdot a_3 = 0$ and $x \cdot a_4 = -\mathrm{c}(k\pi_5)$ for some k coprime to 5. As usual note that $\mathrm{c}(\pi_q) \geq |\mathrm{c}(j\pi_q)|$ and $\mathrm{c}(\pi_5) \geq |\mathrm{c}(k\pi_5)|$.

Let $x_0 = \left(\frac{1+\sqrt{5}}{2}\right) a_1 + \left(\frac{1+\sqrt{5}}{4}\right) a_3 - \mathrm{c}(k\pi_5) a_4 + \left(\frac{3+2(1-\sqrt{5})\,\mathrm{c}(j\pi_q)}{2}\right) a_5$.

$$\begin{aligned}
x_0 \cdot a_1 &= \frac{1+\sqrt{5}}{2} - \frac{1}{2}\left(\frac{1+\sqrt{5}}{4}\right) \\
&\qquad + 0 - \frac{1+\sqrt{5}}{4}\left(\frac{3+2(1-\sqrt{5})\,\mathrm{c}(j\pi_q)}{2}\right) \\
&= \frac{1}{8}(4 + 4\sqrt{5} - 1 - \sqrt{5} - 3 - 3\sqrt{5} \\
&\qquad - 2(1+\sqrt{5})(1-\sqrt{5})\,\mathrm{c}(j\pi_1)) \\
&= \mathrm{c}(j\pi_q) = x \cdot a_1 \\
x_0 \cdot a_3 &= -\frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right) + \frac{1+\sqrt{5}}{4} + 0 + 0 \\
&= 0 = x \cdot a_3 \\
x_0 \cdot a_4 &= 0 + 0 - \mathrm{c}(k\pi_5) + 0 \\
&= x \cdot a_4 \\
x_0 \cdot a_5' &= \left(\frac{1+\sqrt{5}}{2}\right) a_1 \cdot a_5' + \left(\frac{1+\sqrt{5}}{4}\right) a_3 \cdot a_5' \\
&\qquad - \mathrm{c}(k\pi_5) a_4 \cdot a_5' + \left(\frac{3+2(1-\sqrt{5})\,\mathrm{c}(j\pi_q)}{2}\right) a_5 \cdot a_5' \\
&= \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{4}\right) + 0 + 0 \\
&\qquad + \left(\frac{3+2(1-\sqrt{5})\,\mathrm{c}(j\pi_q)}{2}\right)\left((1+\sqrt{5})\frac{1+\sqrt{5}}{4} + 0 - \frac{3+\sqrt{5}}{2}\right) \\
&= -\frac{1}{2} + \left(\frac{3+2(1-\sqrt{5})\,\mathrm{c}(j\pi_q)}{2}\right)\left(\frac{3+\sqrt{5}}{2} - \frac{3+\sqrt{5}}{2}\right) \\
&= -\frac{1}{2} = x \cdot a_5'.
\end{aligned}$$

Thus x_0 is the projection of x onto V_2, the space spanned by $\{a_1, a_3, a_4, a_5\}$.

Let

$$x_2 = \bigl(2+\sqrt{5}+2(3+\sqrt{5})\,\mathrm{c}(\pi_q)\bigr)a_1 + \frac{1}{4}\bigl(5+3\sqrt{5}+4(3+\sqrt{5})\,\mathrm{c}(\pi_q)\bigr)a_3 + \frac{1}{2}a_4 + \frac{1}{4}\bigl(7+3\sqrt{5}+8(2+\sqrt{5})\,\mathrm{c}(\pi_q)\bigr)a_5.$$

Then

$$\begin{aligned}
x_2\cdot a_1 &= 2+\sqrt{5}+2(3+\sqrt{5})\,\mathrm{c}(\pi_q) - \frac{1}{8}\bigl(5+3\sqrt{5}+4(3+\sqrt{5})\,\mathrm{c}(\pi_q)\bigr)\\
&\qquad + 0 - \frac{1+\sqrt{5}}{16}\bigl(7+3\sqrt{5}+8(2+\sqrt{5})\,\mathrm{c}(\pi_q)\bigr)\\
&= \mathrm{c}(\pi_q) = -a_2\cdot a_1\\
x_2\cdot a_3 &= -\frac{1}{2}\bigl(2+\sqrt{5}+2(3+\sqrt{5})\,\mathrm{c}(\pi_q)\bigr)\\
&\qquad + \frac{1}{4}\bigl(5+3\sqrt{5}+4(3+\sqrt{5})\,\mathrm{c}(\pi_q)\bigr) + 0 + 0\\
&= \frac{1+\sqrt{5}}{4} = -\,\mathrm{c}(\pi_5) = -a_2\cdot a_3\\
x_2\cdot a_4 &= \frac{1}{2} = -a_2\cdot a_4\\
x_2\cdot a_5 &= -\frac{1+\sqrt{5}}{4}\bigl(2+\sqrt{5}+2(3+\sqrt{5})\,\mathrm{c}(\pi_q)\bigr) + 0 + 0\\
&\qquad + \frac{1}{4}\bigl(7+3\sqrt{5}+8(2+\sqrt{5})\,\mathrm{c}(\pi_q)\bigr)\\
&= 0 = -a_2\cdot a_5.
\end{aligned}$$

Thus x_2 is the projection of $-a_2$ onto V_2. As before this tells us that $u_2 = x_2 + a_2$ is orthogonal to V_2.

$$\begin{aligned}
u_2\cdot u_2 &= (x_2+a_2)\cdot u_2 = a_2\cdot u_2 = a_2\cdot x_2 + 1\\
&= -\frac{1+\sqrt{5}}{4}\bigl(2+\sqrt{5}+2(3+\sqrt{5})\,\mathrm{c}(\pi_q)\bigr)\\
&\qquad - \left(\frac{1+\sqrt{5}}{16}\right)\bigl(5+3\sqrt{5}+4(3+\sqrt{5})\,\mathrm{c}(\pi_q)\bigr) - \frac{1}{4} + 0 + 1\\
&= -\frac{1+\sqrt{5}}{2} - 2(2+\sqrt{5})\,\mathrm{c}(\pi_q) - 2(3+\sqrt{5})\,\mathrm{c}^2(\pi_q) < 0,
\end{aligned}$$

as $\mathrm{c}(\pi_q) > 0$. Together with that fact that W_2 is positive definite, this implies that W is non-degenerate.

We find that

$$\begin{aligned}
x_0\cdot a_5 &= -\left(\frac{1+\sqrt{5}}{4}\right)\left(\frac{1+\sqrt{5}}{2}\right) + 0 + 0 + \frac{3+2(1-\sqrt{5})\,\mathrm{c}(j\pi_q)}{2}\\
&= \frac{1}{4}\bigl(3-\sqrt{5}+4(1-\sqrt{5})\,\mathrm{c}(j\pi_q)\bigr)
\end{aligned}$$

and hence it follows that

$$\begin{aligned}
x_0\cdot x_0 &= \frac{1+\sqrt{5}}{2}a_1\cdot x_0 + \left(\frac{1+\sqrt{5}}{4}\right)a_3\cdot x_0 - \mathrm{c}(k\pi_5)a_4\cdot x_0\\
&\qquad + \left(\frac{3+2(1-\sqrt{5})\,\mathrm{c}(j\pi_q)}{2}\right)a_5\cdot x_0\\
&= \mathrm{c}^2(k\pi_5) + \frac{9-3\sqrt{5}}{8} + 2(2-\sqrt{5})\,\mathrm{c}(j\pi_q) + 2(3-\sqrt{5})\,\mathrm{c}^2(j\pi_q).
\end{aligned}$$

Note that $c^2(\pi_5) = (3+\sqrt{5})/8$. Now

$$\begin{aligned}
1 - x_0 \cdot x_0 - u_2 \cdot u_2 = 1 - c^2(k\pi_5) &- \frac{9-3\sqrt{5}}{8} - 2(2-\sqrt{5})\,c(j\pi_q) - 2(3-\sqrt{5})\,c^2(j\pi_q)\\
&+ \frac{1+\sqrt{5}}{2} + 2(2+\sqrt{5})\,c(\pi_q) + 2(3+\sqrt{5})\,c^2(\pi_q)\\
&+ c^2(\pi_5) - \frac{3+\sqrt{5}}{8}\\
= \big(c^2(\pi_5) - c^2(k\pi_5)\big) &+ \big(2(2+\sqrt{5})\,c(\pi_q) - 2(2-\sqrt{5})\,c(j\pi_q)\big)\\
&+ \big(2(3+\sqrt{5})\,c^2(\pi_q) - 2(3-\sqrt{5})\,c^2(j\pi_q)\big).
\end{aligned}$$

We claim that this last expression is strictly positive. First observe that $c^2(\pi_5) - c^2(k\pi_5) \geq 0$. Next we have

$$2(3+\sqrt{5})\,c^2(\pi_q) > 2(3-\sqrt{5})\,c^2(\pi_q) \geq 2(3-\sqrt{5})\,c^2(j\pi_q).$$

Finally, since $c(j\pi_q) \geq -c(\pi_q)$ and $2-\sqrt{5} < 0$,

$$2(2-\sqrt{5})\,c(j\pi_q) \leq 2(\sqrt{5}-2)\,c(\pi_q) < 2(2+\sqrt{5})\,c(\pi_q).$$

Thus

$$2(2+\sqrt{5})\,c(\pi_q) - 2(2-\sqrt{5})\,c(j\pi_q) > 0 \quad \text{and} \quad \big(2(3+\sqrt{5})\,c^2(\pi_q) - 2(3-\sqrt{5})\,c^2(j\pi_q)\big) > 0$$

establishing our claim. Therefore $\frac{1-x_0\cdot x_0}{u_2\cdot u_2} < 1$, as $u_2 \cdot u_2 < 0$.

Following the proof of Theorem 3.16, if we write $x = x_0 + \omega u_2$ then either $\omega \geq 1$ (if $x \in \Phi^+$) or $\omega \leq -1$ (if $x \in \Phi^-$), and hence $\omega^2 \geq 1$. We know that x is a root, thus

$$\begin{aligned}
1 = x \cdot x &= (x_0 + \omega u_2) \cdot (x_0 + \omega u_2)\\
&= x_0 \cdot x_0 + \omega^2 u_2 \cdot u_2
\end{aligned}$$

and hence

$$\omega^2 = \frac{1 - x_0 \cdot x_0}{u_2 \cdot u_2} < 1$$

which is a contradiction. So this case cannot arise and we conclude that all automorphisms are inner by graph. This completes the proof of Theorem 3.16.

If there are many maximal finite standard parabolic subgroups then, as the above example illustrates, there is a chance that all automorphisms preserve reflections. At the other extreme the following result looks at some nearly finite Coxeter groups where it is only assumed that one standard parabolic subgroup is finite.

3.20 Corollary *Suppose W is a non-degenerate nearly finite Coxeter group such that W_i is of type A_{n-1} for $n \neq 6$, D_{2k+1}, E_6 or E_7. If there is an $l \neq i$ such that m_{il} is odd, then all automorphisms of W are inner by graph.*

Proof If m_{il} is odd then in the Coxeter diagram of W there is an edge with an odd label incident with the vertex corresponding to r_i. By assumption, all the edges in the diagram of W_i have odd labels. Thus, by Lemma 3.18, W has only one conjugacy class of reflections.

Let $\alpha \in \mathrm{Aut}(W)$ be any automorphism of W. Up to inner automorphisms we may assume that $\alpha(W_i) = W_j$ where W_j has the same type as W_i, by Theorem 3.14. Thus there is a permutation $\sigma \in \mathrm{Sym}_n$ such that $\tilde{\sigma} : W_j \to W_i$ given by $\tilde{\sigma}(r_t) = r_{\sigma t}$ is an isomorphism. Then $\alpha\tilde{\sigma}$ is an automorphism of a group of type A_{n-1} for $n \neq 6$, D_{2k+1}, E_6 or E_7. By Propositions 2.4, 2.8 and 2.9 this automorphism is inner. In particular this implies that α maps the reflections in W_i to reflections in W_j. But W only has one class of reflections and hence α preserves reflections and the result follows by Theorem 3.16. ∎

Chapter 4

Affine Weyl Groups, Hyperbolic Groups and Other Infinite Coxeter Groups

In the last chapter we showed that if W is a nearly finite Coxeter group, then any reflection-preserving automorphism is inner by graph. The question remains whether the reflection preserving hypothesis is necessary. In this chapter it will be shown that all automorphisms of hyperbolic Coxeter groups and affine Weyl groups are inner by graph. The proofs in this chapter will follow the same plan. First it will be shown that any automorphism must preserve $\mathrm{Ref}(W)$, the set of reflections. In cases where the diagram of the group is a forest with no unusual labels this will finish the proof, by Corollary 1.44. In all but four of the remaining cases the group is a non-degenerate nearly finite group, and the result follows by Theorem 3.16. The last four cases are dealt with by separate arguments.

§4.1 Preliminaries

4.1 Proposition *Suppose that W is a Coxeter group, with $\Pi = \{ a_1, \ldots, a_n \}$ the set of simple roots. Suppose that α is an automorphism of W and*

$$\alpha(r_i) = r_{b_i}$$

for all $r_i \in \Pi$. Let M and N be the $n \times n$ matrices whose (i, j) entries are (respectively) $a_i \cdot a_j$ and $b_i \cdot b_j$, then M and N have the same signature.

Proof Identify elements of V with their coordinate vectors (written as column vectors) relative to the basis Π, and let $X = (b_1 | \cdots | b_n)$ be the matrix with the b_i as columns. We have $N = X^t M X$. If X is nonsingular we see that N and M have the same signature.

If X is singular then $\{ b_1, \ldots, b_n \}$ span a proper subspace U of V. However, if $g = r_{v_1} r_{v_2} \ldots r_{v_l}$ for some $r_{v_i} \in \Phi$, and if $v \in V$ is arbitrary, then

$$gv - v = \sum_{i=2}^{l} (r_{v_i}) \big(r_{v_{i+1}} \ldots r_{v_l}(v) \big)$$

is in the space spanned by $v_1, v_2, \ldots, v_l$. Since $r_{b_1}, r_{b_2}, \ldots, r_{b_n}$ generate W it follows that U contains $gv - v$ for all $g \in W$ and $v \in V$. In particular, U contains $r_i(a_i) - a_i = 2a_i$ for all i, contradicting the fact that U is a proper subspace of V. ∎

4.2 Corollary *Let $\alpha : W \to W$ be a homomorphism, with $\alpha(r_i) = r_{b_i}$ for all i. If the matrices $(a_i \cdot a_j)$ and $(b_i \cdot b_j)$ have different signatures, then α is not an automorphism.*

4.3 Lemma *If a reflection is conjugate to a simple reflection r_i with the property that $\langle r_i \rangle$ can be written as an intersection of maximal finite subgroups, then all automorphisms of W map that reflection to a reflection.*

Proof Using Lemma 1.32 we know that r_i is mapped to a reflection by all automorphisms and the rest is clear. ∎

4.4 Lemma *If W is a Coxeter group such that W_i is finite for all i, then any automorphism of W maps reflections to reflections.*

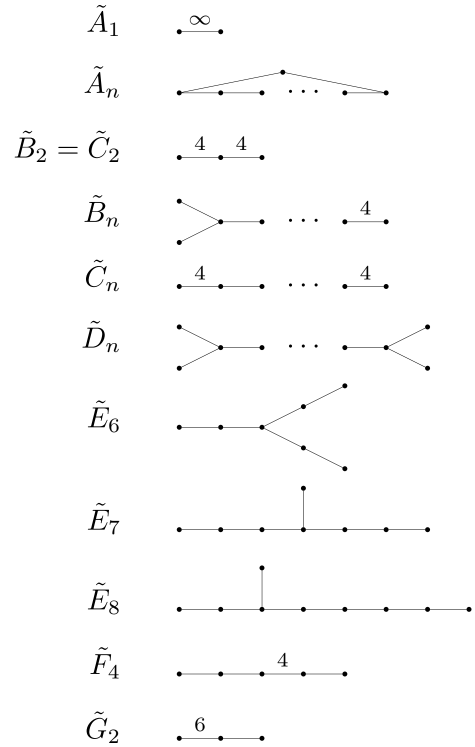
Proof If r_i is any simple root, then $\langle r_i \rangle = \bigcap_{j \neq i} W_i$ is the intersection of a collection of maximal finite standard parabolic subgroups. By Corollary 1.31, if α is any automorphism of W, then $\alpha\langle r_i \rangle$ is a parabolic subgroup of order 2. Thus $\langle r_i \rangle$ is conjugate to a parabolic subgroup of the form $\langle r_k \rangle$ for some simple reflection r_k. Hence $\alpha(r_i)$ is conjugate to r_k and so is a reflection. ∎

This lemma applies exactly to the compact hyperbolic Coxeter groups and the affine Weyl groups.

§4.2 Affine Weyl Groups

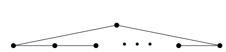
In the Chapter 2 the automorphisms of the finite irreducible Coxeter groups were classified. It was mentioned, in Lemma 1.1, that a Coxeter group is finite if and only if it is positive definite. The obvious next class to consider is the class of positive semi-definite Coxeter groups; it is well-known that these are isomorphic to the affine Weyl groups.

The following is a list of the positive semi-definite Coxeter groups.



By convention the $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ and $\tilde{D}_n$ diagrams have $n+1$ vertices. Note that if W is a Coxeter group of one of the above types then W_i is finite for all i. Thus, by Lemma 4.4, if α is any automorphism of a group W of one of the above types, then α preserves $\mathrm{Ref}(W)$. Therefore except possibly for $\tilde{A}_1$ and $\tilde{A}_n$, α is inner by graph, by Corollary 1.44.

$\tilde{A}_n$: Let W be a Coxeter group of type $\tilde{A}_n$, where $n > 1$, with diagram



First observe that $a_1 + a_2 + \cdots + a_{n+1}$ is orthogonal to a_i for all i and so B is degenerate. Letting $\alpha(r_i) = r_{b_i}$ where $b_i \in \Phi$ we may define $\phi_\alpha : V \to V$ by $\phi_\alpha(a_i) = b_i$. Noting that there are two choices for b_i and replacing b_{i+1} with $-b_{i+1}$ at need we may ensure $b_i \cdot b_{i+1} = a_i \cdot a_{i+1}$

for $i = 1$ to $n-1$. If $b_n \cdot b_1 = a_n \cdot a_1$ then ϕ_α is orthogonal and α is inner by graph by Theorem 1.44. If $\phi_\alpha(a_n) \cdot \phi_\alpha(a_1) = -a_n \cdot a_1 = 1/2$ instead of $-1/2$ then

$$\det(b_i \cdot b_j) = \det \begin{bmatrix} 1 & -1/2 & 0 & \cdots & 0 & 1/2 \\ -1/2 & 1 & -1/2 & \cdots & 0 & 0 \\ 0 & -1/2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1/2 \\ 1/2 & 0 & 0 & \cdots & -1/2 & 1 \end{bmatrix} = \frac{1}{2^{n-2}} \neq 0.$$

This means that the matrices $(a_i \cdot a_j)$ and $(b_i \cdot b_j)$ have different signatures and hence, by Proposition 4.1, this case cannot arise. Hence $\mathrm{Aut}(W) \cong \mathrm{Inn}(W) \rtimes \mathrm{Gr}(W) \cong W \rtimes I_2(n)$, using Lemma 2.16.

$\tilde{A}_1$: Let W be a Coxeter group of type $\tilde{A}_1$ with diagram

$$\bullet \overset{\infty}{\text{---}} \bullet$$

Modulo inner by graph automorphisms we may assume that $\alpha(r_1) = r_1$. Now $r_1 r_2$ generates an infinite cyclic subgroup that is characteristic (being the unique cyclic subgroup of index 2). So $\alpha(r_1 r_2)$ is $r_1 r_2$ or $r_2 r_1$, whence $\alpha(r_2)$ is r_2 or $r_1 r_2 r_1$, and so α is the identity or conjugation by r_1. Thus we have proved the following.

4.5 Proposition *If W is a Coxeter group with B positive semi-definite, then all automorphisms of W are inner by graph, and hence*

$$\mathrm{Aut}(W) \cong W \rtimes \mathrm{Gr}(W).$$

§4.3 Hyperbolic Coxeter Groups

In this section we follow the terminology of Humphrey's book [Hum90]. The following definitions are taken almost directly from §6.8 of [Hum90].

4.6 Definition Denote by ω_s, $s \in \Pi$, the basis dual to the basis a_s, $s \in \Pi$, relative to B. The cone C is defined as follows.

$$C = \{\, \lambda \in V \mid B(\lambda, a_s) > 0 \text{ for all } s \in \Pi \,\} = \left\{ \sum c_s \omega_s \mid c_s > 0 \right\}.$$

In particular, all ω_s lie in the closure D of C, which is a fundamental domain for the action of W on $\bigcup_{w \in W} w(C)$, a subset of the dual space.

4.7 Definition Define the irreducible Coxeter group W, with simple roots Π, to be *hyperbolic* if B has signature $(n-1, 1)$ and $B(\lambda, \lambda) < 0$ for all $\lambda \in C$.

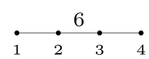
Humphreys' definition, while being common, is not universally used. In §6.9 of [Hum90] Humphreys provides a list of the hyperbolic Coxeter groups. The only infinite classes of hyperbolic Coxeter groups are the rank three cases. If we exclude infinite bonds it is clear that if W is an infinite rank three non-degenerate Coxeter group then W_i is finite for all i and so W is nearly finite. Thus by Lemma 4.4 and Theorem 3.16 all automorphisms of W are inner by graph. The degenerate cases are dealt with by Proposition 4.5, while rank 3 Coxeter groups with infinite bonds are dealt with in Chapter 5.

We now concentrate on the rank ≥ 4 case. See Table I on page 70 for a complete list of the groups mentioned in Humphreys. Immediately following Table I on page 74 is a full explanation of how to read this table. For later reference we have numbered the hyperbolic Coxeter groups listed in [Hum90]. Table I gives this numbering, which roughly follows the ordering of the diagrams by size in [Hum90] but is otherwise arbitrary.

It should be noted at this stage that not all hyperbolic groups are nearly finite. For example, a group W of type $\mathcal{H}_{17}$ is hyperbolic but no parabolic subgroup of the form W_i is finite. Note that all non-degenerate nearly finite groups have signature $(n-1,1)$ and so satisfy part of the definition of hyperbolic Coxeter groups.

In view of Corollary 1.44 and Theorem 3.16 there are several ways of showing that all automorphisms of W are inner by graph. First we show that all automorphisms preserve reflections. If the Coxeter diagram of W is a forest with no unusual labels, then we are finished. Similarly if W is nearly finite. In other cases a different argument is used.

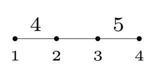
In some cases all reflections are conjugate to a simple reflection with the property mentioned in Lemma 4.3 and so all automorphisms preserve $\mathrm{Ref}(W)$. For example, if W is a Coxeter group of type $\mathcal{H}_{32}$ on Humphreys' list with diagram



the maximal finite standard parabolic subgroups of W are W_2, W_3 and $W_{\{a_2,a_3\}}$. Now $\langle r_2\rangle = W_{\{a_2,a_3\}} \cap W_3$ and $\langle r_3\rangle = W_{\{a_2,a_3\}} \cap W_2$. Furthermore $r_1 = r_2r_1r_2r_1r_2$ is conjugate to r_2 and similarly r_4 is conjugate to r_3. Any reflection in W is conjugate to a simple reflection and so by Lemma 4.3 all automorphisms of W map reflections to reflections.

4.8 Notation If W is a Coxeter group with Π the set of simple roots, then the parabolic subgroup $W_{\Pi\setminus\{a_{i_1},a_{i_2},\ldots\}}$ will be denoted by $W_{i_1 i_2 \ldots}$.

$\mathcal{H}_{01}$: Let W be a Coxeter group of type $\mathcal{H}_{01}$ with diagram

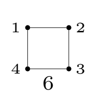


It is easily checked that the maximal finite standard parabolic subgroups are: W_1 of type H_3, W_2 of type $A_1 \times I_2(5)$, W_3 of type $I_2(4) \times A_1$ and W_4 of type B_3. It is easy to see that

$$\langle r_1\rangle = W_2 \cap W_3 \cap W_4$$

and so, by Lemma 1.32, r_1 is mapped to a reflection by all automorphisms. A similar argument can be used for the remaining reflections. A similar argument can be used for the groups: $\mathcal{H}_{01}$–$\mathcal{H}_{14}$ (the compact hyperbolic groups), $\mathcal{H}_{17}$, $\mathcal{H}_{21}$, $\mathcal{H}_{36}$, $\mathcal{H}_{37}$ and $\mathcal{H}_{55}$. Except for the compact groups the arguments are summarized in Table II on page 74. The first 14 groups listed there are the compact hyperbolic groups and we may use Lemma 4.4 to show that $\mathrm{Ref}(W)$ is preserved.

$\mathcal{H}_{18}$: Let W be a Coxeter group of type $\mathcal{H}_{18}$ with diagram



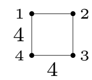
The maximal finite standard parabolic subgroups are: $W_3 \cong W_4$ of type A_3 and W_{12} of type $I_2(6)$. Now

$$\langle r_3\rangle = W_{12} \cap W_4$$

and all simple reflections are conjugate to r_3. So, by Lemma 4.3, all simple reflections are mapped to reflections by any automorphism. Hence the set of reflections is preserved by all automorphisms.

A similar argument can be used for the groups: $\mathcal{H}_{18}-\mathcal{H}_{20}$, $\mathcal{H}_{22}$, $\mathcal{H}_{23}$, $\mathcal{H}_{25}$, $\mathcal{H}_{26}$, $\mathcal{H}_{32}$, $\mathcal{H}_{34}$, $\mathcal{H}_{42}$, $\mathcal{H}_{44}$, $\mathcal{H}_{45}$, $\mathcal{H}_{48}$, $\mathcal{H}_{50}$, $\mathcal{H}_{52}-\mathcal{H}_{54}$, $\mathcal{H}_{56}-\mathcal{H}_{58}$, $\mathcal{H}_{60}$, $\mathcal{H}_{61}$, $\mathcal{H}_{63}-\mathcal{H}_{66}$, $\mathcal{H}_{68}-\mathcal{H}_{70}$ and $\mathcal{H}_{72}$. These arguments are summarized in Table III on page 75.

$\mathcal{H}_{15}$: Let W be a Coxeter group of type $\mathcal{H}_{15}$ with diagram



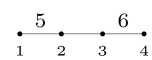
The maximal finite standard parabolic subgroups are: $W_1 \cong W_3$ of type B_3 and W_4 of type A_3. Now

$$\langle r_2 \rangle = W_1 \cap W_3 \cap W_4$$

and so r_2 is always mapped to a reflection. Using the argument from $\mathcal{H}_{18}$ we can see that r_1 and r_3 are conjugate to r_2 and so are also always mapped to reflections. Now look at W_1 of type B_3. Suppose that α is an automorphism of W. Then $\alpha(W_1)$ is a maximal finite subgroup of W of order 48. Thus $\alpha(W_1)$ is a conjugate of either W_1 or W_3, and so, up to inner automorphisms, we may assume that $\alpha(W_1) = W_1$ or W_3. Now there is a graph automorphism which interchanges W_1 and W_3 and hence, up to inner by graph automorphisms, we can assume that α fixes W_1. Thus $\alpha|_{W_1}$ is an automorphism of a group of type B_3, and so r_4 is mapped to a reflection, by Proposition 2.8. As inner by graph automorphisms preserve the set of reflections the original automorphism must also map reflections to reflections. (Note that it is conceivable that the image of a parabolic subgroup of type B_{2k+1} could be a parabolic subgroup of type $A_1 \times D_{2k+1}$, but for this diagram there are no parabolics of type $A_1 \times D_3$.)

A similar argument can be used for the groups: $\mathcal{H}_{15}$, $\mathcal{H}_{35}$, $\mathcal{H}_{51}$, $\mathcal{H}_{62}$ and $\mathcal{H}_{71}$. The Table IV on page 76 summarizes the arguments.

$\mathcal{H}_{30}$: Let W be a Coxeter group of type $\mathcal{H}_{30}$ with diagram



The maximal finite standard parabolic subgroups are: W_2 of type $A_1 \times I_2(6)$, W_3 of type $I_2(5) \times A_1$ and W_4 of type H_3. The simple reflections r_1, r_2 and r_3 can be dealt with by noting that they are all conjugate and

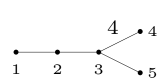
$$\langle r_1 \rangle = W_2 \cap W_3 \cap W_4.$$

Suppose that α is an automorphism and look at $\alpha(W_3)$ which must be conjugate to a maximal finite standard parabolic subgroup. A consideration of the orders shows that, up to inner automorphisms, we may assume that α fixes W_3 setwise. Now the centre of W_3 is $\langle r_4 \rangle$ and so $\alpha(r_4)$ is the non-identity element in

$$Z\big(\alpha(W_3)\big) = Z(W_3) = \langle r_4 \rangle$$

and so $\alpha(r_4) = r_4$. Thus all automorphisms preserve the set of reflections.

In some cases this argument must be applied to a parabolic subgroup which is the intersection of several maximal finite standard parabolic subgroups. For example consider the argument for a group W of type $\mathcal{H}_{39}$ with diagram



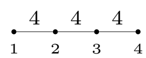
There are two classes of reflections with r_1 and r_4 as representatives. The maximal finite standard parabolics are W_2, W_3, W_4 and W_5 and it is easy to deal with r_1 as

$$\langle r_1 \rangle = W_2 \cap W_3 \cap W_4 \cap W_5.$$

Now observe that $\langle r_4 \rangle$ is the centre of the parabolic subgroup $W_3 \cap W_5$ which is of type $A_1 \times A_2$. If α is any automorphism, then by Corollary 1.31 $\alpha(W_3 \cap W_5)$ is a parabolic subgroup of order 12. Hence, up to inner automorphisms, we may assume that $\alpha(W_3 \cap W_5)$ is a standard parabolic subgroup. There are only two types of Coxeter group with order 12, namely $A_1 \times A_2$ and $I_2(6)$, but W does not have a standard parabolic subgroup of type $I_2(6)$ and so $\alpha(W_3 \cap W_5)$ is a standard parabolic subgroup of type $A_1 \times A_2$. In any case the centre of such a subgroup has the form $\langle r_i \rangle$ for some simple reflection r_i and hence r_4 is mapped to a reflection.

Similar arguments can be used for the groups: $\mathcal{H}_{30}$, $\mathcal{H}_{31}$, $\mathcal{H}_{38} - \mathcal{H}_{41}$, $\mathcal{H}_{43}$, $\mathcal{H}_{46}$, $\mathcal{H}_{47}$, $\mathcal{H}_{49}$, $\mathcal{H}_{59}$ and $\mathcal{H}_{67}$. Table V on page 76, summarizes the arguments.

$\mathcal{H}_{28}$: Let W be a Coxeter group of type $\mathcal{H}_{28}$ with diagram



The maximal finite standard parabolic subgroups are: $W_2 \cong W_3$ of type $A_1 \times I_2(4)$ and finally W_{14} of type $I_2(4)$. The simple reflections r_2 and r_3 are easily dealt with as $\langle r_2 \rangle = W_3 \cap W_{14}$ and similarly for r_3. Suppose that $\alpha(r_4)$ is not a reflection. Modifying α by inner by graph automorphisms if necessary, we may assume that $\alpha(W_2) = W_2$. Now

$$\langle r_1 \rangle = Z(W_2) \cap W_3$$

and so

$$\begin{aligned} \langle \alpha(r_1) \rangle &= \alpha\big(Z(W_2)\big) \cap \alpha(W_3) \\ &= Z(W_2) \cap \alpha(W_3) \end{aligned}$$

Now $W_2 \cap \alpha(W_3) = \alpha(W_2 \cap W_3)$ is a parabolic subgroup of W_2 of type $A_1 \times A_1$, and hence equals $w\langle r_1, r_i \rangle w^{-1}$ for some $w \in W_2$ and $i = 3$ or 4. So

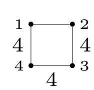
$$\begin{aligned} Z(W_2) \cap \alpha(W_3) &= w\langle r_1, r_i \rangle w^{-1} \cap Z(W_2) \\ &= w\big(\langle r_1, r_i \rangle \cap Z(W_2)\big)w^{-1} \\ &= \langle w r_1 w^{-1} \rangle. \end{aligned}$$

Hence $\alpha(r_1) = w r_1 w^{-1}$, a reflection. A symmetrical argument shows that $\alpha(r_4)$ is also a reflection.

A similar argument can be used for the groups: $\mathcal{H}_{28}$, $\mathcal{H}_{29}$ and $\mathcal{H}_{33}$.

The remaining arguments apply to only one group on the list.

$\mathcal{H}_{16}$: Let W be a Coxeter group of type $\mathcal{H}_{16}$ with diagram



The maximal finite standard parabolic subgroups are: $W_3 \cong W_4$ of type B_3 and W_{12} of type $I_2(4)$. It is easily shown that r_3 and r_4 map to reflections. Now

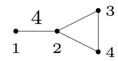
$$\langle r_1, r_2 \rangle = W_3 \cap W_4$$

is of type A_2 and so $\alpha\langle r_1, r_2 \rangle$ is, up to inner automorphisms, a standard parabolic subgroup of order 6. Hence we may assume

$$\alpha\langle r_1, r_2 \rangle = \langle r_1, r_2 \rangle.$$

But all the automorphisms of A_2 fix the set of reflections, and so r_1 and r_2 are mapped to reflections.

$\mathcal{H}_{24}$: Let W be a Coxeter group of type $\mathcal{H}_{24}$ with diagram

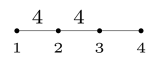


The maximal finite standard parabolic subgroups are: $W_3 \cong W_4$ of type B_3 and W_2 of type $A_1 \times A_2$. For r_1 note that $\langle r_1 \rangle = W_2 \cap W_3 \cap W_4$. Up to inner automorphisms, $\alpha(W_3 \cap W_4)$ is a standard parabolic subgroup of order 8 and so we may assume

$$\alpha(W_2 \cap W_3) = W_2 \cap W_3.$$

As r_2 is a non-central involution in $W_2 \cap W_3$, $\alpha(r_2)$ must also be a non-central involution in this subgroup. Hence $\alpha(r_2)$ is a reflection, and the remaining simple reflections are conjugate to r_2.

$\mathcal{H}_{27}$: Let W be a Coxeter group of type $\mathcal{H}_{27}$ with diagram



The maximal finite standard parabolic subgroups are: W_1 of type B_3, W_2 of type $A_1 \times A_2$ and W_3 of type $I_2(4) \times A_1$. The reflections r_3 and r_4 are conjugate and

$$\langle r_4 \rangle = W_1 \cap W_2 \cap W_3.$$

Up to inner automorphisms W_1, of type B_3, is fixed since it is the unique standard parabolic subgroup of type $A_1 \times A_2$. Therefore, by Proposition 2.8, r_2 is mapped to a reflection. Finally

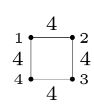
$$\langle r_1 \rangle = Z(W_2)$$

and up to inner automorphisms we may assume W_2 is fixed and therefore r_1 is also fixed; in particular, it is mapped to a reflection.

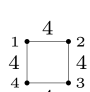
§4.4 Automorphisms

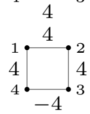
We have just seen that if W is a hyperbolic Coxeter group, then any automorphism of W preserves reflections. Thus if the Coxeter diagram of W is a forest with no unusual labels or if W is nearly finite, then all automorphisms of W are inner by graph. These arguments cover all but 4 of the types listed in Table I. The exceptions are $\mathcal{H}_{17}$, $\mathcal{H}_{21}$, $\mathcal{H}_{57}$ and $\mathcal{H}_{37}$. In the first three cases a simple argument involving the signature of the form can be used; the final case is a little more complicated.

$\mathcal{H}_{17}$: Let W be a Coxeter group of type $\mathcal{H}_{17}$ with diagram



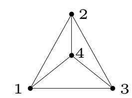
Suppose we have an automorphism α. As shown in the previous section α maps reflections to reflections. Choose $b_i \in \Phi$ with $\alpha(r_i) = r_{b_i}$, noting that there are two choices for each b_i, one positive the other negative. Replacing b_i with $-b_i$, if necessary, we may choose b_1 such that $b_1 \cdot b_4 = a_1 \cdot a_4$. Now choose the sign of b_2 so that $b_1 \cdot b_2 = a_1 \cdot a_2$, and finally choose the sign of b_3 so that $b_2 \cdot b_3 = a_2 \cdot a_3$. We are now left with 2 possible values for $b_3 \cdot b_4$, namely $b_3 \cdot b_4 = \pm\sqrt{2}/2$. In the following discussion these choices will be denoted as follows


$b_3 \cdot b_4 = a_3 \cdot a_4$


$b_3 \cdot b_4 = \sqrt{2}/2 = -a_3 \cdot a_4$

In the first of these two cases ϕ_α preserves the bilinear form, and so by Corollary 1.44 it follows that α is inner by graph. The second case is impossible since the matrix $(b_i \cdot b_j)$ turns out to be singular, contradicting Lemma 4.1. Virtually identical arguments to this apply for the groups $\mathcal{H}_{21}$ and $\mathcal{H}_{57}$: in both cases ϕ_α must preserve the form since the alternative Gram matrix turns out to be singular.

Finally the exceptional case: let W be a Coxeter group of type $\mathcal{H}_{37}$ with diagram



To save time label an edge with a minus sign if $b_i \cdot b_j = -a_i \cdot a_j$. If a vertex has two incident edges labelled with a minus sign then changing the choice of b_i corresponding to that vertex removes two minus signs and adds one. If all three edges are labelled with minuses then this removes them all. Thus we may reduce the number of minus signs until each vertex is incident with at most one minus. Hence we can have at most two minus signs and they cannot be incident. The case where there are two minus signs contradicts Lemma 4.1, since $(b_i \cdot b_j)$ turns out to be singular.

We are left with the possibility that there is exactly one minus. The standard parabolic subgroup $W_{\{a_1,a_2\}}$ is a maximal finite subgroup and so, up to inner by graph automorphisms, we may assume that it is fixed elementwise. Now suppose that $\alpha(r_3) = x$ and $\alpha(r_4) = y$. We may assume that

$$a_1 \cdot x = a_2 \cdot x = a_1 \cdot y = a_2 \cdot y = -\frac{1}{2} \tag{4.9}$$

while $x \cdot y = 1/2$. It is worthwhile noting at this point that as $r_i a_j = a_i + a_j$ for all i, j the roots are all integral linear combinations of the simple roots, and hence x and y must also be integer combinations of the a_i. In view of the equations 4.9 it follows that x and y both lie in the set $\{\, v_{r,s} \mid r, s \in \mathbb{Z} \,\}$ where

$$v_{r,s} = (-1 + r + s)a_1 + (-1 + r + s)a_2 + ra_3 + sa_4$$

Now

$$v_{r,s} \cdot v_{r,s} = 1 - 3rs$$

and so if $v_{r,s}$ is a root we must have $1 - 3rs = 1$ where $r = 0$ or $s = 0$. Thus

$$\begin{aligned} x &= (-1 + r)a_1 + (-1 + r)a_2 + ra_3 \\ y &= (-1 + s)a_1 + (-1 + s)a_2 + sa_4, \end{aligned}$$

for some integers r, s. This gives $x \cdot y = 1 - 3rs/2$. But $x \cdot y = 1/2$, and so

$$rs = \frac{1}{3}$$

which implies that r and s are not both integers. Hence there is no such automorphism.

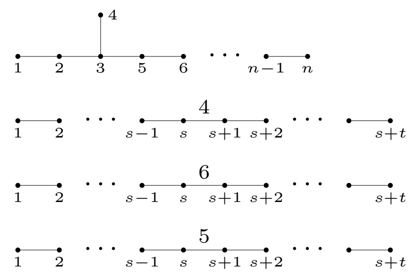
Thus we have proved the following theorem.

4.10 Theorem *If W is a hyperbolic Coxeter group (in the sense of [Hum90]), then all automorphisms of W are inner by graph and hence*

$$\operatorname{Aut}(W) = \operatorname{Inn}(W) \rtimes \operatorname{Gr}(W).$$

§4.5 Other Simple Diagrams

The results developed so far can clearly be used on a wide variety of Coxeter groups. For example, consider the following obvious generalizations of types, E, F, G and H.



E_n : $n \geq 10$

$F_{r,s}$: $2 \leq s \leq t,\ s+t \geq 6$

$G_{r,s}$: $1 \leq s \leq t,\ s+t \geq 4$

$H_{r,s}$: $2 \leq s \leq t$ or $s = 1$ and $t \geq 4$.

Some of these types have already been covered: Types $E_{10} = \mathcal{H}_{70}$, $F_{3,3} = \mathcal{H}_{48}$, $F_{4,2} = \mathcal{H}_{47}$, $H_{1,4} = \mathcal{H}_{11}$, $H_{2,2} = \mathcal{H}_{02}$, $G_{1,3} = \mathcal{H}_{31}$ and $G_{2,2} = \mathcal{H}_{32}$ correspond to hyperbolic groups.

4.11 Lemma *Suppose that the Coxeter group W is the direct product of two nontrivial irreducible parabolic subgroups W_I and W_J of types A_n and A_m respectively. If N is a normal subgroup of W then one or other of the following two alternatives must hold:*

(1) $N = (W_I \cap N)(W_J \cap N)$,

(2) $N = \{\, w \in W \mid l(w)$ *is even* $\} = W^+$.

Proof Suppose that alternative (1) does not hold, and let $w \in N$ with $w \notin (W_I \cap N)(W_J \cap N)$. Write $w = xy$ with $x \in W_I$ and $y \in W_J$, and observe that neither x nor y is in N. Since N is normal, conjugating by elements of W_I shows that $x'y \in N$ whenever x and x' are conjugate in W_I, and hence $x^{-1}x' = (xy)^{-1}(x'y) \in N$. So modulo $W_I \cap N$ the element x of W_I is equal to all its conjugates; that is, it is central in $W_I/(W_I \cap N)$. Similarly, $y(W_J \cap N)$ is central in in $W_J/(W_J \cap N)$. But all symmetric groups have the property that the only quotient with nontrivial centre is the abelianized group, which has order 2. So $W_I \cap N$ and $W_J \cap N$ are, respectively, the derived groups W_I' and W_J' of W_I and W_J. Now N must be one of the three subgroups of W containing $W_I'W_J'$. But N cannot be $W_I W_J'$ or $W_I' W_J$, since in both these cases we would obtain $N = (W_I \cap N)(W_J \cap N)$. So $N = \{\, w \in W \mid l(w)$ is even $\}$. ∎

4.12 Corollary *Suppose that the Coxeter group W is the direct product of two nontrivial standard parabolic subgroups W_I and W_J of type A_n and A_m respectively. Then the only direct product decompositions of W are as follows.*

(1) $W \cong W_I \times W_J$.

(2) $W \cong W_I \times W^+$ *if* $|I| = 1$.

(3) $W \cong W_J \times W^+$ *if* $|J| = 1$.

Proof Suppose $W = GH$ with $G, H \lhd W$ and $G \cap H = \{1\}$. Suppose that neither G nor H is W^+. By Lemma 4.11 $G = AB$ and $H = CD$ where $A, C \lhd W_I$ and $B, D \lhd W_J$, and it follows that $W_I = AC$ and $W_J = BD$; moreover, these are direct product decomposition of W_I and W_J. But no symmetric group has a nontrivial direct product decomposition; so we deduce that either $A = D = \{1\}$ or $B = C = \{1\}$, and hence that $\{\, G, H \,\} = \{\, W_I, W_J \,\}$. Then $H \neq W^+$, and so we may write $H = CD$ as above. Every proper normal subgroup of a symmetric group is contained in the alternating group; so if $C \neq W_I$ then $C = W^+ \cap C \leq W^+ \cap CD = \{1\}$. If $D \neq W_J$ then $D \leq W^+ \cap D = \{1\}$. If $C = W_I$ and $D = W_J$ then $W^+ \leq W = CD$ and contradiction. So either $C = W_I$ and $D = \{1\}$ or $D = W_J$ and $C = \{1\}$. As W^+ has index 2 in W the result follows. ∎

Now consider the type E_n ($n \geq 10$). The hypotheses of Corollary 3.20 are satisfied: W_4 is of type A_{n-1}, where $n - 1 \neq 5$, and m_{43} is odd. So it follows immediately that all automorphisms are inner (as there are no graph automorphisms).

For $H_{1,n-1}$ ($n \geq 5$) we see that W_2 is finite and is the unique maximal standard parabolic subgroup of type $A_1 \times A_{n-2}$, and r_1 is the unique nonidentity central element of W_2. It follows that any automorphism must map r_1 to a conjugate of itself, and since all reflections are conjugate we deduce that all automorphisms are reflection preserving. Hence by Theorem 3.16 it follows that all automorphisms are inner (since again there are no graph automorphisms).

For $G_{1,n-1}$ ($n \geq 4$) there are two classes of reflections, representatives of which are r_1 and r_2. The argument used for $H_{1,n-1}$ applies to show that each automorphism maps r_1 to a conjugate of itself. Now W_1 is finite and is the unique standard parabolic subgroup of type A_{n-1}. Note that all the reflections in W_1 are conjugate to r_2. If $n \neq 6$ then all automorphisms of A_{n-1} preserve reflections, and it follows that in this case all automorphisms preserve the class of reflections conjugate to r_2. If $n = 6$ we consider instead W_3, which is the unique maximal standard parabolic subgroup of type $G_2 \times A_3$. Without loss of generality we may assume that α is an automorphism of W satisfying $\alpha(W_3) = W_3$. Thus $\alpha(r_1r_2)^3 = (r_1r_2)^3$, since $(r_1r_2)^3$ is the unique nonidentity element of $Z(W_3)$. The only conjugates of r_1 in W_3 are r_1, $r_2r_1r_2$ and $r_1r_2r_1r_2r_1$ and so α must permute these. It follows that

$$\alpha(r_2) = \alpha(r_1r_2r_1r_2r_1)\alpha(r_1r_2)^3 = s(r_1r_2)^3$$

where s is r_1, $r_2r_1r_2$ or $r_1r_2r_1r_2r_1$. In all these cases we find that $\alpha(r_2)$ is conjugate to r_2, and so in this case also the class of reflections conjugate to r_2 is preserved. It follows from Theorem 3.16 that all automorphisms are inner (there being no graph automorphisms).

Now consider the cases $F_{s,t}$, $G_{s,t}$ and $H_{s,t}$ ($2 \leq s \leq t$). Observe that W_s and W_{s+1} are the only standard parabolic subgroups of types $A_{s-1} \times A_t$ and $A_s \times A_{t-1}$. If α is an automorphism of W then, modulo graph automorphisms if $s = t$, we conclude that $\alpha(W_s)$ is conjugate to W_s and $\alpha(W_{s+1})$ is conjugate to W_{s+1}. If $s = 2$ then r_1 is the unique nonidentity central element of W_s, and so $\alpha(r_1)$ is conjugate to r_1. If $s > 2$ then, by Corollary 4.12, each automorphism of a group of type $A_{s-1} \times A_t$ preserves the factors (since $s - 1 < t$). If $s - 1 \neq 5$ then all automorphisms of A_{s-1} preserve reflections, and so it follows that $\alpha(r_1)$ is a reflection. If $s - 1 = 5$ we use W_{s+1} instead of W_s. By Corollary 4.12 any automorphism of a group of type $A_s \times A_{t-1}$ preserves or interchanges the factors, and since all automorphisms of a group of type A_s preserves reflections we again conclude that $\alpha(r_1)$ is a reflection.

If $t = 2$ then $s = 2$ and by symmetry we deduce that $\alpha(r_{s+t})$ is a reflection. If $t > 2$ then W_{s+1} has type $A_s \times A_{t-1}$, and if $t - 1 \neq 5$ then any automorphism of such a group maps the reflections of the A_{t-1} factor to reflections. In particular, $\alpha(r_{s+t})$ is a reflection.

If $t - 1 = 5$ and $s > 2$ then by considering W_s (of type $A_{s-1} \times A_t$) the same argument again proves that $\alpha(r_{s+t})$ is a reflection.

Finally suppose that $s = 2$ and $t = 6$. Without loss of generality we may suppose that $\alpha(W_3) = W_3$ (of type $A_2 \times A_5$). If $\alpha(r_8)$ is not a reflection then α must induce a nontrivial outer automorphism on the A_5 factor; but we can still conclude that $\alpha(r_8)$ has odd length. So $\alpha(r_8) \notin W^+$. Now consider $W_2 = \langle r_1 \rangle \times W_J$, where $J = \{\, a_j \mid 3 \leq j \leq 8 \,\}$. Since $\alpha(W_2)$ is conjugate to W_2 we deduce that $w\alpha(W_J)w^{-1}$ is a direct factor of W_2 for some $w \in W$. Since $w\alpha(W_J)w^{-1} \not\subseteq W^+$ we deduce from Corollary 4.12 that $w\alpha(W_J)w^{-1} = W_J$, and since automorphisms of groups of type A_6 preserve reflections, we deduce that in fact $\alpha(r_8)$ must be a reflection. Thus in any case all automorphisms preserve reflections. Theorem 3.16 then completes the proof that all automorphisms are inner by graph.

Note that this argument applies for Coxeter groups with the following diagram.

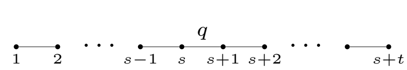
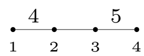
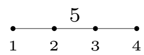
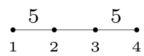
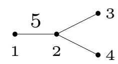
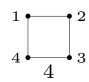
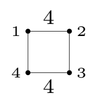
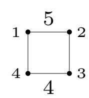
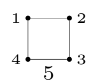
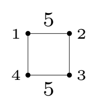
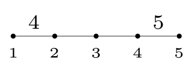
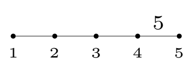
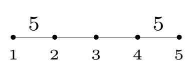
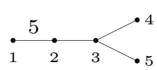
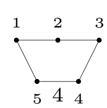


Table I: Automorphisms of Hyperbolic Coxeter Groups

The Compact Hyperbolic Coxeter Groups

$\mathcal{H}_{01}$		W_1 is finite of type H_3
$\mathcal{H}_{02}$		W_1 is finite of type H_3
$\mathcal{H}_{03}$		W_1 is finite of type H_3
$\mathcal{H}_{04}$		W_1 is finite of type A_3
$\mathcal{H}_{05}$		W_1 is finite of type B_3
$\mathcal{H}_{06}$		W_1 is finite of type B_3
$\mathcal{H}_{07}$		W_1 is finite of type B_3
$\mathcal{H}_{08}$		W_1 is finite of type H_3
$\mathcal{H}_{09}$		W_1 is finite of type H_3
$\mathcal{H}_{10}$		W_1 is finite of type H_4
$\mathcal{H}_{11}$		W_1 is finite of type H_4
$\mathcal{H}_{12}$		W_1 is finite of type H_4
$\mathcal{H}_{13}$		W_1 is finite of type D_4
$\mathcal{H}_{14}$		W_1 is finite of type B_4

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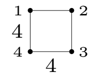
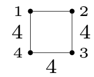
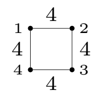
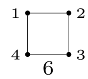
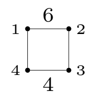
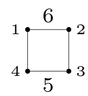
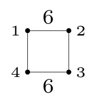
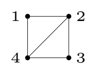
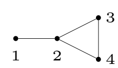
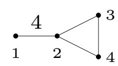
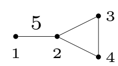
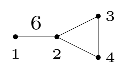
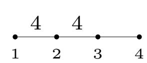
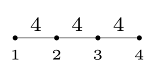
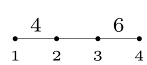
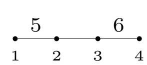
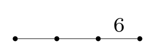
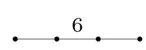
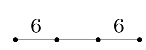
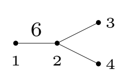
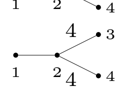
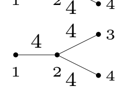
$\mathcal{H}_{15}$		$\mathcal{H}_{15}$	W_1 is finite of type B_3
$\mathcal{H}_{16}$		p. 65	W_4 is finite of type B_3
$\mathcal{H}_{17}$		$\mathcal{H}_{01}$	p. 66
$\mathcal{H}_{18}$		$\mathcal{H}_{18}$	W_4 is finite of type A_3
$\mathcal{H}_{19}$		$\mathcal{H}_{18}$	W_1 is finite of type B_3
$\mathcal{H}_{20}$		$\mathcal{H}_{18}$	W_1 is finite of type H_3
$\mathcal{H}_{21}$		$\mathcal{H}_{01}$	p. 67
$\mathcal{H}_{22}$		$\mathcal{H}_{18}$	W_4 is finite of type A_3
$\mathcal{H}_{23}$		$\mathcal{H}_{18}$	W_4 is finite of type A_3
$\mathcal{H}_{24}$		p. 66	W_4 is finite of type B_3
$\mathcal{H}_{25}$		$\mathcal{H}_{18}$	W_4 is finite of type H_3
$\mathcal{H}_{26}$		$\mathcal{H}_{18}$	W_2 is finite of type $A_1 \times A_2$
$\mathcal{H}_{27}$		p. 66	Γ is a forest
$\mathcal{H}_{28}$		$\mathcal{H}_{28}$	Γ is a forest
$\mathcal{H}_{29}$		$\mathcal{H}_{28}$	Γ is a forest
$\mathcal{H}_{30}$		$\mathcal{H}_{30}$	W_4 is finite of type H_3
$\mathcal{H}_{31}$		$\mathcal{H}_{30}$	Γ is a forest
$\mathcal{H}_{32}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{33}$		$\mathcal{H}_{28}$	Γ is a forest
$\mathcal{H}_{34}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{35}$		$\mathcal{H}_{15}$	Γ is a forest
$\mathcal{H}_{36}$		$\mathcal{H}_{01}$	Γ is a forest

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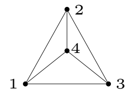
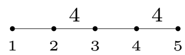
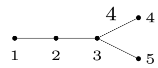
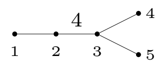
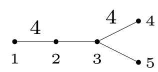
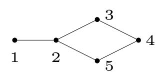
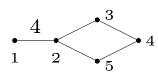
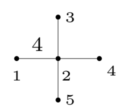
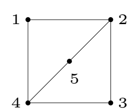
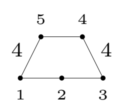
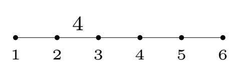
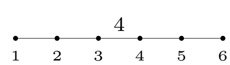
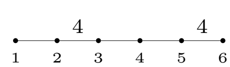
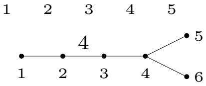
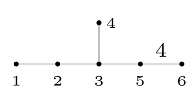
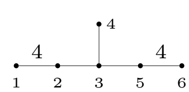
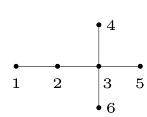
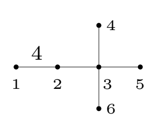
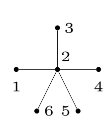
$\mathcal{H}_{37}$		$\mathcal{H}_{01}$	p. 67
$\mathcal{H}_{38}$		$\mathcal{H}_{30}$	Γ is a forest
$\mathcal{H}_{39}$		$\mathcal{H}_{30}$	Γ is a forest
$\mathcal{H}_{40}$		$\mathcal{H}_{30}$	Γ is a forest
$\mathcal{H}_{41}$		$\mathcal{H}_{30}$	Γ is a forest
$\mathcal{H}_{42}$		$\mathcal{H}_{18}$	W_3 is finite of type A_4
$\mathcal{H}_{43}$		$\mathcal{H}_{30}$	W_3 is finite of type B_4
$\mathcal{H}_{44}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{45}$		$\mathcal{H}_{18}$	W_4 is finite of type D_4
$\mathcal{H}_{46}$		$\mathcal{H}_{30}$	W_1 is finite of type F_4
$\mathcal{H}_{47}$		$\mathcal{H}_{30}$	Γ is a forest
$\mathcal{H}_{48}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{49}$		$\mathcal{H}_{30}$	Γ is a forest
$\mathcal{H}_{50}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{51}$		$\mathcal{H}_{15}$	Γ is a forest
$\mathcal{H}_{52}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{53}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{54}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{55}$		$\mathcal{H}_{01}$	Γ is a forest

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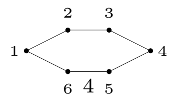
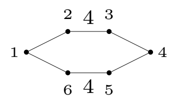
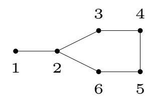
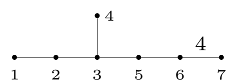
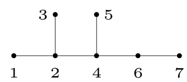
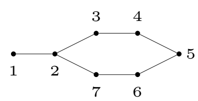
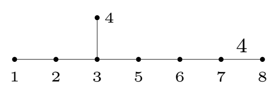
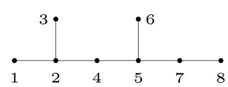
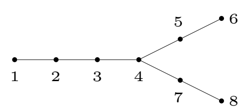
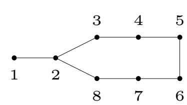
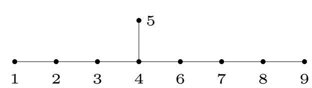
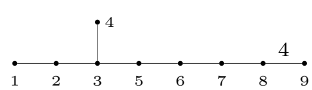
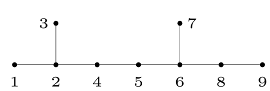
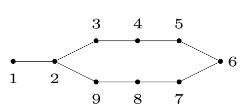
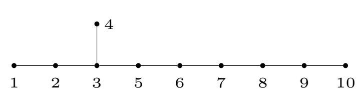
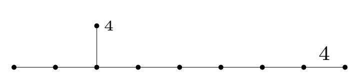
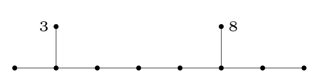
$\mathcal{H}_{56}$		$\mathcal{H}_{18}$	W_1 is finite of type B_5
$\mathcal{H}_{57}$		$\mathcal{H}_{18}$	p. 67
$\mathcal{H}_{58}$		$\mathcal{H}_{18}$	W_6 is finite of type A_5
$\mathcal{H}_{59}$		$\mathcal{H}_{30}$	Γ is a forest
$\mathcal{H}_{60}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{61}$		$\mathcal{H}_{18}$	W_3 is finite of type A_6
$\mathcal{H}_{62}$		$\mathcal{H}_{15}$	Γ is a forest
$\mathcal{H}_{63}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{64}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{65}$		$\mathcal{H}_{18}$	W_3 is finite of type A_7
$\mathcal{H}_{66}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{67}$		$\mathcal{H}_{30}$	Γ is a forest
$\mathcal{H}_{68}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{69}$		$\mathcal{H}_{18}$	W_3 is finite of type A_8
$\mathcal{H}_{70}$		$\mathcal{H}_{18}$	Γ is a forest
$\mathcal{H}_{71}$		$\mathcal{H}_{15}$	Γ is a forest
$\mathcal{H}_{72}$		$\mathcal{H}_{18}$	Γ is a forest

Reading Table I The first column lists the name we will use, the second column gives the Coxeter Diagram. In the compact hyperbolic case all maximal parabolic subgroups are finite and all automorphisms preserve reflections. The final column gives the type of the parabolic subgroup W_1.

With the non-compact hyperbolic Coxeter groups some work must be done to show that all automorphism preserve reflections. In those cases the third column lists the argument used. For example, the group $\mathcal{H}_{26}$ has an $\mathcal{H}_{18}$ in the third column; this indicates that the style of argument used for $\mathcal{H}_{18}$ in the text applies to this group. Other arguments are referred to by a page number. For example, the p. 66 listed against $\mathcal{H}_{27}$ indicates that the proof that all automorphisms of a Coxeter group of type $\mathcal{H}_{27}$ preserve reflections is on p. 66. For ease of reference the standard arguments and the pages on which they appear are as follows.

$\mathcal{H}_{01}$	p. 63
$\mathcal{H}_{18}$	p. 63
$\mathcal{H}_{15}$	p. 64
$\mathcal{H}_{30}$	p. 64
$\mathcal{H}_{28}$	p. 65

The final column either specifies a finite maximal standard parabolic subgroup, in the cases where W is nearly finite, or indicates that the diagram is a forest (with no unusual labels), or again lists the page on which the proof is given.

Table II

The first column names the group being dealt with. The second column lists the simple reflections in the order in which they are dealt with in subsequent columns. The third column lists the maximal finite standard parabolics. The final column shows the intersections of maximal finite subgroups which give subgroups of the form $\langle r_i \rangle$ for r_i from the list of class representatives, in the order in which they are listed.

Group	Reflections	Maximal finite subgroups	$\langle r_i \rangle =$
$\mathcal{H}_{17}$	r_1, r_2, r_3, r_4	$W_{12}, W_{13}, W_{14}, W_{23}, W_{24}, W_{34}$	$W_{23} \cap W_{34}, W_{13} \cap W_{34}, W_{12} \cap W_{14}, W_{12} \cap W_{23}$
$\mathcal{H}_{21}$	r_1, r_2, r_3, r_4	$W_{12}, W_{13}, W_{14}, W_{23}, W_{24}, W_{34}$	$W_{23} \cap W_{34}, W_{13} \cap W_{34}, W_{12} \cap W_{14}, W_{12} \cap W_{23}$
$\mathcal{H}_{36}$	r_1, r_2, r_3, r_4	$W_2, W_{13}, W_{14}, W_{34}$	$W_2 \cap W_{34}, W_{13} \cap W_{14}, W_2 \cap W_{14}, W_2 \cap W_{13}$
$\mathcal{H}_{37}$	r_1, r_2, r_3, r_4	$W_{12}, W_{13}, W_{14}, W_{23}, W_{24}, W_{34}$	$W_{23} \cap W_{34}, W_{13} \cap W_{34}, W_{12} \cap W_{14}, W_{12} \cap W_{23}$
$\mathcal{H}_{55}$	$r_1, r_2, r_3, r_4, r_5, r_6$	$W_2, W_{ij},\ i \neq j,\ \ i, j \neq 2$	$W_2 \cap W_{34} \cap W_{56}, W_{13} \cap W_{45} \cap W_{46}, W_2 \cap W_{14} \cap W_{56}, W_2 \cap W_{13} \cap W_{56}, W_2 \cap W_{13} \cap W_{46}, W_2 \cap W_{13} \cap W_{45}$

Table III

The columns are the same as those in the previous table except that the second column only lists representatives of the classes of reflections.

Group	Classes	Maximal finite subgroups	$\langle r_i \rangle =$
$\mathcal{H}_{18}$	r_3	W_3, W_4, W_{12}	$W_{12} \cap W_4$
$\mathcal{H}_{19}$	r_1, r_2	W_1, W_2, W_{34}	$W_2 \cap W_{34}, W_1 \cap W_{34}$
$\mathcal{H}_{20}$	r_1	W_1, W_2, W_{34}	$W_2 \cap W_{34}$
$\mathcal{H}_{22}$	r_2	W_2, W_4, W_{13}	$W_{13} \cap W_4$
$\mathcal{H}_{23}$	r_1	W_2, W_3, W_4	$W_2 \cap W_3 \cap W_4$
$\mathcal{H}_{25}$	r_1	W_2, W_3, W_4	$W_2 \cap W_3 \cap W_4$
$\mathcal{H}_{26}$	r_1, r_2	$W_2, W_{13}, W_{14}, W_{34}$	$W_2 \cap W_{34}, W_{13} \cap W_{14}$
$\mathcal{H}_{32}$	r_2, r_3	W_2, W_3, W_{14}	$W_3 \cap W_{14}, W_2 \cap W_{14}$
$\mathcal{H}_{34}$	r_1, r_2	W_1, W_2, W_{34}	$W_2 \cap W_{34}, W_1 \cap W_{34}$
$\mathcal{H}_{42}$	r_1	W_2, W_3, W_4, W_5	$W_2 \cap W_3 \cap W_4 \cap W_5$
$\mathcal{H}_{44}$	r_1, r_2	$W_1, W_2, W_{34}, W_{35}, W_{45}$	$W_2 \cap W_{34} \cap W_{35}, W_1 \cap W_{34} \cap W_{35}$
$\mathcal{H}_{45}$	r_1	$W_2, W_4, W_{13}, W_{15}, W_{35}$	$W_2 \cap W_4 \cap W_{35}$
$\mathcal{H}_{48}$	r_3, r_4	$W_2, W_3, W_4, W_5, W_{16}$	$W_2 \cap W_4 \cap W_5 \cap W_{16}, W_2 \cap W_3 \cap W_5 \cap W_{16}$
$\mathcal{H}_{50}$	r_2, r_3	$W_2, W_3, W_4, W_{15},$ W_{16}, W_{56}	$W_3 \cap W_4 \cap W_{15} \cap W_{16},$ $W_2 \cap W_4 \cap W_{15} \cap W_{16}$
$\mathcal{H}_{52}$	r_1, r_4, r_6	$W_2, W_3, W_5, W_{14},$ W_{16}, W_{46}	$W_2 \cap W_3 \cap W_5 \cap W_{46}, W_2 \cap W_3 \cap W_5 \cap W_{16}$ $W_2 \cap W_3 \cap W_5 \cap W_{14}$
$\mathcal{H}_{53}$	r_1	W_2, W_3, W_4, W_5, W_6	$W_2 \cap W_3 \cap W_4 \cap W_5 \cap W_6$
$\mathcal{H}_{54}$	r_1, r_2	$W_2, W_3, W_{14}, W_{15}, W_{16},$ W_{45}, W_{46}, W_{56}	$W_2 \cap W_3 \cap W_{45} \cap W_{56},$ $W_3 \cap W_{14} \cap W_{56}$
$\mathcal{H}_{56}$	r_1	$W_1, W_4, W_5, W_6, W_{23}$	$W_4 \cap W_5 \cap W_6 \cap W_{23}$
$\mathcal{H}_{57}$	r_1	$W_{ij},\ i \neq j$	$W_{23} \cap W_{45} \cap W_{56}$
$\mathcal{H}_{58}$	r_1	W_2, W_3, W_4, W_5, W_6	$W_2 \cap W_3 \cap W_4 \cap W_5 \cap W_6$
$\mathcal{H}_{60}$	r_7	$W_1, W_2, W_3, W_4, W_5, W_6$	$W_1 \cap W_2 \cap \cdots \cap W_6$
$\mathcal{H}_{61}$	r_1	$W_2, W_3, W_4, W_5, W_6, W_7$	$W_2 \cap W_3 \cap \cdots \cap W_7$
$\mathcal{H}_{63}$	r_8	$W_1, W_2, W_3, W_4, W_5, W_6, W_7$	$W_1 \cap W_2 \cap \cdots \cap W_7$
$\mathcal{H}_{64}$	r_1	$W_2, W_3, W_4, W_5, W_6, W_7, W_8$	$W_2 \cap W_3 \cap \cdots \cap W_8$
$\mathcal{H}_{65}$	r_1	$W_2, W_3, W_4, W_5, W_6, W_7, W_8$	$W_2 \cap W_3 \cap \cdots \cap W_8$
$\mathcal{H}_{66}$	r_1	$W_2, W_3, W_4, W_5,$ W_6, W_7, W_8, W_9	$W_2 \cap W_3 \cap \cdots \cap W_9$
$\mathcal{H}_{68}$	r_9	$W_1, W_2, W_3, W_4,$ W_5, W_6, W_7, W_8	$W_1 \cap W_2 \cap \cdots \cap W_8$
$\mathcal{H}_{69}$	r_1	$W_2, W_3, W_4, W_5,$ W_6, W_7, W_8, W_9	$W_2 \cap W_3 \cap \cdots \cap W_9$
$\mathcal{H}_{70}$	r_{10}	$W_1, W_2, W_3, W_4, W_5,$ W_6, W_7, W_8, W_9	$W_1 \cap W_2 \cap \cdots \cap W_9$
$\mathcal{H}_{72}$	r_1	$W_2, W_4, W_5, W_6, W_7, W_8,$ $W_9, W_{13}, W_{1,10}, W_{3,10}$	$W_2 \cap W_4 \cap \cdots \cap W_9 \cap W_{3,10}$

Table IV

As well as the columns used in the previous tables the fifth column lists the simple reflection which corresponds to 'r_1' in a parabolic subgroup of type B_{2k+1}. The final column lists that standard parabolic subgroup of type B_{2k+1}.

Group	Classes	Maximal fin. sbgps.	$\langle r_i \rangle =$	'r_1'	B_{2k+1}
$\mathcal{H}_{15}$	r_2, r_4	W_1, W_3, W_4	$W_1 \cap W_3 \cap W_4$	r_4	W_1
$\mathcal{H}_{35}$	r_1, r_3, r_4	W_2, W_3, W_4	$W_2 \cap W_3 \cap W_4$	r_3, r_4	W_4, W_3 (resp.)
$\mathcal{H}_{51}$	r_1, r_6	W_2, W_3, W_4, W_5, W_6	$W_2 \cap \cdots \cap W_6$	r_6	W_4
$\mathcal{H}_{62}$	r_1, r_8	$W_2, \ldots, W_8$	$W_2 \cap W_8$	r_8	W_4
$\mathcal{H}_{71}$	r_2, r_{10}	$W_2, \ldots, W_9, W_{1,10}$	$W_3 \cap \cdots \cap W_9 \cap W_{1,10}$	r_{10}	W_4

Table V

In addition to the usual first four columns there is a column which shows the centre of a parabolic subgroup written as the intersection of maximal finite standard parabolic subgroups.

Group	Classes	Maximal fin. sbgps.	$\langle r_i \rangle =$	Centre
$\mathcal{H}_{30}$	r_1, r_4	W_2, W_3, W_4	$W_2 \cap W_3 \cap W_4$	$\langle r_4 \rangle = Z(W_3)$
$\mathcal{H}_{31}$	r_1, r_4	W_2, W_3, W_4	$W_2 \cap W_3 \cap W_4$	$\langle r_4 \rangle = Z(W_3)$
$\mathcal{H}_{38}$	r_1, r_4, r_5	W_2, W_3, W_4, W_5	$W_2 \cap W_3 \cap W_4 \cap W_5$	$\langle r_4 \rangle = Z(W_3 \cap W_5)$, $\langle r_5 \rangle = Z(W_3 \cap W_4)$
$\mathcal{H}_{39}$	r_1, r_4	W_2, W_3, W_4, W_5	$W_2 \cap W_3 \cap W_4 \cap W_5$	$\langle r_4 \rangle = Z(W_3 \cap W_5)$
$\mathcal{H}_{40}$	r_1, r_4	W_2, W_3, W_4, W_5	$W_2 \cap W_3 \cap W_4 \cap W_5$	$\langle r_4 \rangle = Z(W_3 \cap W_5)$
$\mathcal{H}_{41}$	r_2, r_4, r_1	W_2, W_3, W_4, W_{15}	$W_3 \cap W_4 \cap W_{15}$, $W_2 \cap W_3 \cap W_{15}$	$\langle r_1 \rangle = Z(W_2 \cap W_4)$
$\mathcal{H}_{43}$	r_2, r_1	W_2, W_3, W_5, W_{14}	$W_3 \cap W_5 \cap W_{14}$	$\langle r_1 \rangle = Z(W_2 \cap W_3)$
$\mathcal{H}_{46}$	r_2, r_4	W_1, W_3, W_4, W_5	$W_1 \cap W_3 \cap W_4 \cap W_5$	$\langle r_4 \rangle = Z(W_3 \cap W_5)$
$\mathcal{H}_{47}$	r_6, r_1	W_1, W_2, W_3, W_4, W_5	$W_1 \cap \cdots \cap W_5$	$\langle r_1 \rangle = Z(W_2 \cap W_3 \cap W_4)$
$\mathcal{H}_{49}$	r_2, r_3, r_6	$W_2, W_3, W_4, W_5, W_{16}$	$W_3 \cap W_4 \cap W_5 \cap W_{16}$, $W_2 \cap W_4 \cap W_5 \cap W_{16}$	$\langle r_6 \rangle = Z(W_3 \cap W_4 \cap W_6)$
$\mathcal{H}_{59}$	r_1, r_7	$W_2, W_3, W_4, W_5, W_6, W_7$	$W_2 \cap \cdots \cap W_7$	$\langle r_7 \rangle = Z(W_3 \cap \cdots \cap W_6)$
$\mathcal{H}_{67}$	r_1, r_9	$W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9$	$W_2 \cap \cdots \cap W_9$	$\langle r_9 \rangle = Z(W_3 \cap \cdots \cap W_8)$

Chapter 5

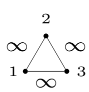
Rank 3 Coxeter Groups

§5.1 Groups with Finite Bonds

As mentioned in the last chapter if W is an infinite rank 3 Coxeter group with finite labels then all automorphisms of W are inner by graph. It is clear that W is nearly finite and that all automorphisms preserve reflections (by Lemma 4.4) and the result follows by Theorem 3.16 if W is non-degenerate. If W is degenerate then the result is part of Proposition 4.5. The rank 3 finite Coxeter groups are dealt with in Chapter 2.

§5.2 Groups With Infinite Bonds

The bulk of this chapter we deal with the rank three Coxeter groups with one or more edges labelled with an infinity. If W is a Coxeter group with r_i and r_j reflections corresponding to vertices in the diagram joined by an edge labelled with an infinity, then r_ir_j has infinite order and we set $a_i \cdot a_j = -1$. Looking at rank three irreducible Coxeter groups with at least one infinite bond there are five cases to consider:



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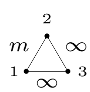

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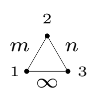

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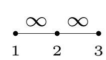
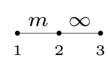


and


where $m, n \geq 3$. We will call these $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ and $\mathcal{I}_5$ respectively.

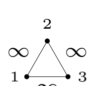
The automorphism group of a Coxeter group where $o(r_ir_j) \in \{2, \infty\}$ for all i and j has been found by Mühlherr in [Mü98], where he completes the work started by James, [Jam88], and Tits, [Tit88]. These papers could be used to find the automorphisms of groups of type $\mathcal{I}_1$ or $\mathcal{I}_4$; instead we present an alternative approach. We will use the relations given in [Mü98] to describe the structure of $\mathrm{Aut}(W)$ when W is of type $\mathcal{I}_1$.

5.1 Definition Let W be a Coxeter group with Π the set of simple roots. If $a_i \in \Pi$ and $K \subseteq \Pi$, then denote by $\sigma_{i,K}$ the function $\{ r_i | a_i \in \Pi \} \to W$ defined by

$$\sigma_{i,K}(r_j) = \begin{cases} r_ir_jr_i & \text{if } a_j \in K \text{ and} \\ r_j & \text{otherwise.} \end{cases}$$

In circumstances where $\sigma_{i,K}$ extends to an endomorphism of W we also use $\sigma_{i,K}$ to denote that endomorphism. If $K = \{ a_j \}$ or $\{ a_j, a_k \}$ then we write $\sigma_{i,j}$ or $\sigma_{i,jk}$, respectively, for $\sigma_{i,K}$.

$\mathcal{I}_1$: Let W be a Coxeter group of type $\mathcal{I}_1$ with diagram



Observe that $W = \mathrm{gp}\langle r_1, r_2, r_3 \,|\, r_1^2 = r_2^2 = r_3^2 = 1 \rangle$ is the free product of three groups of order 2. Every element of W has a unique reduced expression and an expression is reduced if and only if no two consecutive terms are equal. Taking any reduced expression and replacing each r_i, for some fixed i, by a reduced expression starting and ending with r_i results in a longer reduced expression.

5.2 Lemma *If $i \neq j$, then $\sigma_{i,j}$ is an automorphism of W.*

Proof As we have seen the group W has the presentation

$$W = \mathrm{gp}\langle\, r_1, r_2, r_3 \,|\, r_1^2 = r_2^2 = r_3^2 = 1 \,\rangle.$$

For all k it is clear that $\sigma_{i,j}(r_k)$ is an involution and so $\sigma_{i,j}$ is an endomorphism of W. The observation that $\sigma_{i,K}^2 = 1$ for all i and K completes the proof. ∎

The maximal finite standard parabolic subgroups are $\langle r_i \rangle$ for $i = 1$, 2 and 3, therefore all automorphisms of W preserve reflections. Let α be an automorphism of W. Then $\alpha(r_1)$ is a conjugate of r_i for some i and up to graph automorphisms we may assume that $i = 1$. Similarly we can ensure that for all j, $\alpha(r_j)$ is a conjugate of r_j. The following lemma helps classify those elements that are conjugates of the r_i and together generate W.

5.3 Lemma *Let W be a Coxeter group of type $\mathcal{I}_1$.*

(1) *Suppose that $x_3 = r_3 w_3 r_3 w_3^{-1} r_3$ is a reduced expression in W. Then*

$$r_3 \notin \langle\, r_1, r_2, x_3 \,\rangle.$$

(2) *Suppose that $x_2 = r_j w_2 r_2 w_2^{-1} r_j$ and $x_3 = r_k w_3 r_3 w_3^{-1} r_k$ are reduced expressions in W where $\{\, j, k \,\} = \{\, 2, 3 \,\}$. Then*

$$r_3 \notin \langle\, r_1, x_2, x_3 \,\rangle.$$

(3) *Suppose that $x_1 = r_i w_1 r_1 w_1^{-1} r_i$, $x_2 = r_j w_2 r_2 w_2^{-1} r_j$ and $x_3 = r_k w_3 r_3 w_3^{-1} r_k$ are reduced expressions in W where $\{\, i, j, k \,\} = \{\, 1, 2, 3 \,\}$. Then*

$$r_3 \notin \langle\, x_1, x_2, x_3 \,\rangle.$$

Proof For part (1) observe that since x_3 has order 2 there is an epimorphism

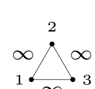
$$f : W \to H = \langle r_1, r_2, x_3 \rangle \leq W$$

given by $r_1 \mapsto r_1$, $r_2 \mapsto r_2$ and $r_3 \mapsto x_3$. For each $w \in W$ the element $f(w)$ is computed by replacing each r_3 in the reduced expression for w by x_3. Since the reduced expression for x_3 begins and ends with r_3, the resulting expression for $f(w)$ must also be reduced: no two consecutive factors can be equal. So f is an isomorphism.

If the expression for w involves at least one r_3 then $l\big(f(w)\big) \geq l(x_3) \geq 3$, and so $f(w) \neq r_3$. If w does not involve r_3 then $f(w) = w \neq r_3$. So $r_3 \notin H$.

The proofs for (2) and (3) are essentially the same as for (1). ∎

5.4 Proposition *Let W be a Coxeter group of type $\mathcal{I}_1$ with diagram*



then

$$\begin{aligned} \mathrm{Aut}(W) &= \langle \sigma_{1,2}, \mathrm{Gr}(W) \rangle \\ &= \big(\mathrm{Inn}(W) \rtimes \langle \sigma_{1,2}, \sigma_{2,3}, \sigma_{3,1} \rangle\big) \rtimes \mathrm{Gr}(W). \end{aligned}$$

Proof If $\rho \in \mathrm{Sym}_3$ let γ_ρ be the corresponding graph automorphism. If $i \neq j$ then $\sigma_{i,j} = \rho_{(1i)(2j)}\sigma_{1,2}\rho_{(1i)(2j)}^{-1}$. Furthermore if $\{i,j,k\} = \{1,2,3\}$ then $\sigma_{i,j}\sigma_{i,k} = \sigma_{i,jk}$ is conjugation by r_i and hence is inner. Let

$$A = \langle \sigma_{1,2}, \mathrm{Gr}(W)\rangle = \langle \mathrm{Inn}(W), \sigma_{1,3}, \sigma_{2,3}, \sigma_{3,1}, \mathrm{Gr}(W)\rangle$$

and suppose that α is an automorphism of W with $\alpha \notin A$. Replacing α by a suitable element of $\alpha\,\mathrm{Gr}(W) \subset \alpha A$, we may assume

$$\begin{aligned} \alpha(r_1) &= x r_1 x^{-1} \\ \alpha(r_2) &= y r_2 y^{-1} \\ \alpha(r_3) &= z r_3 z^{-1} \end{aligned}$$

where each of these expressions is reduced. Assume that α is chosen such that $l(x)+l(y)+l(z)$ is minimal. Since $\alpha \neq 1$ at least one of x, y, z is nontrivial and without loss of generality we may assume that $z \neq 1$. Let $z = r_k z'$ where $l(z') < l(z)$. If $l(r_k y) < l(y)$ then following α by conjugation by r_k yields the automorphism α' where

$$\begin{aligned} \alpha'(r_1) &= (r_k x) r_1 (r_k x)^{-1} \\ \alpha'(r_2) &= (r_k y) r_2 (r_k y)^{-1} \\ \alpha'(r_3) &= z' r_3 z'^{-1}. \end{aligned}$$

Now $l(r_k x) + l(r_k y) + l(z') \leq l(x) + 1 + l(y) - 1 + l(z) - 1 < l(x) + l(y) + l(z)$ and so $\alpha' \in A$ contradicting the assumption that $\alpha \notin A$. Thus $l(r_k y) > l(y)$ and similarly $l(r_k x) > l(x)$.

Suppose that $x = y = 1$. If $k = 1$ then we find that

$$\begin{aligned} \alpha\sigma_{1,3} : r_1 &\mapsto r_1 \\ r_2 &\mapsto r_2 \\ r_3 &\mapsto \alpha(r_1 r_3 r_1) \\ &\qquad = r_1(z r_3 z^{-1}) r_1 \\ &\qquad = z' r_3 z'^{-1}. \end{aligned}$$

The total length has again been reduced and we are led to a contradiction as before, since $\sigma_{1,3} \in A$. A similar argument applies for $k = 2$ and hence $k = 3$. But then we have $W = \langle r_1, r_2, r_3 z' r_3 z'^{-1} r_3\rangle$ contradicting Lemma 5.3, and this case cannot occur.

Now suppose that $x = 1$ but that $y \neq 1$; say $y = r_j y'$, where $l(y') < l(y)$. Observe that $j \neq k$ as $l(r_k y) > l(y)$. If $j = 1$ or $k = 1$ then considering either $\alpha\sigma_{1,j}$ or $\alpha\sigma_{i,k}$ leads to a contradiction. Thus $\{j,k\} = \{2,3\}$, and $W = \langle r_1, r_j y' r_2 y'^{-1} r_j, r_k z' r_3 z'^{-1} r_k\rangle$ contradicting Lemma 5.3. Similarly $x \neq 1$ and $y = 1$ is impossible.

Finally suppose that $x \neq 1$ and $y \neq 1$. Let $x = r_i x'$ and $y = r_j y'$ where $l(x') < l(x)$ and $l(y') < l(y)$. If any two of r_i, r_j, r_k are equal then following α by conjugation by this element yields a contradiction while the case $\{i,j,k\} = \{1,2,3\}$ contradicts Lemma 5.3. Hence there is no such α and we have $\mathrm{Aut}(W) = A$.

Following [Mü98] let $\mathrm{Spe}(W) = \langle \mathrm{Inn}(W), \sigma_{1,2}, \sigma_{2,3}, \sigma_{3,1}\rangle$. Then $\mathrm{Spe}(W)$ is the group of automorphisms of W that preserve the conjugacy classes of reflections. Thus

$$\mathrm{Aut}(W) = \mathrm{Spe}(W) \rtimes \mathrm{Gr}(W).$$

The proof to this point has shown that $\mathrm{Spe}(W)$ is generated by the automorphisms of the form $\sigma_{i,j}$ and $\sigma_{i,jk}$. The theorem in [Mü98] shows that a presentation of $\mathrm{Spe}(W)$ on these generators is given by the relations

$$
\begin{aligned}
\sigma_{1,2}^2 &= 1 \\
\sigma_{1,23}^2 &= 1 \\
\sigma_{1,2}\sigma_{2,3} &= \sigma_{1,23} \\
\sigma_{1,2}\sigma_{2,13} &= \sigma_{1,23}\sigma_{2,13}\sigma_{1,23}\sigma_{1,2} \\
\sigma_{1,2}\sigma_{3,12} &= \sigma_{3,12}\sigma_{1,2} \\
\sigma_{1,2}\sigma_{1,23} &= \sigma_{1,23}\sigma_{1,2}
\end{aligned}
$$

together with the relations found by applying any permutation in Sym_3 to the above.

Modifying our notation, let us write $i_1 = \sigma_{1,23}$, $i_2 = \sigma_{2,13}$ and $i_3 = \sigma_{3,12}$; these in fact correspond to the inner automorphisms of W given by r_1, r_2, r_3. Written in terms of the generators $\sigma_{1,2}$, $\sigma_{2,3}$, $\sigma_{3,1}$, i_1, i_2, i_3 the defining relations become

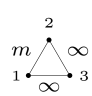
$$
\begin{gathered}
\sigma_{1,2}^2 = \sigma_{2,3}^2 = \sigma_{3,1}^2 = 1, \\
i_1^2 = i_2^2 = i_3^2 = 1, \\
\sigma_{1,2} i_2 \sigma_{1,2} = i_1 i_2 i_1, \quad \sigma_{2,3} i_3 \sigma_{2,3} = i_2 i_3 i_2, \quad \sigma_{3,1} i_1 \sigma_{3,1} = i_3 i_1 i_3, \\
\sigma_{1,2} i_3 \sigma_{1,2} = i_3, \quad \sigma_{2,3} i_1 \sigma_{2,3} = i_1, \quad \sigma_{3,1} i_2 \sigma_{3,1} = i_2, \\
\sigma_{1,2} i_1 \sigma_{1,2} = i_1, \quad \sigma_{2,3} i_2 \sigma_{2,3} = i_2, \quad \sigma_{3,1} i_3 \sigma_{3,1} = i_3.
\end{gathered}
$$

Thus i_1, i_2, i_3 generate a normal subgroup $\overline{W}$ isomorphic to W, and the action of the subgroup S generated by $\sigma_{1,2}$, $\sigma_{2,3}$, and $\sigma_{3,1}$ on $\overline{W}$ corresponds to the action of $\sigma_{1,2}$, $\sigma_{2,3}$, and $\sigma_{3,1}$ on W. Since the only further relations between $\sigma_{1,2}$, $\sigma_{2,3}$, and $\sigma_{3,1}$ are $\sigma_{1,2}^2 = \sigma_{2,3}^2 = \sigma_{3,1}^2 = 1$ we see that S is the free product of three groups of order 2, and $\mathrm{Spe}(W)$ is isomorphic to the semidirect product $\overline{W} \rtimes S$. Thus

$$
\begin{aligned}
\mathrm{Aut}(W) &= \big(\mathrm{Inn}(W) \rtimes \langle \sigma_{1,2}, \sigma_{2,3}, \sigma_{3,1} \rangle\big) \rtimes \mathrm{Gr}(W) \\
&= (\overline{W} \rtimes S) \rtimes \mathrm{Gr}(W) \\
&\cong (W \rtimes W) \rtimes \mathrm{Sym}_3 \,.
\end{aligned}
$$

∎

$\mathcal{I}_2$: Let W be a Coxeter group of type $\mathcal{I}_2$ with diagram



where $m \geq 3$. Observe that W is the free product of a group of order 2 with a group of type $I_2(m)$. The maximal finite subgroups are W_3 and $\langle r_3 \rangle$ and so any automorphism of W maps r_3 to a conjugate of itself. Up to inner automorphisms we may assume that if α is any automorphism, then $\alpha(W_3) = W_3$. Thus $\alpha|_{W_3}$ is an automorphism of a group of type $I_2(m)$ with $m \geq 3$, and hence preserves reflections. So α preserves $\mathrm{Ref}(W)$, by Proposition 2.13.

5.5 Definition Let $\zeta : W_3 \to W_3$ be any automorphism of W_3 and define α_ζ from $\{ r_i | a_i \in \Pi \}$ to W by

$$
\begin{aligned}
\alpha_\zeta(r_1) &= \zeta(r_1) \\
\alpha_\zeta(r_2) &= \zeta(r_2) \\
\alpha_\zeta(r_3) &= r_3.
\end{aligned}
$$

The next lemma shows that α_ζ extends to an automorphism of W. We also use α_ζ to denote that automorphism.

5.6 Lemma *For all ζ the function α_ζ is an automorphism of W. If $\zeta \neq 1$ then α_ζ is not an inner automorphism of W.*

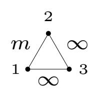
Proof It is clear that α_ζ is an endomorphism of W and that $\alpha_\zeta \alpha_{\zeta^{-1}} = 1$. If α_ζ were conjugation by $w \in W$ then w would centralize r_3. If we recall that W is the free product of $\langle r_3 \rangle$ and a dihedral group it is clear that $\mathcal{C}_W(r_3) = \{\,1, r_3\,\}$. As $\zeta \neq 1$ we must have $w = r_3$. But conjugation by r_3 does not fix W_3 setwise; hence α_ζ is not inner. ∎

Let μ be the automorphism of W_3 which is conjugation by r_1, and let α be the automorphism of W which is conjugation by r_1. Then

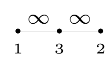
$$\begin{aligned} \alpha\alpha_\mu(r_1) &= r_1 \\ \alpha\alpha_\mu(r_2) &= \alpha(r_1 r_2 r_1) = r_2 \\ \alpha\alpha_\mu(r_3) &= \alpha(r_3) = r_1 r_3 r_1. \end{aligned}$$

Thus $\alpha\alpha_\mu = \sigma_{1,3}$. Similarly if ν is the automorphism of W_3 which is conjugation by r_2 and β is the automorphism of W which is conjugation by r_2 then $\beta\alpha_\nu = \sigma_{2,3}$.

5.7 Lemma *Suppose that W is a Coxeter group of type $\mathcal{I}_2$ or $\mathcal{I}_4$ with diagram*



or



If $x_3 = w r_3 w^{-1}$ is a reduced expression, where $w \in W$, then

$$\langle\, r_1, r_2, x_3 \,\rangle = W$$

if and only if $w \in \langle\, r_1, r_2 \,\rangle$.

Proof Let $\rho_i = (r_3 r_2)^i r_3$. Then as $w \in W$ we can find a unique expression for w of the form

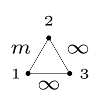
$$w = p_0 \rho_{i_0} p_1 \rho_{i_1} \cdots p_n \rho_{i_n} p_{n+1}$$

where $p_0, p_{n+1} \in \langle r_1, r_2 \rangle$ and $p_1, \dots p_n \in \langle r_1, r_2 \rangle \backslash \langle r_2 \rangle$. Note that the case $n = -1$ corresponds to $w \in \langle r_1, r_2 \rangle$. Since we have assumed that $w r_3 w^{-1}$ is reduced, it follows that $p_{n+1} \neq 1$ (unless $w = 1$). If $n \geq 0$ then

$$\langle r_1, r_2, w r_3 w^{-1} \rangle = \langle r_1, r_2, r_3 w' r_3 {w'}^{-1} r_3 \rangle$$

(for some w' such that $r_3 w' r_3 {w'}^{-1} r_3$ is reduced) and no term including an r_3 can have length 1. Thus if $\langle r_1, r_2, x_3 \rangle = W$ then $n = -1$, and so $w \in \langle r_1, r_2 \rangle$. ∎

5.8 Proposition *Let W be a Coxeter group of type $\mathcal{I}_2$ with diagram*



then $\mathrm{Aut}(W) = \mathrm{Inn}(W) \rtimes \langle\, \alpha_\zeta \mid \zeta \in \mathrm{Aut}(W_3) \,\rangle$.

Proof Let α be an automorphism of W, a Coxeter group of type $\mathcal{I}_2$, and suppose, for a contradiction, that $\alpha \notin \mathrm{Inn}(W)\langle\, \alpha_\zeta \mid \zeta \in \mathrm{Aut}(W_3) \,\rangle$. Replacing α by a suitable element of $\big(\mathrm{Inn}(W)\big)\alpha$ we may assume that $\alpha(W_3) = W_3$. If $\zeta^{-1} = \alpha|_{W_3}$ then $\alpha\alpha_\zeta$ is an automorphism of W fixing r_1 and r_2. Thus we may assume that α fixes r_1 and r_2 and (since α preserves the conjugacy class of r_3) that $\alpha(r_3) = w r_3 w^{-1}$. By Lemma 5.7, as α is an automorphism, we must have $w \in W_3$. If

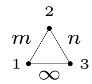
$$w = r_{i_1} r_{i_2} \dots r_{i_n}$$

where $i_j \in \{\,1, 2\,\}$ then

$$\alpha = \sigma_{i_n,3} \sigma_{i_{n-1},3} \cdots \sigma_{i_1,3} \in \langle\, \mathrm{Inn}(W), \alpha_\zeta \mid \zeta \in \mathrm{Aut}(W_3) \,\rangle,$$

a contradiction. Since Lemma 5.6 gives $\mathrm{Inn}(W) \cap \langle\, \alpha_\zeta \mid \zeta \in \mathrm{Aut}(W_3) \,\rangle = \{1\}$ the result follows. ∎

$\mathcal{I}_3$: Let W be a Coxeter group of type $\mathcal{I}_3$ with diagram



where $m, n \geq 3$. The maximal finite subgroups are W_1 and W_3. Thus $\langle r_2 \rangle = W_1 \cap W_3$ and hence any automorphism of W maps r_2 to a reflection. Up to inner automorphisms $\alpha|_{W_1}$ is an automorphism of a group of type $I_2(m)$ with $m \geq 3$ and hence r_1 is always mapped to a reflection, similarly for r_3.

Let w_i be the longest element of W_i and let $c_i = w_i r_2$, for $i = 1, 3$. If m is even then $c_3 = r_a$ where $a = \frac{1}{\mathrm{s}(\pi_m)}(a_1 + \mathrm{c}(\pi_m)a_2)$ is the unique root in $\Phi^+_{\{a_1,a_2\}}$ that is orthogonal to a_2. Similarly, if n is even then $c_1 = r_b$ where $b = \frac{1}{\mathrm{s}(\pi_n)}(a_3 + \mathrm{c}(\pi_n)a_2)$. It follows from Theorem B of [BH99] that

$$C_W(r_2) = \begin{cases} \langle r_2 \rangle & \text{if } m \text{ and } n \text{ are odd,} \\ \langle r_2 \rangle \times \langle r_a \rangle & \text{if } m \text{ is even and } n \text{ is odd,} \\ \langle r_2 \rangle \times \langle r_b \rangle & \text{if } m \text{ is odd and } n \text{ is even,} \\ \langle r_2 \rangle \times \langle r_a, r_b \rangle & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

where $C_W(r_2)$ is the centralizer of r_2 in W. (See also [Bri96].) If m and n are both even then

$$a \cdot b = \frac{-\big(1 + \mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n)\big)}{\mathrm{s}(\pi_m)\,\mathrm{s}(\pi_n)} < -1,$$

and it follows that $\{\, a, b \,\}$ is a root basis in the sense of Definition 1.38. Thus r_a and r_b are simple reflections for the reflection subgroup of W they generate. The root system of this reflection subgroup is the set $\{\, x \in \Phi \mid x \cdot a_2 = 0 \,\}$.

5.9 Definition Let μ be an automorphism of W_3 which fixes r_2 and ν an automorphism of W_1 which fixes r_2. Define $\alpha_{\mu,\nu} : W \to W$ to be the endomorphism satisfying

$$\begin{aligned} \alpha_{\mu,\nu}(r_1) &= \mu(r_1) \\ \alpha_{\mu,\nu}(r_2) &= r_2 \\ \alpha_{\mu,\nu}(r_3) &= \nu(r_3), \end{aligned}$$

provided such an endomorphism exists.

In fact $\alpha_{\mu,\nu}$ exists in all cases. Clearly $\alpha_{\mu,\nu}(r_i)$ is an involution for all i. Now

$$\alpha_{\mu,\nu}(r_1 r_2) = \alpha_{\mu,\nu}(r_1)\alpha_{\mu,\nu}(r_2) = \mu(r_1)r_2 = \mu(r_1 r_2)$$

and hence $\big(\alpha_{\mu,\nu}(r_1)\alpha_{\mu,\nu}(r_2)\big)^m = 1$. Similarly $\big(\alpha_{\mu,\nu}(r_2)\alpha_{\mu,\nu}(r_3)\big)^n = 1$, and thus the required relations hold. and similarly for the rest. Thus $\alpha_{\mu,\nu}$ defines an endomorphism of W. It is clear that if $\mu, \mu' \in \mathrm{Aut}(W_3)$ and $\nu, \nu' \in \mathrm{Aut}(W_1)$ all fix r_2 then

$$\alpha_{\mu,\nu}\alpha_{\mu',\nu'} = \alpha_{\mu\mu',\nu\nu'}.$$

Furthermore, $\alpha_{\mu,\nu} = 1$ if and only if $\mu = 1$ and $\nu = 1$. Thus

$$\mathcal{A} = \{\, \alpha_{\mu,\nu} \mid \mu \in \mathrm{Aut}(W_3), \nu \in \mathrm{Aut}(W_1) \text{ and } \mu(r_2) = \nu(r_2) = r_2 \,\}$$

is a subgroup of $\mathrm{Aut}(W)$ isomorphic to the direct product of the stabilizers of r_2 in $\mathrm{Aut}(W_3)$ and $\mathrm{Aut}(W_1)$.

5.10 Lemma *With μ, ν as above, if $\alpha_{\mu,\nu}$ is an inner automorphism then μ and ν are both identity automorphisms or both conjugation by r_2.*

Proof If $\alpha_{\mu,\nu}$ is an inner automorphism, say $\alpha_{\mu,\nu}(x) = wxw^{-1}$, then we have

$$\begin{aligned} wr_2w^{-1} &= r_2 \\ wr_1w^{-1} &\in \langle r_1, r_2 \rangle \\ wr_3w^{-1} &\in \langle r_2, r_3 \rangle. \end{aligned}$$

Thus we have $wa_2 = \pm a_2$ and, using 1.12, $wa_1 = \frac{\mathrm{s}((h-1)\pi_m)}{\mathrm{s}(\pi_m)} a_1 + \frac{\mathrm{s}\left(h\pi_m\right)}{\mathrm{s}(\pi_m)} a_2$ for some h and $wa_3 = \frac{\mathrm{s}((k-1)\pi_n)}{\mathrm{s}(\pi_n)} a_1 + \frac{\mathrm{s}\left(k\pi_n\right)}{\mathrm{s}(\pi_n)} a_2$ for some k. Identifying elements of W with their matrices relative to $\{\, a_1, a_2, a_3 \,\}$,

$$w = \begin{bmatrix} \mathrm{s}\big((h-1)\pi_m\big)/\,\mathrm{s}(\pi_m) & 0 & 0 \\ \mathrm{s}(h\pi_m) & \pm 1 & \mathrm{s}\big((k-1)\pi_n\big)/\,\mathrm{s}(\pi_n) \\ 0 & 0 & \mathrm{s}(k\pi_n)/\,\mathrm{s}(\pi_n) \end{bmatrix}$$

As w can be written as a product of reflections $\det(w) = \pm 1$. Hence

$$\frac{\mathrm{s}\big((h-1)\pi_m\big)}{\mathrm{s}(\pi_m)} \times \frac{\mathrm{s}(k\pi_n)}{\mathrm{s}(\pi_n)} = \pm 1.$$

Each of the two factors on the left of this equation is 0 or is at least 1 in absolute value; thus we must have

$$\begin{aligned} \mathrm{s}\big((h-1)\pi_m\big) &= \pm\,\mathrm{s}(\pi_m) \\ \mathrm{s}(k\pi_n) &= \pm\,\mathrm{s}(\pi_n). \end{aligned}$$

Looking at each possibility we find that wa_1 is $\pm a_1$ or $\pm r_2(a_1)$ and wa_3 is $\pm a_3$ or $r_2(a_3)$. Thus

$$\begin{aligned} wr_1w^{-1} &= r_1 \quad \text{or} \quad r_2r_1r_2 \\ wr_2w^{-1} &= r_2 \\ wr_3w^{-1} &= r_3 \quad \text{or} \quad r_2r_3r_2. \end{aligned}$$

If $w \neq 1$ or r_2, then renumbering (swapping r_1 and r_2 if necessary) gives

$$\begin{aligned} wr_1w^{-1} &= r_1 \\ wr_2w^{-1} &= r_2 \\ wr_3w^{-1} &= r_2r_3r_2. \end{aligned}$$

But then

$$\begin{aligned} -1 &= a_1 \cdot a_3 \\ &= wa_1 \cdot wa_3 \\ &= \pm(a_1 \cdot r_2a_3) \\ &= \pm\big(a_1 \cdot (a_3 + 2\,\mathrm{c}(\pi_m)a_2)\big) \\ &= \pm\big(-1 - 2\,\mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n)\big). \end{aligned}$$

But this forces $\mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n) = 0$ or $|\,\mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n)| = 1$ contradicting $3 \leq m, n < \infty$. Therefore $\alpha_{\mu,\nu}$ is inner only when μ and ν are both the identity or both conjugation by r_2. ∎

We shall prove that $\mathrm{Aut}(W)$ is generated by inner automorphisms, graph automorphisms (if $m = n$) and the automorphisms $\alpha_{\mu,\nu}$ described above. Let α be an arbitrary element of $\mathrm{Aut}(W)$. Up to graph automorphisms we may assume that $\alpha(W_3)$ is conjugate to W_3 and $\alpha(W_1)$ is conjugate to W_1; so up to inner and graph automorphisms we may assume that $\alpha(W_3) = W_3$ and $\alpha(W_1) = uW_1u^{-1}$ for some $u \in W$. At the expense of modifying α by a further inner automorphism, we may assume that u is the minimal length element in W_3uW_1. Now

$$\alpha\big(\langle r_2\rangle\big) = \alpha(W_3) \cap \alpha(W_1) = W_3 \cap uW_1u^{-1},$$

and by Corollary 1.28 we deduce that $\alpha(r_2) = r_i = ur_ju^{-1}$ for some $i \in \{1, 2\}$ and $j \in \{2, 3\}$. Furthermore $a_i = u(a_j)$, since $u(a_2)$ and $u(a_3)$ are both positive.

If m and n are both odd then the only $w \in W$ with $w(a_2) = a_2$ is $w = 1$, and it follows that the equation $u(a_i) = a_j$ determines u uniquely (since a_1, a_2, a_3 all lie in the same W-orbit). If $j = 3$ then either $u = r_3r_2\overset{n-1}{\cdots}$ (if $i = 2$) or $u = (r_2r_1\overset{m-1}{\cdots})(r_3r_2\overset{n-1}{\cdots})$ (if $i = 1$). However, both of these contradict the fact that u has minimal length in W_3uW_1. Similarly, $j = 2$ and $i = 1$ leads to $u = r_2r_1\overset{m-1}{\cdots}$, which again contradicts the minimality of u. So $i = j = 2$ and $u = 1$. This gives $\alpha(W_3) = W_3$, $\alpha(W_1) = W_1$ and $\alpha(r_2) = r_2$, and it follows that $\alpha = \alpha_{\mu,\nu}$, where $\mu = \alpha|_{W_3}$ and $\nu = \alpha|_{W_1}$.

If n is odd and m even then r_1 is not conjugate to either r_2 or r_3, and so we must have $i = 2$. There are two elements $w \in W$ with $w(a_2) = a_2$, namely 1 and $r_1r_2\overset{m-1}{\cdots}$. So if $j = 2$ then either $u = 1$ or $u = r_1r_2\overset{m-1}{\cdots}$, and if $j = 3$ then either $u = r_3r_2\overset{n-1}{\cdots}$ or $u = (r_1r_2\overset{m-1}{\cdots})(r_3r_2\overset{n-1}{\cdots})$. Again, only the case $u = 1$ is consistent with the fact that u is minimal length in W_1uW_3, and as above we deduce that $\alpha = \alpha_{\mu,\nu}$.

If m is odd and n is even then r_3 is not conjugate to either r_2 or r_1, and so we must have $j = 2$. The only elements of W with $w(a_2) = a_2$ are $w = 1$ and $w = r_3r_2\overset{n-1}{\cdots}$. We deduce that if $i = 2$ then either $u = 1$ or $u = r_3r_2\overset{n-1}{\cdots}$, and if $i = 1$ then either $u = r_1r_2\overset{m-1}{\cdots}$ or $u = (r_1r_2\overset{m-1}{\cdots})(r_3r_2\overset{n-1}{\cdots})$. Only $u = 1$ is possible, and again $\alpha = \alpha_{\mu,\nu}$.

We are left with the case in which m and n are both even. For the next calculation we allow $n = 2$ as we shall refer to this calculation again in case $\mathcal{I}_5$. Since r_1, r_2, r_3 lie in separate conjugacy classes we deduce that $i = j = 2$, and u lies in the subgroup $\{ w \mid w(a_2) = a_2 \}$. As explained above, this group is generated by the reflections $r_a = r_1r_2\overset{m-1}{\cdots}$ and $r_b = r_3r_2\overset{n-1}{\cdots}$, and so u is an alternating product of r_a's and r_b's. Note that if the expression for u ends with r_b then $u(b)$ is negative, and since b is in the root system of W_1 this contradicts the minimality of u in W_3uW_1. Similarly, u cannot begin with r_a. So

$$u = (r_br_a)^k$$

for some k.

Thus we have $W = \langle\alpha(r_1), \alpha(r_2), \alpha(r_3)\rangle = \langle r_1, r_2, r_x\rangle$, where $x = (r_br_a)^k a_3$. Now

$$\begin{aligned}
x &= (r_br_a)^k a_3 \\
&= (r_br_a)^k\big(\,\mathrm{s}(\pi_n)b - \mathrm{c}(\pi_n)a_2\big) \\
&= \mathrm{s}(\pi_n)(r_br_a)^k b - \mathrm{c}(\pi_n)(r_br_a)^k a_2 \\
&= \mathrm{s}(\pi_n)(r_br_a)^k b - \mathrm{c}(\pi_n)a_2
\end{aligned}$$

since a and b are orthogonal to a_2. But $\{ a, b \}$ forms a root basis where

$$a \cdot b = \frac{-\big(1 + \mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n)\big)}{\mathrm{s}(\pi_m)\,\mathrm{s}(\pi_n)} = -\cosh(t)$$

for some t (since $a \cdot b < -1$). The elements of the (infinite dihedral) root system generated by a and b can be found by calculations analogous to those used to establish 1.12 and 1.13(the finite case). We find that

$$(r_br_a)^k b = \frac{\sinh\big((k+1)t\big)}{\sinh(t)}b + \frac{\sinh(kt)}{\sinh(t)}a.$$

We are given $a_1 \cdot a_2 = -\,\mathrm{c}(\pi_m)$ and as $(r_b r_a)^k a_2 = a_2$ we have

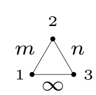
$$a_2 \cdot x = a_2 \cdot (r_b r_a)^k a_3 = (r_b r_a)^k a_2 \cdot (r_b r_a)^k a_3 = a_2 \cdot a_3 = -\,\mathrm{c}(\pi_n).$$

Finally

$$\begin{aligned}
x \cdot a_1 &= \mathrm{s}(\pi_n)\frac{\sinh\big((k+1)t\big)}{\sinh(t)} b \cdot a_1 \\
&\qquad + \mathrm{s}(\pi_n)\frac{\sinh(kt)}{\sinh(t)} a \cdot a_1 + \mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n) \\
&= \mathrm{s}(\pi_n)\frac{\sinh\big((k+1)t\big)}{\sinh(t)} \frac{\big(-1-\mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n)\big)}{\mathrm{s}(\pi_n)} \\
&\qquad + \mathrm{s}(\pi_n)\frac{\sinh(kt)}{\sinh(t)}\,\mathrm{s}(\pi_m) + \mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n) \\
&= \frac{\sinh\big((k+1)t\big)}{\sinh(t)}\big(-1-\mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n)\big) + \frac{\sinh(kt)}{\sinh(t)}\,\mathrm{s}(\pi_m)\,\mathrm{s}(\pi_n) + \mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n) \\
&\le -\frac{\sinh\big((k+1)t\big)}{\sinh(t)} + \frac{\sinh(kt)}{\sinh(t)} + \mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n) < 1
\end{aligned}$$

If $x \cdot a_1 > -1$ then it follows that the restriction of our bilinear form to the space V' spanned by a_1 and x is positive definite. By the main theorem of [Deo89], or alternatively Theorem 3.3 of [Dye90], the reflection subgroup $W' = \langle r_1, r_x \rangle$ is a Coxeter group on V'. This implies, by Lemma 1.1, that W' is finite, a contradiction as $r_x r_1 = \alpha(r_3 r_1)$ is infinite, and so $|x \cdot a_1| \ge 1$. Hence $x \cdot a_1 < -1$. Thus $\{\, a_1, a_2, x \,\}$ is a root basis for $\langle r_1, r_2, r_x \rangle = W$. Hence every root in Φ^+ can be expressed as a positive linear combination of a_1, a_2 and x. But it is clearly impossible to express a_3 as a positive linear combination of a_1, a_2 and x unless the coefficient of a_1 in x is zero. This coefficient is $\frac{\mathrm{s}(\pi_n)}{\mathrm{s}(\pi_m)} \frac{\sinh(kt)}{\sinh(t)}$, which is zero if and only if $k = 0$. This gives $x = a_3$ and so α is the identity. Thus we have proved the following.

5.11 Proposition *Let W be a Coxeter group of type $\mathcal{I}_3$ with diagram*



and $\mathcal{A} = \{\, \alpha_{\mu,\nu} \mid \mu \in \mathrm{Aut}(W_3), \nu \in \mathrm{Aut}(W_1)$ and $\mu(r_2) = \nu(r_2) = r_2 \,\}$. Then

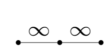
$$\mathrm{Aut}(W) = \mathrm{Inn}(W)\mathcal{A} \qquad \textit{(if } m \ne n\textit{)}$$

or

$$\mathrm{Aut}(W) = \mathrm{Inn}(W)\mathcal{A}\,\mathrm{Gr}(W) \qquad \textit{(if } m = n\textit{)}$$

where in the latter case we have $|\,\mathrm{Gr}(W)| = 2$. Further $\mathrm{Inn}(W) \cap \mathcal{A}$ has order 2, generated by conjugation by r_2.

$\mathcal{I}_4$: Let W be a Coxeter group of type $\mathcal{I}_4$ with diagram



The maximal finite standard parabolic subgroups are W_2 and $\langle r_2 \rangle$. Thus any automorphism of W maps r_2 to a conjugate of itself. The finite standard parabolics include the two just mentioned together with $\langle r_1 \rangle$ and $\langle r_3 \rangle$. So, by Proposition 2.3, representatives of the conjugacy classes of involutions are r_1, r_2, r_3 and $r_1 r_3$.

5.12 Definitions Define $\psi : W \to W$ by

$$\begin{aligned}\psi(r_1) &= r_1\\ \psi(r_2) &= r_2\\ \psi(r_3) &= r_1 r_3.\end{aligned}$$

Let γ be the obvious graph automorphism of W.

5.13 Lemma *The function ψ is an automorphism of W which does not preserve reflections.*

Proof The observations that r_1, r_2 and $r_1 r_3$ are involutions and that

$$\psi(r_1)\psi(r_3) = r_3 = \psi(r_3)\psi(r_1)$$

show that ψ is an endomorphism of W which does not preserve reflections. Noting that $\psi^2 = 1$ completes the proof. ∎

Let $\alpha \in \mathrm{Aut}(W)$. If α does not preserve reflections then either $\alpha(r_1)$ or $\alpha(r_3)$ belongs to the class of $r_1 r_3$. Up to graph automorphisms we may assume that $\alpha(r_3)$ is in the class of $r_1 r_3$. If $\alpha(r_1)$ is conjugate to r_1, then $\psi\alpha$ preserves reflections, and if $\alpha(r_1)$ is conjugate to r_3 then $\psi\gamma\alpha$ preserves reflections. So we may suppose that α preserves reflections. Furthermore, up to inner automorphisms $\alpha(W_2) = W_2$ and hence up to inner by graph automorphisms we may assume

$$\begin{aligned}\alpha(r_1) &= r_1\\ \alpha(r_3) &= r_3.\end{aligned}$$

We know that $\alpha(r_2) = w r_2 w^{-1}$ and (by Lemma 5.7) as α is surjective we must have $w \in W_2$. Hence α is inner, being conjugation by w. We have therefore proved most of the following.

5.14 Proposition *If W is a Coxeter group of type $\mathcal{I}_4$ with diagram*

$$\underset{1}{\bullet}\overset{\infty}{\text{——}}\underset{2}{\bullet}\overset{\infty}{\text{——}}\underset{3}{\bullet}$$

then $\mathrm{Aut}(W) = \mathrm{Inn}(W) \rtimes \langle \psi, \gamma \rangle$.

Proof It only remains to show that none of the automorphisms in $\langle \psi, \gamma \rangle$ are inner. As $\langle \psi, \gamma \rangle \cong \mathrm{Sym}_3$ we check the five non-identity automorphisms. Since

$$\begin{aligned}\psi(r_3) &= r_1 r_3\\ \gamma(r_1) &= r_3\\ \psi\gamma(r_3) &= r_1\\ \gamma\psi\gamma(r_1) &= r_1 r_3\\ \psi\gamma\psi\gamma(r_1) &= r_3\end{aligned}$$

none of the non-identity automorphisms preserve the conjugacy classes of reflections. So they cannot be inner. ∎

$\mathcal{I}_5$: Let W be a Coxeter group of type $\mathcal{I}_5$ with diagram

$$\underset{1}{\bullet}\overset{m}{\text{——}}\underset{2}{\bullet}\overset{\infty}{\text{——}}\underset{3}{\bullet}$$

The maximal finite standard parabolics are W_2 and W_3. Given that $\langle r_1 \rangle = W_2 \cap W_3$ we know that any automorphism of W maps r_1 to a reflection. If $\alpha(r_1)$ is conjugate to r_2 but not r_1 then m must be even; however, in this case r_2 is not conjugate to the reflections in W_2, whereas $\alpha(r_1)$ is in subgroups from both conjugacy classes of maximal finite parabolics. Thus $\alpha(r_1)$ is a conjugate of r_1.

Representatives of the classes of involutions are r_1, r_2, r_3, $r_1 r_3$ and the longest element of W_3; this last only occurs if m is even and if m is odd then r_1 and r_2 are conjugate. Now r_3 is not conjugate to any element in a parabolic subgroup of order $2m$ and so cannot be mapped to an element in W_3 by any automorphism. Thus, if $\alpha \in \mathrm{Aut}(W)$ then $\alpha(r_3)$ is conjugate to r_3 or $r_1 r_3$.

5.15 Definitions Let $\psi : W \to W$ be the automorphism given by

$$\psi(r_1) = r_1, \qquad \psi(r_2) = r_2 \qquad \text{and} \qquad \psi(r_3) = r_1 r_3,$$

and for any automorphism ζ of W_3 such that $\zeta(r_1) = r_1$ let $\alpha_\zeta : W \to W$ be the automorphism given by

$$\alpha_\zeta(r_1) = r_1, \qquad \alpha_\zeta(r_2) = \zeta(r_2) \qquad \text{and} \qquad \alpha_\zeta(r_3) = r_3.$$

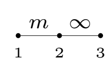
(Thus α_ζ is analogous to $\alpha_{\zeta,1}$ for the $\mathcal{I}_3$ case.) Let

$$\mathcal{B} = \mathrm{gp}\langle \psi, \{\, \alpha_\zeta \mid \zeta \in \mathrm{Aut}(W_3) \text{ such that } \zeta(r_1) = r_1 \,\}\rangle.$$

Notice that $\alpha_\zeta \psi = \psi \alpha_\zeta$ for all ζ. Notice also that if $\zeta \in \mathrm{Aut}(W_3)$ is conjugation by r_1 then α_ζ is conjugation by r_1 on W. It is not hard to see that this is the only nontrivial inner automorphism in the group $\mathcal{B}$. For suppose that $w \in W$ induces an inner automorphism that lies in $\mathcal{B}$. Since all elements of $\mathcal{B}$ map r_3 to r_3 or $r_1 r_3$, it follows that $w r_3 w^{-1} = r_3$, since $r_1 r_3$ is not a reflection. Furthermore, $w r_1 w^{-1} = r_1$, and $w r_2 w^{-1} = r_x$ for some x in the root system of W_3. So $w(a_2)$ is in the root system of W_3, and since $w(a_2) \cdot w(a_1) = a_2 \cdot a_1$ it follows that $w(a_2)$ is either a_2 or $r_1(a_2)$. The former possibility gives the identity automorphism and the latter gives conjugation by r_1.

Now let α be any automorphism of W. Then up to the automorphism ψ we may assume that α preserves reflections. Up to inner automorphisms we may assume that $\alpha(W_3) = W_3$ and $\alpha(r_1) = r_1$. Then $\mu = \alpha|_{W_3}$ is an automorphism of W_3 that fixes r_1, and so replacing α with $\alpha_{\mu^{-1}}\alpha$ we get an automorphism of W that fixes both r_1 and r_2. As $\alpha(r_3)$ is a conjugate of r_3 the methods used for $\mathcal{I}_3$ finish the proof of the following.

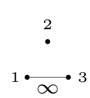
5.16 Proposition *If W is a Coxeter group of type $\mathcal{I}_5$ with diagram*



then $\mathrm{Aut}(W) = \mathrm{Inn}(W)\mathcal{B}$ *and* $|\,\mathrm{Inn}(W) \cap \mathcal{B}| = 2$.

Note that if ζ is conjugation by r_1 then α_ζ is inner, being conjugation by r_1. All other ζ give rise to outer automorphisms.

There is one case still outstanding: $\tilde{A}_1 \times A_1$. Let W be a Coxeter group of type $\tilde{A}_1 \times A_1$ with diagram



The finite standard parabolic subgroups are $\langle r_1 \rangle$, $\langle r_2 \rangle$, $\langle r_3 \rangle$, $\langle r_1, r_2 \rangle$ and $\langle r_2, r_3 \rangle$. It is clear that $\langle r_2 \rangle = Z(W)$ thus we have $\alpha(r_2) = r_2$ for all automorphisms α of W.

5.17 Definition Define $\alpha_{12} : W \to W$ by

$$\begin{aligned}\alpha_{12}(r_1) &= r_1 r_2\\ \alpha_{12}(r_2) &= r_2\\ \alpha_{12}(r_3) &= r_3.\end{aligned}$$

It is easily seen that α_{12} is an automorphism of W. Let γ denote the obvious graph automorphism.

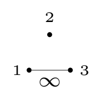
Let α be any automorphism of W. Following α with an inner by graph automorphism if necessary we may assume that $\alpha(W_3) = W_3$. Now $\alpha(r_2) = r_2$ and so either $\alpha(r_1) = r_1$ or $\alpha(r_1) = r_1 r_2$. If $\alpha(r_1) = r_1 r_2$, then $\alpha_{12}\alpha(r_1) = \alpha_{12}(r_1 r_2) = r_1 r_2 r_2 = r_1$. Thus, replacing α with $\alpha_{12}\alpha$ we may assume that $\alpha(r_1)$ is conjugate to r_1. Then $\alpha(r_3)$ is conjugate to

r_3 or r_2r_3. If $\alpha(r_3) = w'r_2r_3{w'}^{-1}$ then $\gamma\alpha_{12}\gamma\alpha(r_1) = r_1$ and $\gamma\alpha_{12}\gamma\alpha(r_3) = wr_3w^{-1}$, where $w = \gamma\alpha_{12}\gamma(w')$. So we may assume up to inner, γ and α_{12}, that

$$\begin{aligned}\alpha(r_1) &= r_1\\ \alpha(r_2) &= r_2\\ \alpha(r_3) &= wr_3w^{-1}.\end{aligned}$$

As r_2 is central we may assume that $w \in W_2$, and hence $\alpha|_{W_2}$ is an automorphism of a Coxeter group of type $\tilde{A}_1$ with $r_1 \mapsto r_1$. Thus $\alpha|_{W_2}$ is inner, by Proposition 4.5, and therefore α is inner.

5.18 Proposition *If W is a Coxeter group of type $\tilde{A}_1 \times A_1$ with diagram*



then $\mathrm{Aut}(W) = \mathrm{Inn}(W) \rtimes \langle\alpha_{12}, \gamma\rangle$.

Proof It is easily checked that $\langle\alpha_{12}, \gamma\rangle$ is a group of type $I_2(4)$ and that none of the non-identity automorphisms in this group preserves the classes of reflections. Thus

$$\mathrm{Inn}(W) \cap \langle\alpha_{12}, \gamma\rangle = \{\, 1 \,\}\,.$$

∎

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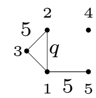
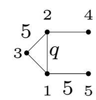
Index and Notation

Errata

Page 56 line 14. Replace the diagram



with the diagram



Page 56, line 21. Replace '$q \leq 5$' with '$q = 5$'.

Page 56, line 24. Replace the sentence 'Thus $\mathrm{Aut}(W) = R(W)$.' with: Thus $\mathrm{Aut}(W) = R(W)$ if $q \geq 5$. If $q = 3$ or 4 then W_3 is the unique maximal finite standard parabolic subgroup of its type. The general argument in the proof of Theorem 3.16 then shows that all automorphisms are inner by graph if $q = 3$ or 4. We now concentrate on the $q \geq 5$ case.