Sydney University Mathematical Society
Problems Competition 2005

Solutions.

1. Suppose that we look at the set X_n of strings of 0's and 1's of length n. Given a string $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in X_n$, we are allowed to change ϵ according to the following rules:

(i) We are allowed to change ϵ_1 to $1 - \epsilon_1$;

(ii) if $2 \le k \le n$, we are allowed to change ϵ_k to $1 - \epsilon_k$ if and only if $\epsilon_{k-1} = 1$ and $\epsilon_j = 0$ for all $j < k - 1$.

Find a formula for the least number of such changes needed to turn the string $(0, 0, \ldots, 0)$ into $(1, 1, \ldots, 1)$.

Solution. Let x_n denote the required number of changes. The answer is

$$x_n = \lfloor 2^{n+1}/3 \rfloor = \begin{cases} \frac{2^{n+1}-1}{3} & \text{if } n \text{ is odd} \\ \frac{2^{n+1}-2}{3} & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

To see this, we also introduce, if $n \ge 2$, the smallest number y_n of changes needed to turn the string $(0, 0, \ldots, 0)$ into $(0, \ldots, 0, 1)$. We also set $y_1 = x_1$ $(= 1)$.

By the symmetric nature of the steps, the smallest number of them needed to turn $(0, \ldots, 0, 1)$ into $(0, \ldots, 0, 0)$ is also y_n.

Notice that $y_2 = 3$, because we must make the moves $(0, 0) \to (1, 0) \to (1, 1) \to (0, 1)$. Now assume that $n \ge 3$.

In going from $(0, 0, \ldots, 0)$ into $(0, \ldots, 0, 1)$, the first time that a 1 appears in the n-th position must be in a move $(0, \ldots, 0, 1, 0) \to (0, \ldots, 0, 1, 1)$. Thereafter, the string $(0, \ldots, 0, 1, 1)$ must be transformed into the string $(0, \ldots, 0, 1)$, and if we want no unnecessary steps, then the 1 in the n-th place must remain unchanged. So the number of steps needed to change $(0, \ldots, 0, 1, 1)$ to $(0, \ldots, 0, 1)$ is y_{n-1}, because of the symmetry mentioned above.

So in going from $(0, 0, \ldots, 0)$ into $(0, \ldots, 0, 1)$, we first need at least y_{n-1} steps to arrive at $(0, \ldots, 0, 1, 0)$, then one step from $(0, \ldots, 0, 1, 0)$ to $(0, \ldots, 0, 1, 1)$, then y_{n-1} more steps to go from $(0, \ldots, 0, 1, 1)$ to $(0, \ldots, 0, 0, 1)$.

Hence $y_n = 2y_{n-1} + 1$. This is true for all $n \ge 2$. Therefore $y_n + 1 = 2(y_{n-1} + 1)$, and it follows by induction that $y_n + 1 = 2^n$ for all $n \ge 1$.

We now get a recurrence relation for x_n. Notice that $x_2 = 2$, because the moves required are $(0, 0) \to (1, 0) \to (1, 1)$. Now assume that $n \ge 3$. In going from $(0, 0, \ldots, 0)$ into $(1, \ldots, 1, 1)$, the first time that a 1 appears in the n-th position must be in a move $(0, \ldots, 0, 1, 0) \to (0, \ldots, 0, 1, 1)$. So first y_{n-1} steps are needed to move from $(0, 0, \ldots, 0)$ to $(0, \ldots, 1, 0)$. Then there is the one step from $(0, \ldots, 0, 1, 0) \to (0, \ldots, 0, 1, 1)$. Thereafter, the string $(0, \ldots, 0, 1, 1)$ is transformed into $(1, \ldots, 1, 1)$ in x_{n-2} moves. Hence $x_n = y_{n-1} + 1 + x_{n-2}$, so that $x_n = 2^{n-1} + x_{n-2}$ for all $n \ge 3$. It is now a routine induction to show that (1) holds.

2. Determine all positive integers n for which

$$\lfloor 2\sqrt{n} \rfloor = 1 + \lfloor \sqrt{n-1} + \sqrt{n+1} \rfloor.$$

Solution. Label the given equation (1). We first check that

$$2\sqrt{n} - 1 < \sqrt{n-1} + \sqrt{n+1} < 2\sqrt{n} \tag{2}$$

for $n = 1, 2, \ldots$. To see the right hand inequality, just notice that $f(x) = \sqrt{x}$ is strictly convex downwards on $(0, \infty)$, since $f''(x) < 0$. So $(f(a) + f(b))/2 < f((a+b)/2)$ for any distinct $a, b > 0$. Applying this to $a = n - 1$ and $b = n + 1$ we get the right hand side.

The left hand inequality in (2) holds for $n = 1$, since $1 < \sqrt{2}$. Suppose that $n \geq 2$. Then

$$2\sqrt{n} - 2\sqrt{n-1} = \frac{2}{\sqrt{n} + \sqrt{n-1}} < 1,$$

and so $2\sqrt{n} - 1 < 2\sqrt{n-1} < \sqrt{n-1} + \sqrt{n+1}$.

Now suppose that $n = m^2$ is a perfect square. Then (2) tells us that $2m - 1 < \sqrt{n-1} + \sqrt{n+1} < 2m$, and so $\lfloor \sqrt{n-1} + \sqrt{n+1} \rfloor = 2m - 1$. Therefore

$$1 + \lfloor \sqrt{n-1} + \sqrt{n+1} \rfloor = 1 + (2m - 1) = \lfloor 2\sqrt{n} \rfloor,$$

so that equality holds in (1).

Now suppose that n is not a perfect square. The interval $[2\sqrt{n} - 1, 2\sqrt{n}]$ has length 1, and its endpoints are not integers. So it contains exactly one integer, k say. Notice that $k = \lfloor 2\sqrt{n} \rfloor$.

If $\sqrt{n-1} + \sqrt{n+1} < k$, then $\sqrt{n-1} + \sqrt{n+1} < k < 2\sqrt{n}$. Squaring this and rearranging, we get $2\sqrt{n^2 - 1} < k^2 - 2n < 2n$. Again squaring and rearranging, we get $0 < k^2(4n - k^2) < 4$, which can only hold for positive integers k, n, if $k = n = 1$, a case we are excluding, since n must be a nonsquare here.

So $\sqrt{n-1} + \sqrt{n+1} \geq k$ must hold, and so

$$1 + \lfloor \sqrt{n-1} + \sqrt{n+1} \rfloor \geq 1 + k = 1 + \lfloor 2\sqrt{n} \rfloor.$$

So (1) does not hold when n is not a perfect square. It therefore holds if and only if n is a perfect square.

3. Let $P(x)$ be a polynomial with real coefficients such that, for some constants a, b,

$$P(x) + aP'(x) + bP''(x) \geq 0 \quad \text{for all } x \in \mathbb{R}.$$

Suppose that $a^2 \geq 4b$. Show that $P(x) \geq 0$ for all $x \in \mathbb{R}$.

Solution. Step 1. We first prove the result when $b = 0$, showing that for any polynomial $Q(x)$ with real coefficients,

$$\text{if } Q(x) + aQ'(x) \geq 0 \text{ for all } x, \text{ then } Q(x) \geq 0 \text{ for all } x. \tag{1}$$

We do this by showing that $Q(x)$ has a minimum, and then showing that the minimum value is at least 0.

For suppose $Q(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ (with $c_n \neq 0$), then $Q(x) + aQ'(x)$ has degree n and leading coefficient c_n. The hypothesis in (1) implies that n is even and that $c_n > 0$. This in turn implies that $Q(x) \to \infty$ as $x \to \infty$ and as $x \to -\infty$. So there is a number M so that $Q(x) \geq Q(0)$ once $|x| > M$. Since Q is continuous on the closed and bounded interval $[-M, M]$, the set of values taken by $Q(x)$, as x varies over this interval, has a minimum m, say, attained at some point $x_0 \in [-M, M]$. In particular, $Q(0) \geq m$, and so $Q(x) \geq m$ for all $x \in \mathbb{R}$ outside $[-M, M]$ as well as for all $x \in [-M, M]$. So m is the minimum value taken by $Q(x)$ for any $x \in \mathbb{R}$.

Since $Q(x)$ is differentiable, $Q'(x_0) = 0$ must hold. Therefore

$$m = Q(x_0) = Q(x_0) + aQ'(x_0),$$

which is ≥ 0 because $Q(x) + aQ'(x) \geq 0$ for all x. This means that $Q(x) \geq 0$ for all x, and therefore Step 1 is completed.

Step 2. We now deduce the stated result. So assume that $P(x) + aP'(x) + bP''(x) \geq 0$ for all $x \in \mathbb{R}$, where $a, b \in \mathbb{R}$ and $a^2 \geq 4b$. There are numbers α and β so that $Q(x) = P(x) + \beta P'(x)$ satisfies

$$P(x) + aP'(x) + bP''(x) = Q(x) + \alpha Q'(x) \tag{2}$$

for all x. Assume this for a moment. Then our hypothesis implies that $Q(x) + \alpha Q'(x) \geq 0$ for all x, and so $Q(x) \geq 0$ for all x, by Step 1. Since $Q(x) = P(x) + \beta P'(x)$, another application of Step 1 shows that $P(x) \geq 0$ for all x.

Finally, we show that α and β can be chosen so that $Q(x) = P(x) + \beta P'(x)$ satisfies (2). The hypothesis that $a^2 \geq 4b$ means that the quadratic equation $1 + ax + bx^2 = 0$ has a real solution, which of course is nonzero. If we call this solution $-1/\alpha$, then we can write $1 + ax + bx^2 = (1 + \alpha x)(1 + \beta x)$ for some $\beta \in \mathbb{R}$. This implies that $a = \alpha + \beta$ and $b = \alpha\beta$. It is now routine to see that for this α and β, (2) holds for $Q(x) = P(x) + \beta P'(x)$.

Generalization. More generally, suppose that $P(x)$ is a polynomial with real coefficients, and suppose that

$$P(x) + a_1 P'(x) + a_2 P''(x) + \cdots + a_r P^{(r)}(x) \geq 0 \quad \text{for all } x \in \mathbb{R},$$

where $1 + a_1\lambda + a_2\lambda^2 + \cdots + a_r\lambda^r$ is a polynomial having r real roots (so that it may be factored

$$1 + a_1\lambda + a_2\lambda^2 + \cdots + a_r\lambda^r = (1 + \alpha_1\lambda)(1 + \alpha_2\lambda)\cdots(1 + \alpha_r\lambda),$$

for some $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$). Then $P(x) \geq 0$ for all $x \in \mathbb{R}$. This may be proved by induction on r. The induction step involves writing

$$P(x) + a_1 P'(x) + a_2 P''(x) + \cdots + a_r P^{(r)}(x) = Q(x) + \alpha_1 Q'(x),$$

where $Q(x) = P(x) + b_1 P'(x) + \cdots + b_{r-1} P^{(r-1)}(x)$, the coefficients b_j coming from the factorization

$$1 + a_1\lambda + a_2\lambda^2 + \cdots + a_r\lambda^r = (1 + \alpha_1\lambda)(1 + b_1\lambda + \cdots + b_{r-1}\lambda^{r-1}).$$

These manipulations are made more natural by working with polynomials in the differentation operator $D = \frac{d}{dx}$, though this is not essential.

4. Suppose that a, b and k are integers, such that

$$0 < a^2 + b^2 - kab \le k$$

Prove that $a^2 + b^2 - kab$ is a perfect square.

Solution. We prove a slightly stronger statement: if $0 < a^2 + b^2 - kab \le k + 1$, then $a^2 + b^2 - kab$ is a perfect square.

First we show that $ab \ge 0$. Suppose, by way of a contradiction, that $ab < 0$. Then $a^2 + b^2 - kab = a^2 + b^2 + k|ab| \ge 1 + 1 + k > k + 1$, contrary to hypothesis.

Since $ab \ge 0$, by replacing (a, b) with $(-a, -b)$ if necessary, we may assume that $a, b \ge 0$.

Let $a^2 + b^2 - kab = d$. For a fixed k and d, consider the minimal solution of the equation $a^2 + b^2 - kab = d$ in non-negative integers a and b, where by minimal, we mean that with a minimal value of $\min(a, b)$. Suppose without loss of generality that $a \ge b$.

The quadratic equation $x^2 - kbx + b^2 - d = 0$ has one integer root, a, so must have 2 integer roots. Let c be this second integer root. Then $c^2 + b^2 - kcb = d$. So $bc \ge 0$ by the first paragraph of this proof. Suppose $b > 0$. Then $c \ge 0$ because $bc \ge 0$. But $(x - a)(x - c) = x^2 - kbx + b^2 - d$, and so $ac = b^2 - d < b^2$. Therefore $c < b^2/a \le b$, contradicting our assumption of minimality.

So our minimal solution must have $b = 0$ and thus $d = a^2$ is a perfect square, as required.

5. Show that there is no four term arithmetic progression, whose terms are distinct square numbers.

Solution. Suppose a^2, b^2, c^2, d^2 form an arithmetic progression in that order. We may suppose that a, b, c and d are non-negative, and so $0 \le a < b < c < d$. Then by hypothesis,

$$b^2 - a^2 = c^2 - b^2 = d^2 - c^2. \tag{1}$$

Let $\gcd(a, b, c, d) = k$. Then the squares of a/k, b/k, c/k and d/k also form an arithmetic progression, and these four numbers have greatest common divisor 1.

So we may assume from the outset that $k = 1$. This implies that a, b, c and d are all odd. For if a is even, then by (1), $c^2 = 2b^2 - a^2$ is also even, so that c is even. Since $2b^2 = a^2 + c^2$, and the right hand side is divisible by 4, b is even. Finally, $d^2 = 2c^2 - b^2$ shows that d is even too. But then $k \ge 2$, contrary to assumption. So a is odd. Similarly, b, c and d are odd.

Now (1) implies in particular that $b^2 - a^2 = d^2 - c^2$, and so

$$(b - a)(b + a) = (d - c)(d + c).$$

If $x' = \gcd(b - a, d + c)$, then we can write $b - a = wx'$ and $c + d = x'z$ for some positive integers w, z such that $\gcd(w, z) = 1$. Then $w(b + a) = z(d - c)$. Since z divides $w(b + a)$

and $\gcd(z, w) = 1$, z must divide $b+a$, and so we can write $b+a = y'z$ for some integer y'. Substituting this into the equation $w(b+a) = z(d-c)$ and cancelling z, we get $d-c = wy'$.

Notice that x' is even, because both $b - a$ and $d + c$ are even. Both $wy' = d - c$ and $y'z = b + a$ are even, and $\gcd(w, z) = 1$. It follows that y' is also even. Write $x' = 2x$ and $y' = 2y$. Then $a = (y'z - wx')/2 = yz - wx$, $b = (y'z + wx')/2 = yz + wx$, $c = (x'z - wy')/2 = xz - wy$ and $d = (x'z + wy')/2 = xz + wy$. These equations imply that $\gcd(x, y) = 1$, because $\gcd(a, b, c, d) = 1$.

If we now substitute these expressions for a, b, c and d in terms of x, y, z and w into the further condition that $c^2 - b^2 = b^2 - a^2$, we obtain, after rearranging:

$$(z^2 - w^2)(x^2 - y^2) = 8wxyz. \tag{2}$$

Notice that x, y, z, w are all positive, and recall that $\gcd(x, y) = \gcd(w, z) = 1$.

If $z^2 - w^2$ and $x^2 - y^2$ are both even, then they are both divisible by four, so $wxyz$ is even. If w is even, then $z^2 - w^2$ even implies z even, contradicting $\gcd(w, z) = 1$. Similar contradictions occur if any of the other variables are assumed even. Hence one of $z^2 - w^2$ and $x^2 - y^2$ is odd. Moreover, (2) implies that $z^2 - w^2$ and $x^2 - y^2$ have the same sign.

Case 1. Suppose that $x^2 - y^2$ is odd and positive.

Then (2) implies that $z^2 - w^2$ is divisible by 8. Now $\gcd(x, y) = 1$ implies that $\gcd(xy, x^2 - y^2) = 1$. Since $x^2 - y^2$ is odd, this implies that $\gcd(8xy, x^2 - y^2) = 1$. From (2), we see that $8xy$ divides $(z^2 - w^2)(x^2 - y^2)$. Therefore $8xy \mid z^2 - w^2$. Similarly, $\gcd(zw, z^2 - w^2) = 1$, and since (2) implies that zw divides $(z^2 - w^2)(x^2 - y^2)$, we see that $zw \mid x^2 - y^2$. Now writing $z^2 - w^2 = 8xys$ and $x^2 - y^2 = zwt$, we see from (2) that $st = 1$, and so $s = t = 1$ because $x^2 - y^2$ is positive. Hence $z^2 - w^2 = 8xy$ and $x^2 - y^2 = zw$.

Arguing just as we did after Equation (1), for any solution to $(x - y)(x + y) = zw$ we can write $x - y = AB$, $x + y = CD$, $z = AC$ and $w = BD$ for some positive integers A, B, C and D with $\gcd(B, C) = 1$. Substituting into $z^2 - w^2 = 8xy$ gives after rearranging:

$$C^2(A^2 - 2D^2) = B^2(D^2 - 2A^2). \tag{3}$$

If p is prime and p divides both $A^2 - 2D^2$ and $D^2 - 2A^2$, then $A^2 \equiv 2D^2 \equiv 4A^2 \pmod{p}$, so that p divides $3A^2$. If $p \mid A$, then $p \mid D$ because p divides $D^2 - 2A^2$, again contradicting $\gcd(z, w) = 1$. So $p = 3$. It is easily seen that there are no nonzero solutions to $x^2 - 2y^2 = 0$ in $\mathbb{Z}_3$. So the fact that $p = 3$ divides $D^2 - 2A^2$ implies that it divides both A and D, and this again contradicts $\gcd(w, z) = 1$. So $\gcd(A^2 - 2D^2, D^2 - 2A^2) = 1$. We know that $\gcd(B, C) = 1$, and therefore $\gcd(B^2, C^2) = 1$ too.

Hence we have $A^2 - 2D^2 = \pm B^2$ and $D^2 - 2A^2 = \pm C^2$ (with the same sign). If the plus sign occurs, then $A^2 \geq 2D^2 \geq 4A^2$ giving $A = 0$, which is impossible, because z is positive. Hence we must have $A^2 + B^2 = 2D^2$ and $D^2 + C^2 = 2A^2$. Therefore, either the quadruple (C^2, A^2, D^2, B^2) or the quadruple (B^2, D^2, A^2, C^2) is an arithmetic progression of squares. Recalling that $0 < a < b < c < d$, we have

$$ABCD = wz \leq wxyz = \frac{1}{4}wx'y'z = \frac{1}{4}(b^2 - a^2) < \frac{b^2}{4} < \frac{abcd}{4} < abcd.$$

But we could have assumed from the beginning that our arithmetic progression was chosen with the properties that $abcd$ was minimal (as well as satisfying $\gcd(a,b,c,d)=1$). So we have a contradiction.

Case 2. Suppose that x^2-y^2 is odd and negative.

In this case, we find as above that $y^2-x^2=zw$, $w^2-z^2=8xy$, and from the first of these equations we find positive integers A, B, C and D so that $y-x=AB$, $y+x=CD$, $z=AC$, $w=BD$ and $\gcd(A,D)=1$. The second equation then leads to $A^2(C^2-2B^2)=D^2(B^2-2C^2)$, and we get a contradiction as in Case 1.

Case 3. Suppose that x^2-y^2 is even and positive.

In this case, we find as above that $x^2-y^2=8zw$ and $z^2-w^2=xy$, and from the first of these equations we find positive integers A, B, C and D so that $z-w=AB$, $z+w=CD$, $x=AC$, $y=BD$ and $\gcd(B,C)=1$. The second equation then leads to $C^2(A^2-2D^2)=B^2(D^2-2A^2)$, and we get a contradiction as in Case 1.

Case 4. Suppose that x^2-y^2 is even and negative.

In this case, we find as above that $w^2-z^2=xy$ and $y^2-x^2=8zw$, and from the first of these equations we find positive integers A, B, C and D so that $w-z=AB$, $w+z=CD$, $x=AC$, $y=BD$ and $\gcd(A,D)=1$. The second equation then leads to $A^2(C^2-2B^2)=D^2(B^2-2C^2)$, and we get a contradiction as in Case 1.

Hence each of the four cases leads to a contradiction, and so there can be no arithmetic progression made up of four distinct square integers.

6. Form a graph with vertex set $\mathbb{Z}$ in which there is an edge from m to n if and only if $|m-n|\in\{1,2,4,8,\ldots\}$. If $m,n\in\mathbb{Z}$, let $\mathrm{dist}(m,n)$ denote the distance from m to n in this graph, that is, the length of a shortest path joining m and n. For example, $\mathrm{dist}(0,6)=2$ because there is an edge from 0 to 4 and one from 4 to 6, but there is no edge from 0 to 6. Now let $B=\{n\in\mathbb{Z}:\mathrm{dist}(n,0)\le r\}$ be the "ball of radius r about 0" in this graph. Show that the complement of B is connected. That is, show that it is possible to join any two points outside B by a path all of whose vertices lie outside B.

Solution. Suppose that r is a positive integer, and that $n\in\mathbb{Z}$ satisfies $\mathrm{dist}(n,0)\le r$ in the graph. Then we can write

$$n=\epsilon_1 2^{m_1}+\cdots+\epsilon_r 2^{m_r}, \tag{1}$$

where $\epsilon_1,\ldots,\epsilon_r\in\{-1,1\}$ and $m_1,\ldots,m_r\in\mathbb{N}$ (where $\mathbb{N}=\{0,1,2,\ldots\}$). For there is a path $x_0,x_1,\ldots,x_r$ from 0 to n. So $x_0=0$, and $x_1=2^m$ or -2^m for some $m\in\mathbb{N}$. Therefore $x_1=\epsilon_1 2^{m_1}$ for $\epsilon=1$ or -1 and for $m_1=m$. Then x_2 is a neighbour of x_1, and so equals $x_1+2^{m'}$ or $x_1-2^{m'}$ for some $m'\in\mathbb{N}$. So $x_2=x_1+\epsilon_2 2^{m_2}$ for $\epsilon=1$ or -1 and $m_2=m'$. Continuing in this way, we find that $x_r=n$ is expressible in the form (1). Conversely, starting from an expression (1) for n, we get a path $x_0,x_1,\ldots,x_r$ from 0 to n by setting x_j equal to the sum of the first j terms in the sum on the right in (1).

Because of the last sentence, we may assume that $m_1\le m_2\le\cdots\le m_r$, and we always do so below.

Now suppose that in (1), r is equal to $\mathrm{dist}(n,0)$. Then no two m_j's are equal. For if $m_j=m_{j+1}$ and ϵ_j and ϵ_{j+1} are distinct, then the two terms $\epsilon_j 2^{m_j}$ and $\epsilon_{j+1}2^{m_{j+1}}$ cancel, and then n could be expressed as a sum (1) with two fewer terms, so that $\mathrm{dist}(n,0) \le r-2$, contrary to assumption. If $m_j = m_{j+1}$ and $\epsilon_j = \epsilon_{j+1}$, then $\epsilon_j 2^{m_j} + \epsilon_{j+1} 2^{m_{j+1}} = \epsilon_j 2^{m_j+1}$, so n could be expressed as a sum (1) with one fewer term.

Moreover, assuming still that $\mathrm{dist}(n,0) = r$, it cannot happen that $m_j = k$ and $m_{j+1} = k+1$ with ϵ_j and ϵ_{j+1} are distinct. For if $\epsilon_j = 1$ and $\epsilon_{j+1} = -1$, say, then $\epsilon_j 2^{m_j} + \epsilon_{j+1} 2^{m_{j+1}} = 2^k - 2^{k+1} = -2^k$, so that n could be expressed as a sum (1) with one fewer term.

Moreover, it cannot happen that $m_j = k$, $m_{j+1} = k+1$ and $m_{j+2} = k+2$ for some k. For if so, then ϵ_j, ϵ_{j+1} and ϵ_{j+2} would be equal by the previous paragraph, and then

$$\epsilon_j 2^{m_j} + \epsilon_{j+1} 2^{m_{j+1}} + \epsilon_{j+2} 2^{m_{j+2}} = \epsilon_j (2^k + 2^{k+1} + 2^{k+2}) = -\epsilon_j 2^k + \epsilon_j 2^{k+3}$$

so that n could be expressed as a sum (1) with one fewer term.

Now if $\mathrm{dist}(n,0) = r$, we can choose a representation (1) of n in which $m_{j+1} \ge m_j + 2$ for each $j < r$. For suppose not. Then we choose a representation (1) for n in which $m_{i+1} = m_i + 1$ for the smallest possible number of i's. We may suppose that, amongst all such representations, the largest index $i \le r-1$ for which $m_{i+1} = m_i + 1$ is as large as possible, and we write j for this index. Now write $m_j = k$ and $m_{j+1} = k+1$. Then by the above we have $\epsilon_j = \epsilon_{j+1}$ and (if $j \le r-2$) $m_{j+2} \ne k+2$. Then we can re-write

$$\epsilon_j 2^{m_j} + \epsilon_{j+1} 2^{m_{j+1}} = \epsilon_j (2^k + 2^{k+1}) = -\epsilon_j 2^k + \epsilon_j 2^{k+2}.$$

In the sum (1), in which $\epsilon_j 2^{m_j} + \epsilon_{j+1} 2^{m_{j+1}}$ has been replaced by $-\epsilon_j 2^k + \epsilon_j 2^{k+2}$, either the number of i for which $m_{i+1} = m_i + 1$ is decreased by 1 (if $j = r-1$ or if $j \le r-2$ and $m_{j+2} \ne k+3$) or the number of such i remains the same (if $j \le r-2$ and $m_{j+2} = k+3$), but with the largest such i now equal to $j+1$. But this contradicts the choice of j.

Suppose that n is written in the form (1), with $m_{j+1} \ge m_j + 2$ for each $j < r$. Then

$$|\epsilon_1 2^{m_1} + \cdots + \epsilon_{r-1} 2^{m_{r-1}}| \le 2^{m_{r-1}} + \cdots + 2^{m_1} < 2^{m_r - 2} + 2^{m_r - 4} + \cdots = \frac{1}{3} 2^{m_r}.$$

Hence

$$\frac{2}{3} 2^{m_r} < |n| < \frac{4}{3} 2^{m_r}. \tag{2}$$

Now suppose that $n \in \mathbb{Z}$ and that $k \in \mathbb{N}$, with $2^k \ge 3|n|$. We will now show that $\mathrm{dist}(n + 2^k, 0) = \mathrm{dist}(n,0) + 1$. To see this, notice that $\frac{2}{3} 2^k \le |n + 2^k| \le \frac{4}{3} 2^k$. So if (1) is a representation of $n + 2^k$ in which $m_{j+1} \ge m_j + 2$ for each $j < r$, then by (2), with n replaced by $n + 2^k$, we have

$$\frac{2}{3} 2^{m_r} < |n + 2^k| \le \frac{4}{3} 2^k,$$

so that $2^{m_r} < 2^{k+1}$, which implies that $m_r \le k$. Also,

$$\frac{2}{3} 2^k \le |n + 2^k| < \frac{4}{3} 2^{m_r},$$

so that $2^k < 2^{m_r + 1}$, which implies that $k \le m_r$. Hence $m_r = k$.

Moreover, ϵ_r must equal 1. For if $\epsilon_r = -1$, then

$$n = \epsilon_1 2^{m_1} + \cdots + \epsilon_{r-1} 2^{m_{r-1}} - 2^{k+1},$$

and then $\frac{2}{3} 2^{k+1} \leq |n|$, which contradicts $2^k \geq 3|n|$.

So we have shown the following:

Lemma. *If* $n \in \mathbb{Z}$, *and if* $2^k \geq 3|n|$, *then* $\operatorname{dist}(n + 2^k, 0) = \operatorname{dist}(n, 0) + 1$.

We now solve the stated problem. Suppose that $n, n^* \in \mathbb{Z} \setminus B$. Choose any path

$$n_0, n_1, \ldots, n_\ell$$

from n to n^* in $\mathbb{Z}$ (for example, if $n < n^*$, let $n_i = n + i$ for $i = 0, \ldots, n^* - n$). Let $d_i = \operatorname{dist}(n_i, 0)$ for $i = 0, \ldots, \ell$. Then $d_0, d_\ell > r$, but $d_i \leq r$ could happen for some $i \in \{1, \ldots, \ell - 1\}$. By the lemma, there is a k_1 so that $\operatorname{dist}(n_i + 2^{k_1}, 0) = \operatorname{dist}(n_i, 0) + 1 = d_i + 1$ for $i = 0, \ldots, \ell$ (just choose k_1 so that $2^{k_1} \geq 3|n_i|$ for $i = 0, \ldots, \ell$). For the same reasons, there is a k_2 so that $\operatorname{dist}(n_i + 2^{k_1} + 2^{k_2}, 0) = \operatorname{dist}(n_i + 2^{k_1}, 0) + 1 = d_i + 2$ for $i = 0, \ldots, \ell$. Continuing in this way, we get integers $k_1, k_2, \ldots, k_s$ so that

$$\operatorname{dist}(n_i + 2^{k_1} + \cdots + 2^{k_j}, 0) = d_i + j \quad \text{for } i = 0, \ldots, \ell$$

for $j = 1, \ldots, s$. Choose s so large that $d_i + s > r$ for $i = 0, \ldots, \ell$. Let $n_i' = n_i + 2^{k_1} + \cdots + 2^{k_s}$ for $i = 0, \ldots, \ell$. Then $n_i' \in \mathbb{Z} \setminus B$ for each i, and $n_0', \ldots, n_\ell'$ is a path in $\mathbb{Z} \setminus B$ because $n_0, \ldots, n_\ell$ is a path.

Moreover, $n_0, n_0 + 2^{k_1}, \ldots, n_0 + 2^{k_1} + \cdots + 2^{k_s} = n_0'$ is a path in $\mathbb{Z} \setminus B$ from $n = n_0$ to n_0', and $n_\ell, n_\ell + 2^{k_1}, \ldots, n_\ell + 2^{k_1} + \cdots + 2^{k_s} = n_\ell'$ is a path in $\mathbb{Z} \setminus B$ from $n^* = n_\ell$ to n_ℓ'. Combining these three paths, we get a path from n to n^* in $\mathbb{Z} \setminus B$.

7. Define a function $f : \mathbb{R}^3 \to \mathbb{C}$ by

$$f(\theta_1, \theta_2, \theta_3) = e^{i(\theta_1 + \theta_2 + \theta_3)} + e^{i(\theta_1 - \theta_2 - \theta_3)} + e^{i(-\theta_1 + \theta_2 - \theta_3)} + e^{i(-\theta_1 - \theta_2 + \theta_3)}.$$

Find the image of f.

Solution. If we write $e^{\pm i\theta_j} = \cos\theta_j \pm i \sin\theta_j$ for $j = 1, 2, 3$, we find after some algebra that

$$f(\theta_1, \theta_2, \theta_3) = 4\cos\theta_1 \cos\theta_2 \cos\theta_3 - 4i \sin\theta_1 \sin\theta_2 \sin\theta_3.$$

To understand the image of this function, it is convenient to use the Open Mapping Theorem. Suppose that $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a function. Then at a point $(x, y) \in \mathbb{R}^2$, $\mathbf{f}(x, y)$ has two components, $f_1(x, y)$ and $f_2(x, y)$, say. We assume that $f_1(x, y)$ and $f_2(x, y)$ have continuous first partial derivatives. Now suppose that at a point (a, b),

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \neq 0, \tag{1}$$

that is,

$$\frac{\partial f_1}{\partial x}(a, b) \frac{\partial f_2}{\partial y}(a, b) - \frac{\partial f_1}{\partial y}(a, b) \frac{\partial f_2}{\partial x}(a, b) \neq 0.$$

Then the Open Mapping Theorem tells us that the point $\mathbf{f}(a,b) = (f_1(a,b), f_2(a,b))$ is an *interior point* of the image of $\mathbf{f}$. That is, there is an $\epsilon > 0$ so that every point (u,v) of $\mathbb{R}^2$ such that

$$\bigl(f_1(a,b) - u\bigr)^2 + \bigl(f_2(a,b) - v\bigr)^2 < \epsilon^2$$

is in the image of $\mathbf{f}$. In other words, for each such (u,v), there is an (x,y) such that $f_1(x,y) = u$ and $f_2(x,y) = v$. So if $\mathbf{f}(a,b) = (f_1(a,b), f_2(a,b))$ is a boundary point of the image of $\mathbf{f}$, then the determinant in (1), evaluated at $(x,y) = (a,b)$, must be 0.

We apply this to $x = \theta_1$ and $y = \theta_2$, and to $f_1(x,y) = 4\cos\theta_1\cos\theta_2\cos\theta_3$ and $f_2(x,y) = -4\sin\theta_1\sin\theta_2\sin\theta_3$. Then the determinant in (1) equals

$$\cos\theta_3\sin\theta_3(\sin^2\theta_1\cos^2\theta_2 - \cos^2\theta_1\sin^2\theta_2) = \cos\theta_3\sin\theta_3(\cos^2\theta_2 - \cos^2\theta_1).$$

Setting this to 0, either $\cos\theta_3 = 0$ (which implies that $\mathbf{f}(a,b)$ at the point (a,b) in question is on the y-axis), $\sin\theta_3 = 0$ (which implies that $\mathbf{f}(a,b)$ is on the x-axis), or

$$\cos\theta_2 = \pm\cos\theta_1.$$

So if $\mathbf{f}(a,b)$ is on the boundary of the image of $\mathbf{f}$, and if $\mathbf{f}(a,b)$ is not on the real or imaginary axis, then

$$e^{i\theta_2} = e^{i\theta_1},\ -e^{i\theta_1},\ e^{-i\theta_1},\ \text{or}\ -e^{-i\theta_1}.$$

For the same reasons, using (θ_1,θ_3) instead of (θ_1,θ_2), we must have

$$e^{i\theta_3} = e^{i\theta_1},\ -e^{i\theta_1},\ e^{-i\theta_1},\ \text{or}\ -e^{-i\theta_1}.$$

For a given $\theta_1 = \theta$, we find that sixteen values of $f(\theta_1,\theta_2,\theta_3)$ corresponding to the four possibilities for θ_2 and the four possibilities for θ_3 yield only four different values:

$$e^{3i\theta} + 3e^{-i\theta},\ e^{-3i\theta} + 3e^{i\theta},\ -e^{3i\theta} - 3e^{-i\theta},\ \text{and}\ -e^{-3i\theta} - 3e^{i\theta}.$$

Since $-e^{3i\theta} - 3e^{-i\theta} = e^{3i(\theta+\pi)} + 3e^{-i(\theta+\pi)}$, we see that the boundary point of the image of f must lie on the curve

$$\gamma(\theta) = e^{3i\theta} + 3e^{-i\theta} = f(\theta,\theta,\theta) = 4\cos^3\theta - 4i\sin^3\theta$$

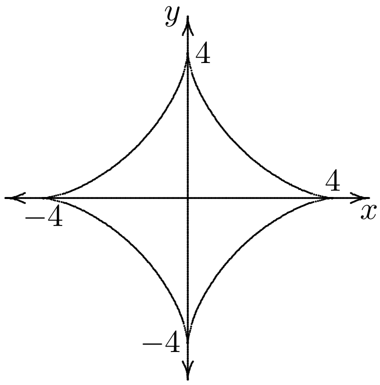
obtained by setting all the θ_j's equal. The real and imaginary parts of $\gamma(\theta)$ are

$$x = 4\cos^3\theta \quad \text{and} \quad y = -4\sin^3\theta,$$

and so the image of the curve has equation

$$x^{2/3} + y^{2/3} = 4^{2/3}. \tag{1}$$

Clearly whenever $f(\theta_1,\theta_2,\theta_3)$ lies on the real axis, its value is $4\cos\theta_1\cos\theta_2\cos\theta_3$, and this lies between -4 and 4, and so $f(\theta_1,\theta_2,\theta_3)$ lies within the curve (1). Similarly, whenever $f(\theta_1,\theta_2,\theta_3)$ lies on the imaginary axis, its value it lies between $-4i$ and $4i$, and so withing the curve (1). So this curve forms the boundary of the image of f, and this image is the region bounded by this curve.



8. Calculate the determinant of the $n \times n$ matrix A with entries

$$a_{i,j} = \frac{(2i+2j-2)!}{2^{2i+2j-2}(i+j-1)!} \quad (i, j = 1, \dots, n).$$

Solution. We prove a more general fact:

Let $\alpha \in \mathbb{R}$, and suppose that $C = (c_{i,j})_{i,j=1}^{n}$ is a matrix such that $c_{i,j+1} = (i+j+\alpha)c_{i,j}$ for all $1 \le i \le n$ and for all $1 \le j \le n-1$. Then $\det(C) = \prod_{i=1}^{n}(i-1)!c_{i,1}$.

To prove this, for each $k = 1, \dots, n$ let

$$C_k = \begin{pmatrix} c_{n-k+1,1} & \cdots & c_{n-k+1,k} \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,k} \end{pmatrix} = (c_{n-k+i,j})_{i,j=1}^{k},$$

so $C_n = C$. We claim that $\det(C_k) = (k-1)!c_{n-k+1,1}\det(C_{k-1})$ for $2 \le k \le n$. To see this, write $\mathbf{c}_1, \dots, \mathbf{c}_k \in \mathbb{R}^k$ for the columns of C_k. For each $1 \le j \le k-1$ perform the column operation $\mathbf{c}_{j+1} \to \mathbf{c}_{j+1} - (n-k+j+\alpha+1)\mathbf{c}_j$, leaving the determinant unchanged. Since

$$c_{n-k+i,j+1} - (n-k+j+\alpha+1)c_{n-k+i,j} = (i-1)c_{n-k+i,j}$$

for each $1 \le i \le k$, we have

$$\det(C_k) = \det \begin{pmatrix} c_{n-k+1,1} & 0 & \cdots & 0 \\ c_{n-k+2,1} & c_{n-k+2,1} & \cdots & c_{n-k+2,k-1} \\ c_{n-k+3,1} & 2c_{n-k+3,1} & \cdots & 2c_{n-k+3,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & (k-1)c_{n,1} & \cdots & (k-1)c_{n,k-1} \end{pmatrix},$$

which equals $(k-1)!c_{n-k+1,1}\det(C_{k-1})$. The statement follows.

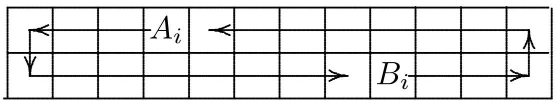
The numbers $a_{i,j}$ satisfy the conditions of the last assertion, with $\alpha = -\frac{1}{2}$.

9. A square $n \times n$ block of government owned land is divided into n^2 square 1×1 lots. There are n farmers, and initially, each farmer buys one lot, so that each row and each column contains only one privatized lot. On each succeeding season, each farmer may acquire another lot if it is not yet privatized and if it has a common side with the lot privatized by this farmer the previous season.

For which n is it possible to privatize n lots initially in such a way that the whole block can be privatized in n seasons?

Solution. We show that it is possible to privatize in this way if and only if n is *not* of the form $4k+3$.

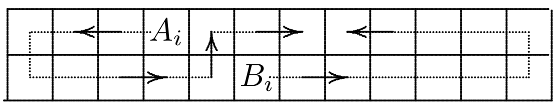
Suppose first that n is even. We start off with a farmer A_i in position $(2i-1, i)$, $i = 1, \ldots, n/2$ and a farmer B_i in position $(2i, n-i+1)$, $i = 1, \ldots, n/2$. It is easy to see that in this initial configuration, there is one farmer in each row, and one in each column. Then the farmers A_i and B_i privatize successive lots as shown, remaining always in rows $2i-1$ and $2i$:



Now suppose that $n = 4k+1$ is congruent to 1, modulo 4. Then down the diagonal of the $n \times n$ block, first place a farmer in the lot in position $(1,1)$, then place k 4×4 sub-blocks, with four farmers A_i, B_i, C_i and D_i in the i-th sub-block as shown:

A_i			
		B_i	
	C_i		
			D_i

Then the farmers A_i and B_i privatize successive lots as shown, remaining always in rows $4i-2$ and $4i-1$:



The farmers C_i and D_i privatize successive lots in a similar way, remaining always in rows $4i$ and $4i+1$. So the privatization can proceed if $n = 4k+1$.

Finally, suppose that $n = 4k+3$. Suppose that a privatization can occur. Colour the lots black and white in checkerboard fashion, with the corner lots coloured black. There are $n^2 = 16k^2 + 24k + 9$ lots in all, $8k^2 + 12k + 5$ of them black, and $8k^2 + 12k + 4$ of them white. Suppose that r farmers start in black lots and s start in white lots. As the privatization proceeds, the lots a farmer acquires alternate in colour. So after the privatization is complete, the r farmers starting on black lots now own $2k+2$ black lots and $2k+1$ white lots each, while the s farmers starting on white lots now own $2k+1$ black lots and $2k+2$ white lots each. Hence $r(2k+2)+s(2k+1)=8k^2+12k+5$ and $s(2k+2)+r(2k+1)=8k^2+12k+4$. Subtracting these two equations, we find that $r-s=1$. Hence $r=2k+2$ and $s=2k+1$.

If a lot is black, the sum $i+j$ of its coordinates (i,j) is even, while if a lot is white, then $i+j$ is odd. So if the coordinates of the lots initially acquired are (i,k_i), $i=1,\ldots,n$, then $i+k_i$ is even for $r=2k+2$ i's, and $i+k_i$ is odd for $s=2k+1$ i's. Hence the sum

$$\sum_{i=1}^{n}(i+k_i)$$

is odd. But the k_i's are a permutation of $1,\ldots,n$, and so

$$\sum_{i=1}^{n}(i+k_i)=\sum_{i=1}^{n}i+\sum_{i=1}^{n}k_i=2\sum_{i=1}^{n}i$$

is even. This contradiction shows that no privatization can occur if $n=4k+3$.

10. To 'triangulate' a convex polygon means to draw straight lines between pairs of non-adjacent vertices, such that no two lines intersect in the interior of the polygon, and the interior is thereby subdivided into triangles. It is well known that the number of triangulations of a convex n-gon is the Catalan number c_{n-2}, where c_n is defined by the recursion $c_n=c_0c_{n-1}+c_1c_{n-2}+\cdots+c_{n-1}c_0$, with initial value $c_0=1$.

Now suppose that P is a regular n-gon. We say that two triangulations are equivalent if one can be obtained from the other by rotating or reflecting P. Find a formula for the number of equivalence classes of triangulations, in terms of the Catalan numbers.

Solution. The group of symmetries of P is the dihedral group of order $2n$: it consists of n rotations (including the identity) and n reflections. We are asked to calculate the number of orbits of this group acting on the set of triangulations of P. By the result somewhat inaccurately known as Burnside's Lemma, this equals $\frac{1}{2n}$ times the number of pairs (σ,T) where σ is in the symmetry group and T is a triangulation which is fixed by σ. We now classify such pairs and count them.

Case 1: σ is the identity. Since every triangulation is fixed by the identity, there are c_{n-2} such pairs.

Case 2a: σ is a nontrivial rotation, and the centre of the polygon lies on one of the lines in T. This is only possible if $2\mid n$. Since σ fixes T, it must also fix this line, and therefore is a half-turn. Having chosen this fixed line in $\frac{n}{2}$ ways, the rest of the triangulation is determined by the triangulation of the $\frac{n+2}{2}$-gon on one side of the line, so there are $\frac{n}{2}c_{\frac{n-2}{2}}$ such pairs.

Case 2b: σ is a nontrivial rotation, and the centre of the polygon lies inside one of the triangles of T. Then σ must fix this triangle, so the triangle must be equilateral and σ must be rotation by $\frac{2\pi}{3}$ clockwise or anti-clockwise. This is only possible if $3\mid n$. Having chosen the fixed triangle in $\frac{n}{3}$ ways and the rotation in 2 ways, the rest of the triangulation is determined by the triangulation of the $\frac{n+3}{3}$-gon on one side of the triangle, so there are $\frac{2n}{3}c_{\frac{n-3}{3}}$ such pairs.

Case 3a: σ is a reflection and $2 \nmid n$. The axis of reflection joins a vertex v with the midpoint of the opposite side. Since this side is fixed by σ, the other vertex with which it forms a triangle in T must be fixed also, hence must be v. Having chosen v in n ways, the rest of the triangulation is determined by the triangulation of the $\frac{n+1}{2}$-gon on one side of this fixed triangle, so there are $nc_{\frac{n-3}{2}}$ such pairs.

Case 3b: σ is a reflection and $2 \mid n$. By the same reasoning as in Case 3a, the reflection cannot fix an edge of the polygon without also fixing a vertex, so the axis cannot be one of the ones joining midpoints of opposite edges; it must join two opposite vertices, say v and v'. There are thus $\frac{n}{2}$ choices for σ. We now have two possibilities. Firstly, the axis could itself be one of the lines of T; then the rest of the triangulation is determined by the triangulation of the $\frac{n+2}{2}$-gon on one side of the axis, so there are $\frac{n}{2}c_{\frac{n-2}{2}}$ such pairs. Secondly, the axis could fail to be one of the lines of T. If this is so, then any line of T which crosses the axis must be perpendicular to it, since otherwise it would intersect its reflection (which would also have to be a line of T). It is then clear that there must be exactly one such line in the triangulation, say ℓ (if there were two, there would be no way to triangulate the region between them subject to this perpendicularity constraint). So v and ℓ belong to a triangle of T, as do v' and ℓ. Now perform the following operation: in the quadrilateral with vertices v, v' and the endpoints of ℓ, erase the diagonal ℓ and draw the other diagonal, the reflection axis. It is clear that this operation gives a bijection between pairs of this second sort and pairs of the first sort. So there are $\frac{n}{2}c_{\frac{n-2}{2}}$ pairs of the second sort also.

Summing up over all the cases, we have the following formulae for the number of orbits:

$$\begin{aligned}
\frac{1}{2n}\left(c_{n-2} + nc_{\frac{n-3}{2}}\right), &\quad \text{if } 2 \not| n,\ 3 \not| n,\\
\frac{1}{2n}\left(c_{n-2} + nc_{\frac{n-3}{2}} + \frac{2n}{3}c_{\frac{n-3}{3}}\right), &\quad \text{if } 2 \not| n,\ 3 \mid n,\\
\frac{1}{2n}\left(c_{n-2} + \frac{3n}{2}c_{\frac{n-2}{2}}\right), &\quad \text{if } 2 \mid n,\ 3 \not| n,\\
\frac{1}{2n}\left(c_{n-2} + \frac{3n}{2}c_{\frac{n-2}{2}} + \frac{2n}{3}c_{\frac{n-3}{3}}\right), &\quad \text{if } 2 \mid n,\ 3 \mid n.
\end{aligned}$$