ALGEBRAIC GEOMETRY AND CONVEX GEOMETRY

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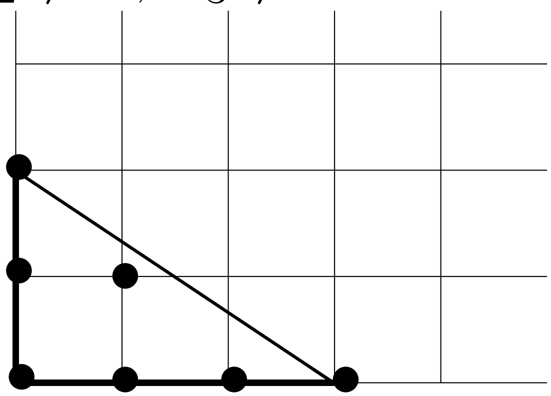
Newton polyhedra generalize the notion of the degree. The **support** $s(P)$ of a Laurent polynomial P is the set of the powers of the monomials appearing in P with nonzero coefficients. The Newton polyhedron $\Delta(P)$ is the convex hull of $s(P)$.

Example. Let P be $y^2 + a_0 + a_1 x + a_2 x^2 + a_3 x^3$, where $a_0 \neq 0$, $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$. Then $\Delta(P)$ is



and $s(P) = \{(0,0),(0,1),(0,2),(0,3),(2,0)\}$.

Discrete invariants of $X \subset (\mathbb{C}^*)^n$ defined by a generic equation $P(x) = 0$ with fixed support $s(P)$ do not depend on the specific choice of the equation; they depend only on Newton polyhedron $\Delta = \Delta(P)$.

Curve $X \subset (\mathbb{C}^*)^2$ defined by a generic equation $P = 0$.

Example 1 (Kh). The **genus** $g(X)$ is equal to the number $B(\Delta)$ of integral points in the interior of $\Delta = \Delta(P)$.

Example 2 (Kh). Let $\bar{X} = X \bigcup A(X)$ be a **smooth compact model of X. Then** $\# A(X)$ equals to the number of integral points in the boundary of Δ.

Example 3 (D.Berstein, Kh). The **Euler characteristic** $\chi(X)$ of X is equal to the volume $V(\Delta)$ of Δ multiplied by $-2!$

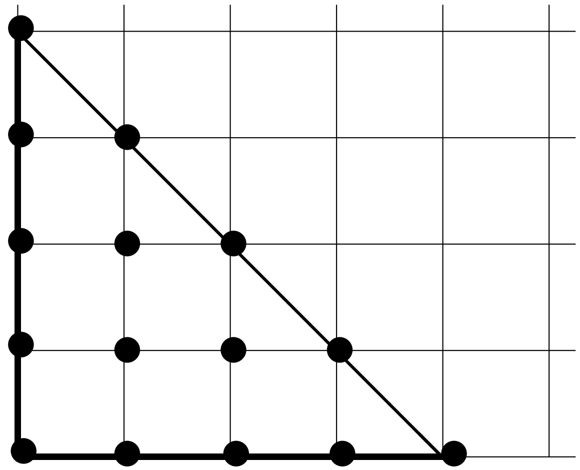
Toy geometric application. The invariants 1)–3) are related:

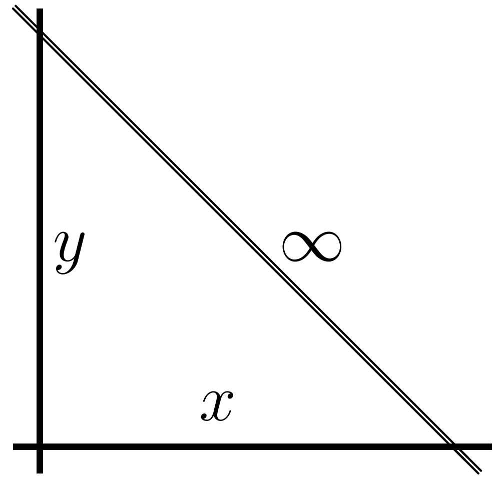
$$\chi(\bar{X}) = \chi(X) + \# A(X) = 2 - 2g(X).$$

It implies the **Pick formula** for an integral polygon Δ:

$$V(\Delta) = \#((\Delta \setminus \partial\Delta) \bigcap \mathbb{Z}^2) + 1/2 \# \partial(\Delta \bigcap \mathbb{Z}^2) - 1.$$

Toric varieties and the combinatorics of polyhedra





Toric variety is a normal connected n-dimensional algebraic variety M on which an $(\mathbb{C}^*)^n$ acts algebraically and has one orbit isomorphic to $(\mathbb{C}^*)^n$. Under the action of $(\mathbb{C}^*)^n$, M is broken up into a finite number of orbits isomorphic to tori of different dimensions. To every Newton polyhedron Δ, we can associate a compact projective toric variety M_Δ in such a way that every k-dimensional face $\Gamma \subset \Delta$ corresponds to a complex k-dimensional orbit $O_\Gamma \subset M_\Delta$. If $\Gamma_1 \subset \Gamma_2$, then $O_{\Gamma_1} \subset \bar{O}_{\Gamma_2}$.

Simple polyhedra and quasismooth toric varieties

A bounded polyhedron is **simple** if it is an intersection of half-spaces in general position. An n-dimensional simple polyhedron has the same structure near each vertex as the positive orthant in $\mathbb{R}^n$ near the origin. In particular, each vertex of a simple n-dimensional polyhedron is incident with n edges, and any k of these edges belong to one k-dimensional face containing the vertex.

The F**-vector** of a simple n-dimensional polyhedron is the vector $(F_0, \dots, F_n)$ where F_k is the number of k-dimensional faces of the polyhedron. The necessary and sufficient conditions for a vector to be F-vector of a simple n-dimensional polyhedron were conjectured by *McMullen*.

Simple polyhedra correspond to **guasismooth toric varieties**. Using topology and algebraic geometry of such varieties *Stanley*, *Billera* and *Lee* proved McMullen's conjecture.

Polyhedra and groups generated by reflections in Lobachevsky spaces

Theorem (Nikulin). *The average number of l-dimensional faces of a k-dimensional face of a simple n-dimensional polyhedron for $0 \leq l < k \leq (n+1)/2$ is $\leq f(l,k,n)$, where f is an explicit function. If $n \to \infty$, f tends to the number of l-dimensional faces of a k-dimensional cube.*

Theorem (Vinberg). *In a Lobachevsky space of dimension > 32, there are no discrete groups generated by reflections with a compact fundamental polyhedron.*

Theorem (Kh). *The bound in Nikulin's Theorem is valid not only for simple polyhedra, but also for edge simple polyhedra.*

Theorem (Prokhorov, Kh). *In a Lobachevsky space of dimension > 995, there are no discrete groups generated by reflections with a fundamental polyhedron of finite volume.*

Problem: *How many solutions in* $(\mathbb{C}^*)^n$ *has a system of equations* $P_1 = \cdots = P_n = 0$*, where* $P_1, \ldots, P_n$ *are generic Laurent polynomials with the fixed supports* $A_1, \ldots, A_n \subset \mathbb{Z}^n$*?*

Let us slightly **reformulate** the problem:

Let $A_i \subset (\mathbb{Z})^n$ *be a finite set; let* L_{A_i} *be the space generated* x^m*, where* $m \in A_i$*. How many solutions in* $(\mathbb{C}^*)^n$ *has a system of equations* $P_1 = \cdots = P_n = 0$ *where* $P_1 \in L_{A_1}, \ldots, P_n \in L_{A_n}$ *is a generic* n*-tuple of functions?*

The Newton polyhedron Δ_i of P_i is the convex hull of A_i.

Theorem (*Kouchnirenko*). *If* $A_1 = \cdots = A_n = A$*, then the number of solutions of the system is equal to the volume* $V(\Delta)$ *of* $\Delta = \Delta_1 = \cdots = \Delta_n$ *multiplied by* $n!$

Theorem (*Bernstein*) (also known as BKK theorem). *The number of solutions of the system is equal to the mixed volume* $V(\Delta_1, \ldots, \Delta_n)$ *of* $\Delta_1, \ldots, \Delta_n$ *multiplied by* $n!$.

Mixed volume

$(\exists\,!)$ $V(\Delta_1,\dots,\Delta_n)$, on n-tuples of convex bodies in $\Delta_i \subset \mathbb{R}^n$, such that:

1. $V(\Delta,\dots,\Delta)$ is the volume of Δ;
2. V is symmetric;
3. V is multi-linear; for example,
   $V(\Delta_1' + \Delta_1'', \Delta_2, \dots) = V(\Delta_1', \Delta_2, \dots) + V(\Delta_1'', \Delta_2, \dots)$;
4. $0 \le V(\Delta_1,\dots,\Delta_n)$;
5. $\Delta_1' \subseteq \Delta_1, \dots, \Delta_n' \subseteq \Delta_n \Rightarrow V(\Delta_1',\dots,\Delta_n') \le V(\Delta_1,\dots,\Delta_n)$;
6. Alexandrov–Fenchel inequality
   $V^2(\Delta_1,\Delta_2,\dots,\Delta_n) \ge V(\Delta_1,\Delta_1,\dots,\Delta_n)V(\Delta_2,\Delta_2,\dots,\Delta_n)$;
7. isoperimetric inequality ($n=2$, Δ_1 is the unite circle, $\Delta = \Delta_2$)

$$(\frac{1}{2}\text{ length of } \partial\Delta)^2 \ge \pi V(\Delta).$$

Intersection index on an irreducible variety X

Let $K(X)$ be the **semigroup of spaces** L of rational functions on X such that: a) $\dim L < \infty$, and b) $L \neq 0$. For $L_1, L_2 \in K(X)$, the **product** is the space $L_1 L_2 \in K(X)$ generated by elements fg, where $f \in L_1$, $g \in L_2$. Assume that $\dim X = n$.

For $L_1, \ldots, L_n \in K(X)$, the intersection index $[L_1, \ldots, L_n]$ is

$$\# x \in X : (f_1(x) = \cdots = f_n(x) = 0),$$

where $f_1 \in L_1, \ldots, f_n \in L_n$ is a generic n-tuple of functions. We neglect roots $x \in X$, such that $\exists\, i : (f \in L_i \Rightarrow f(x) = 0)$, and such that $\exists f \in L_j$ for $1 \leq j \leq n$ having a pole at x.

The intersection index is well-defined. It is **multi-linear** with respect to the product in $K(X)$.

BKK theorem computes the intersection index **for** $X = (\mathbb{C}^*)^n$ and for an n-tuple of **spaces generated by monomials**.

Grothendieck semigroup and group

For **a commutative semigroup** S let

$$a \sim b \Leftrightarrow (\exists c \in S) | (a + c = b + c).$$

The **Grothendieck semigroup** $\mathrm{Gr}_s(S)$ of S is S modulo the equivalents relation $\sim$.

The **Grothendieck group** $\mathrm{Gr}(S)$ of S is the group of formal differences of $\mathrm{Gr}_s(S)$.

Let $\rho : S \to \mathrm{Gr}_s(S)$ be the natural map.

Theorem (*Kh*). *Let $\mathcal{K}$ be the semigroup of finite subsets $\mathbb{Z}^n \subset \mathbb{R}^n$ with respect to addition, then $\mathrm{Gr}_s(\mathcal{K})$ consists of convex polyhedra and $\rho(A)$ is the convex hull $\Delta(A)$ of A.*

We need the following algebraic analog of this theorem.

The Grothendieck semigroup $Gr_s(K(X))$ of $K(X)$

One can describe the relation $\sim$ in $K(X)$ as follows: $f \in \mathbb{C}(X)$ is called integral over L if it satisfies an equation

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0$$

with $m > 0$ and $a_i \in L^i$. The collection of all integral functions over L is a finite-dimensional subspace $\overline{L}$ called the completion of L. In $K(X)$:

1. $L_1 \sim L_2 \Leftrightarrow \overline{L}_1 = \overline{L}_2$;

2. $L \sim \overline{L}$;

3. $L \sim M \Rightarrow M \subset \overline{L}$.

The index $[L_1, \ldots, L_n]$ can be extended to the Grothendieck group $\mathrm{Gr}(K(X))$ of $K(X)$ and considered as **a birationally invariant generalization of the intersection index of divisors** which is applicable to non-complete varieties.

Regularization of a semigroup of integral points

For a **semigroup** $S \subset \mathbb{Z}^n$ **of integral points** let:
1) $G(S) \subset \mathbb{Z}^n$ be the group generated by S;
2) $L(S) \subset \mathbb{R}^n$ be the subspace spanned by S;
3) $C(S)$ be the closure of the convex spanned by S.

The **regularization** $\tilde{S}$ of S is the semigroup $C(S) \cap G(S)$.

Theorem (*Kaveh, Kh*). *Let $C' \subset C(S)$ be a strongly convex cone which intersects the boundary (in the topology of the linear space $L(S)$) of the cone $C(S)$ only at the origin. Then there exists a constant $N > 0$ (depending on C'), such that any point in the group $G(S)$ which lies in C' and whose distance from the origin is bigger than N belongs to S.*

Semigroup of integral points and its NO body

Let M by a hyperplane in $L(S)$. Let M_k be the affine space parallel to M and intersecting $G(S)$ and $C(S)$ which has distance k from the origin (the distance is normalized in such a way that as values it takes all the non-negative integers k).

The **Hilbert function** H_S **of the semigroup** S **in the codirection** M is define by $H_S(k) = \#M_k \cap S$.

The convex body $\Delta(S, M) = C(S) \cap M_1$ is by definition **the Newton–Okounkov of the semigroup** S **in the codirection** M.

Theorem (*Kaveh, Kh*). *The function $H_S(k)$ grows like $a_q k^q$, where q is the dimension of the convex body $\Delta(S)$, and the q-th growth coefficient a_q is equal to the (normalized in the appropriate way) q-dimensional volume of $\Delta(S)$.*

Algebra of almost finite type, its NO body (begging)

Let F be a field of transcendence degree n over $\mathbf{k}$. We deal with graded subalgebras in the algebra $F[t]$ of polynomials over F:

1. $A_L = \bigoplus_{k \geq 0} L^k t^k$, where $L \subset F$ is a subspace, $\dim_{\mathbf{k}} L < \infty$; $L^0 = \mathbf{k}$ and L^k is the span of all the products $f_1 \cdots f_k$ with $f_1, \ldots, f_k \in L$.
2. An **algebra of almost integral type** is a graded subalgebra in some algebra A_L.

We construct a $\mathbb{Z}^{n+1}$-valued valuation v_t on $F[t]$ by extending a $\mathbb{Z}^n$-valuation v on F which takes all the values in $\mathbb{Z}^n$.

By definition the **NO body of an algebra** A of an almost finite type is the NO body of the semigroup $S(A) = v_t(A \setminus \{0\})$ projected to the first factor $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$.

Algebra and its NO body (continuation)

Theorem (*Kaveh, Kh*). *The Hilbert function $H_A(k)$ of an algebra A of almost integral type grows like $a_q k^q$, where $q = \dim_{\mathbb{R}} \Delta(A)$ and a_q is the (normalized in the appropriate way) q-dimensional volume of $\Delta(A)$.*

One defines a componentwise product of graded subalgebras. Consider the class of graded algebras of almost integral type such that, for $k \gg 0$, all their k-th homogeneous components are non-zero. Let A_1, A_2 be algebras of such kind and put $A_3 = A_1 A_2$. It is easy to verify the inclusion $\Delta(A_1) + \Delta(A_2) \subset \Delta(A_3)$.

Brunn–Minkowsky inequality in convex geometry

$$V^{1/n}(\Delta_1) + V^{1/n}(\Delta_2) \leq V^{1/n}(\Delta_1 + \Delta_2).$$

Theorem (*Kaveh, Kh*). $a_n^{1/n}(A_1) + a_n^{1/n}(A_2) \leq a_n^{1/n}(A_3)$.

NO bodies and Intersection theory (beginning)

With a space $L \in K(X)$, we associate the algebra A_L, its integral closure $\overline{A_L}$ in the field $\mathbb{C}(X)[t]$ and their NO bodies $\Delta(A_L) \subseteq \Delta(\overline{A_L})$. For a big space L we have $\Delta(A_L) = \Delta(\overline{A_L})$.

Theorem (*Kaveh, Kh*). *For $L \in K(X)$ we have:* $[L, \dots, L] = n!\mathrm{Vol}(\Delta(\overline{A_L}))$.

Theorem implies the Kušnirenko theorem.

For any $J \subset \{1, \dots, n\}$ let L_J be $\prod_{i \in J} L_i$.

Theorem (*Kaveh, Kh*). *For $L_1, \dots, L_n \in K(X)$ we have:*

$$[L_1, \dots, L_n] = (-1)^n \sum_J (-1)^{\#(J)} [L_J, \dots, L_J] =$$

$$= (-1)^n \sum_J (-1)^{\#(J)} \mathrm{Vol}(\Delta(\overline{A_{L_J}})).$$

NO bodies and Intersection theory (continuation)

The BKK theorem is a special case of the above theorem.

Proof:

1. **The Newton polyhedron of the product of two Laurent polynomials is equal to the sum of the corresponding Newton polyhedra**.

2. Let $\Delta_1, \cdots, \Delta_n$ be convex bodes in $\mathbb{R}^n$. For any $J \subset \{1, \ldots, n\}$ let Δ_J be $\sum_{i \in J} \Delta_i$. The following formula holds:

$$n! V(\delta_1, \ldots, \delta_n) = (-1)^n \sum_J (-1)^{\#(J)} \mathrm{Vol}(\Delta_J).$$

Brunn–Minkowsky inequality in intersection theory

Theorem (*Kaveh, Kh*). *For* $L_1, \dots, L_n \in K(X)$ *we have:*
$\Delta(\overline{A_{L_1L_2}}) \supseteq \Delta(\overline{A_{L_1}}) + \Delta(\overline{A_{L_2}})$.

Brunn–Minkowsky inequality (*Kaveh, Kh*). *Let* $L_1, L_2 \in K(X)$ *and* $L_3 = L_1L_2$, *then:*

1. $[L_1, \dots, L_1]^{1/n} + [L_2, \dots, L_2]^{1/n} \le [L_3, \dots, L_3]^{1/n}$.

2. In particular, for $n = 2$ *we have:*
$[L_1, L_1]^{1/2} + [L_2, , L_2]^{1/2} \le [L_1L_2, L_1L_2]^{1/2}$.

Squaring the last inequality, one obtains the following

Hodge type inequality(*Kaveh, Kh*). *For* $n = 2$, *we have*
$[L_1, L_1][L_2, L_2] \le [L_1, L_2]^2$.

Alexandrov–Fenchel type inequality in algebra

Alexandrov–Fenchel inequality in convex geometry

$$V^2(\Delta_1, \Delta_2, \dots, \Delta_n) \geq V(\Delta_1, \Delta_1, \dots, \Delta_n)V(\Delta_2, \Delta_2, \dots, \Delta_n).$$

Theorem (*Kaveh, Kh*). *Let X, $\dim X = n$, be an irreducible variety, let $L_1, \dots, L_n \in K(X)$ and let $L_3, \dots, L_n$ be big subspaces. Then* $[L_1, L_2, L_3, \dots, L_n]^2 \geq$

$$[L_1, L_1, L_3, \dots, L_n][L_2, L_2, L_3, \dots, L_n].$$

An older version of this theorem dealing with the intersection theory of divisors due to Teissier and Kh.

The Alexandrov–Fenchel inequality in convex geometry follow easily from this theorem via the BKK theorem. This trick has been known. Our elementary proof of the key analogue of the Hodge index inequality which makes all the chain of arguments involved elementary and more natural.

Some other results (begginng)

1. Let $\mathbf{K}_a$ be the set of primary ideals of the ring of regular functions at $\mathbf{a} \in X, \dim X = n$. The local intersection index $[L_1, \ldots, L_n]_a$ for $L_i \in \mathbf{K}_a$ is equal to the multiplicity at $\mathbf{a}$ of a system $f_1 = \cdots = f_n = 0$, where f_i is a generic function from L_i.

Local algebraic Alexandrov–Fenchel type inequality
$[L_1, L_2, \ldots, L_n]_a^2 \leq [L_1, L_1, \ldots, L_n]_a [L_2, L_2, \ldots, L_n]_a$.

2. Let $C \subset \mathbb{R}^n$ be a strongly convex cone. A compact set $A \subset C$ is called C-**co-convex body** if $C \setminus A$ is convex. One can construct a **theory of C- co-convex bodies** analogous to the theory of convex bodies and define the mixed volume $V_C(A_{i_1} \ldots, A_{i_n})$ of an n-tuple of C-co-convex bodies $(A_{i_1} \ldots, A_{i_n})$.

Local geometric Alexandrov–Fenchel inequality
$V_C(A_1, A_2, \ldots, A_n)^2 \leq V_C(A_1, A_1, \ldots, A_n) V_C(A_2, A_2, \ldots, A_n)$.

Some other results (continuation)

3. **NO body and reductive group action**. For $L \in \mathcal{K}(X)$ the NO body $\Delta(\overline{A_L})$ strongly depends on a choice of $\mathbb{Z}^n$-valued valuation v on $\mathbb{C}(X)$. If X is equipped with a reductive group action and if one is interested only in the invariant subspaces $L \in \mathcal{K}(X)$, then one can use this freedom to make all results more precise and explicit

4. **NO body and Fujita approximation theorem**. Another result of the theory: one can prove analogues of Fujita approximation theorem for semigroups of integral points and graded algebras which implies a generalization of this theorem for arbitrary linear series.

THANK YOU!