Selecta Mathematica Sovietica
Vol. 9, No. 2 (1990)

0272-9903/90/020155-15 $1.50 + 0.20/0

Algebraic Structure of a Lie Algebra of Vector Fields*

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1. Introduction

Interest in the study of the algebraic structure of infinite-dimensional Lie algebras has grown in recent years. This is due to the ever-increasing role that infinite-dimensional Lie vector algebras play in the solution of a number of problems in theoretical and mathematical physics. The principal examples of infinite-dimensional Lie algebras include the Lie algebras of vector fields on manifolds. Even the simplest of these examples, the Lie algebra Vect $\mathbf{R}^1$ of all smooth (of class C^∞) vector fields on the real line, is the source of a host of difficult problems of algebra and analysis. In particular, only recently has it been found that nontrivial identities are satisfied in this Lie algebra, the rate of growth of this Lie algebra has been computed, and the investigation of general finite-dimensional subalgebras begun (cf. [**4**], [**6**], [**9**]). In the present paper, we will discuss a simple and elegant hypothesis that arises as a result of our previous studies (cf. Section 4) and present certain lines of reasoning that seem to indicate that it is true.

2. Pauli space

A well-known technique of polarization of polynomials (that are not necessarily commutative) reduces the problem of describing all the identities that are satisfied in this Lie algebra to the description of linear identities, that is, identities in which each variable occurs linearly (cf. [**1**], [**8**]). In order to study these identities, it is best to introduce the concept of a Pauli space for the given

* Originally published as Keldysh Inst. Prikl. Mat. USSR Academy of Sciences, Preprint 168, Moscow, 1985, 22 pp. Translated by Robert H. Silverman.

Lie algebra G. Suppose that $F_\infty(k)$ denotes a free Lie algebra over a field k of characteristic 0 with a countable number of generators $X_1, \ldots, X_n, \ldots$. We let Π denote the subspace in $F(k)$ generated by all the Lie monomials in which each generator X_i occurs at most once (Pauli "exclusion principle"). By Π^n, we denote the subspace in Π generated by monomials of degree n, and by Π_n, the subspace in Π^n generated by the monomials formed from the generators $X_1, \ldots, X_n$. It is clear that

$$\Pi = \bigoplus_{n=1}^{\infty} \Pi^n.$$

Now let $\mathfrak{G}$ denote an arbitrary Lie algebra over a field k. Every set $x = (x_1, \ldots, x_n, \ldots)$ of elements of $\mathfrak{G}$ generates a homomorphism $\mathcal{S}_x\colon F_\infty(k) \to \mathfrak{G}$ which carries generators of X_i into the elements x_i. The kernel I_x of this homomorphism is an ideal in $F_\infty(k)$. The intersection $\bigcap_x I_x$ over all sets x is called the *identity ideal* of the Lie algebra $\mathfrak{G}$ and denoted by $I(\mathfrak{G})$.

The space $\Pi(\mathfrak{G}) = \Pi/(\Pi \cap I(\mathfrak{G}))$ will be called the *Pauli space of the Lie algebra* $\mathfrak{G}$. In an analogous way, we introduce the spaces $\Pi^n(G)$ and $\Pi_n(G)$. The latter space will be called a *homogeneous Pauli space of degree n for* $\mathfrak{G}$, or simply a *Pauli space* if n and $\mathfrak{G}$ are clear from the context.

The group S_∞ of permutations of the indices $(1, 2, \ldots, n, \ldots)$ acts in Π and Π^n in a natural way. This operation extends to both $\Pi(G)$ and $\Pi^n(\mathfrak{G})$. The group S of permutations of the indices $(1, 2, \ldots, n)$ acts in the spaces Π^n_n and $\Pi_n(\mathfrak{G})$ and the following relations are valid:

$$\Pi(\mathfrak{G}) = \bigoplus_{n=1}^{\infty} \Pi^n(\mathfrak{G}), \qquad \Pi^n(\mathfrak{G}) = \operatorname{Ind}_{S_n}^{S_\infty} \Pi_n(\mathfrak{G}). \tag{1}$$

These relations prove that the structure of the S_∞-module of $\Pi(\mathfrak{G})$ is entirely determined by the set of S_n-modules $\Pi_n(\mathfrak{G})$, $n = 1, 2, \ldots$. This set, therefore, encodes all the information about the algebraic structure of the Lie algebra $\mathfrak{G}$. Unfortunately, an explicit description of the modules of $\Pi_n(\mathfrak{G})$ is known only for a few special examples.

If $\mathfrak{G} = F_\infty(k)$ is a free Lie algebra, $\Pi_n(\mathfrak{G}) = \Pi_n$. In this case, the structure of an S_n-module has been described by Klyachko [7]. It turns out that

$$\Pi_n = \operatorname{Ind}_{Z_n}^{S_n}(\chi), \tag{2}$$

where Z_n is the subgroup in S_n consisting of cyclic permutations and χ is an arbitrary primitive character of Z_n (i.e., a character whose values on the generators of Z_n are primitive roots of degree n of unity).

Note that a generalization of Klyachko's theorem to Lie superalgebras has recently been obtained, yielding a new proof in the case of Lie algebras (cf. [**10**]).

We now present certain facts regarding the structure of the module Π_n; these follow from Equation (2).

If $n \le 5$, the spectrum of the module Π_n is simple (i.e., every irreducible component occurs with multiplicity 1). For large n, the multiplicity $\nu(\pi)$ of the component $\pi \in \hat{S}_n$ in Π_n is roughly equal to $(1/n)\dim\pi$, as is clear from the following exact formula:

$$\nu(\pi) = 1/n \sum_{d|n} \mu(d)\chi_\pi([d]^{n/d}), \tag{3}$$

where the sum is taken over all divisors d of the number n, μ is the Möbius function, χ_π is the character of the irreducible representation of π, and $[d]^{n/d}$ denotes the permutation class consisting of n/d cycles of length d.

For the sake of better visualization and in light of references to be cited below, we now present explicit formulas for the spectrum Π_n with $n \le 7$. We will use the standard enumeration of the elements of $\hat{S}_n$ by partitioning the number n and writing out the expression $1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$ for a partition of n into α_1 units, α_2 pairs, and so on.

$$\Pi_1 = \pi_1; \qquad \Pi_2 = \pi_{1^2}; \qquad \Pi_3 = \pi_{21};$$

$$\Pi_4 = \pi_{31} + \pi_{21^2}; \qquad \Pi_5 = \pi_{41} + \pi_{32} + \pi_{31^2} + \pi_{21^3};$$

$$\Pi_6 = \pi_{51} + \pi_{42} + 2\pi_{41^2} + \pi_{3^2} + 3\pi_{321} + \pi_{31^3} + 2\pi_{2^21^2} + \pi_{21^4};$$

$$\Pi_7 = \pi_{61} + 2\pi_{52} + 2\pi_{51^2} + 2\pi_{43} + 5\pi_{421} + 3\pi_{41^3} + 3\pi_{3^21}$$
$$+ 3\pi_{32^2} + 5\pi_{321^2} + 2\pi_{31^4} + 2\pi_{2^31} + 2\pi_{2^21^3} + \pi_{21^5}.$$

3. Harmonic polynomials

Let G be some group of linear transformations of an n-dimensional linear space V over k. As usual, we denote by $S(V)$ and $S(V^*)$ symmetric algebras over V and over the dual space V^*. The elements of $S(V^*)$ may be considered as polynomial functions on V, and the elements of $S(V)$ as differential operators with constant coefficients acting in $S(V^*)$. In this interpretation, the natural pairing of $S(V)$ and $S(V^*)$ assumes the form

$$\langle \mathcal{D}, P \rangle = \mathcal{D}P(0). \tag{4}$$

Suppose that $S(V)^G$ is an algebra of G-invariant differential operators, and let $S(V)^G_+$ be the ideal in this algebra that consists of operators without free term (annihilating constants).

The element of $S(V^*)$ that is orthogonal to $S(V)^G_+$ in the sense of (4) will be called the *harmonic polynomial for G*. The space of harmonic polynomials for the group G is denoted by $\mathscr{H}(G)$.

The classical harmonic polynomials are obtained if $k = R$, $G = SO(n)$; in this case, the algebra $S(V)^G$ is generated by a single Laplacian operator.

The space of harmonic polynomials for a group S_n, which acts by means of permutations of the basis vectors in an n-dimensional space V, is denoted by $\mathscr{H}_n$, and its homogeneous component of degree k, by $\mathscr{H}_n^k$. The following properties of harmonic polynomials for S_n are well known (cf. [**10**]):

(1) The S_n-module of $\mathscr{H}_n$ is equivalent to a regular representation of S_n; in particular, $\dim \mathscr{H}_n = n!$.

(2) The S_n-module of $S(V^*)$ is equivalent to $\mathscr{H}_n \otimes S(V^*)^{S_n}$.

The structure of the S_n-modules of $\mathscr{H}_n^k$ was described in [**3**]. The generating function for the multiplicities of the irreducible components of these modules proved to be a "q-analog" of the well-known "hook formula" for the dimensions of the irreducible S_n-modules. That is, if we denote by $\mu_k(\pi)$ the multiplicity of the component $\pi \in \hat{S}_n$ in $\mathscr{H}_n^k$, then

$$\sum_k \mu_k(\pi) q^k = \prod_{k=1}^{n} \left(q^{f_k} \frac{1-q^k}{1-q^{h_k}} \right), \tag{5}$$

where h_k is the length of a hook, and f_k is the length of the foot of the hook of the kth cell in Young's diagram corresponding to the representation π. As $q \to 1$, this equality becomes the hook formula:

$$\dim \pi = \prod_{k=1}^{n} \frac{k}{h_k}. \tag{6}$$

4. Statement of the basic hypothesis

The experience gained in our studies of identities in Lie algebras demonstrates that in certain cases modules of $\Pi_n(\mathfrak{G})$ may be "formed" from modules of the form of $\mathscr{H}_n^k$. Thus, the relation

$$\Pi_n = \sum_{d \in \mathbb{Z}} \mathscr{H}_n^{a+dn} \tag{7}$$

has been obtained [**10**] for $(a, n) = 1$. We suppose that the equality

$$\Pi_n(\mathrm{Vect}\, \mathbb{R}^1) = \mathscr{H}_n^{n-1} \tag{8}$$

is satisfied for the Lie algebra of vector fields on the real line.[1] Below, we prove that, in every case, the spectra of the left and right sides in (8) possess a host of

[1] This hypothesis was proved in [**11**].

identical properties. Here we may note that equalities such as (7) and (8) are valid for both Lie algebras as well. That is, we denote by $F_m(T_k)$ a free Lie algebra with m generators in which the following identity T_4 is satisfied:

$$\mathrm{Alt}(\mathrm{ad}\, X_1\, \mathrm{ad}\, X_2 \cdots \mathrm{ad}\, X_k) Y = 0, \tag{9}$$

where the symbol Alt denotes the alternating sum over all permutations of indices $1, 2, \dots, k$.

The identity T_1 expresses the fact that the algebra is commutative, while the identity T_3 is equivalent to the claim that G is metabelian, i.e., $[[\mathfrak{G}, \mathfrak{G}], [\mathfrak{G}, \mathfrak{G}]] = 0$. It has been conjectured that the Lie algebra $F_m(T_4)$ is isomorphic to the Lie algebra generated by m general vector fields on the real line (cf. [**6**]). The Pauli space $\Pi_n(F_m(T_k))$ with $m \geq n$ is independent of m and may be denoted simply by $\Pi_n(T_k)$. It is clear that $\Pi_n(T_2) = 0$ if $n \geq 3$. It may easily be verified that

$$\Pi_n(T_3) = \pi_{n-1,1} = \mathscr{H}_n^1.$$

If the hypotheses stated above are true, the equality

$$\Pi_n(T_4) = \mathscr{H}_n^{n-1} \tag{10}$$

must be true. Finally, it is known that the identity T_6 is satisfied by the Lie algebra Ham $\mathbf{R}^2$ of Hamiltonian vector fields. If it is supposed that there are no other identities of degree at most 7, the equalities

$$\Pi_n(\mathrm{Ham}\, \mathbf{R}^2) = \mathscr{H}_n^{n-1} + \mathscr{H}_n^{2n-1} \tag{11}$$

will be valid for $n \leq 7$.

5. The spectrum of $\Pi_n(\mathrm{Vect}\, \mathbf{R}^1)$

The ordinary method of studying the spectrum of an S_n-module M is to compute the wreath numbers of this modulus with distinct "standard" moduli. Young's modulus

$$Y_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n}(1), \qquad Y'_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n}(sgn) \tag{12}$$

may serve as such a standard modulus, where $\lambda = (n_1, \dots, n_r)$ is a partitioning of the number n, and $S_\lambda = S_{n_1} \times \cdots \times S_{n_r}$ is Young's subgroup in S_n. In particular, the irreducible representation Π_λ is characterized by the following properties:

$$c(\pi_\lambda, Y_\mu) = \begin{cases} 0 & \text{if } \lambda < \mu, \\ 1 & \text{if } \lambda = \mu, \end{cases} \qquad c(\pi_\lambda, Y'_\mu) = \begin{cases} 0 & \text{if } \lambda' < \mu, \\ 1 & \text{if } \lambda' = \mu, \end{cases} \tag{13}$$

where λ' denotes the partitioning that is self-adjoint to λ (with transposed

Young's diagram), and the symbol $<$ is to be understood as indicating lexicographic ordering of the partitions.

By the Frobenius principle of duality for an arbitrary S_n-module M, the following equalities are valid:

$$c(M, Y_\lambda) = \dim M^{S_\lambda}, \qquad c(M, Y'_\lambda) = \dim {}^{S_\lambda}M, \tag{14}$$

where we denote by M^G the space of G-invariant vectors in M, and by ${}^GM$ the space of G-anti-invariant vectors (i.e., the vectors multiplied by the substitution symbol). It is well known (and also follows from (13)) that the numbers $c(M, Y_\lambda)$ completely determine the structure of an S_n-module M. The same holds for the numbers $c(M, Y')$. Unfortunately, it has been possible to find these numbers for a module of $\Pi_n(\mathrm{Vect}\,\mathbf{R}^1)$, only under additional constraints on λ. Here we note one interpretation of the number $c(\Pi_n(\mathfrak{G}), Y_\lambda)$, $\lambda = (n_1, \ldots, n_r)$. The technique of polarization demonstrates that it is equal to the number of linearly independent commutators of length n that may be formed from the general elements $x_1, \ldots, x_r$ of $\mathfrak{G}$ such that x_i occur n_i times. In the case $\mathfrak{G} = \mathrm{Vect}\,\mathbf{R}^1$, this number has been found previously [4], [9] for $r = 2$. For $\lambda = (k, l)$, $k + l = n$, it was proved that

$$c(\Pi_n(\mathrm{Vect}\,\mathbf{R}^1), Y_\lambda) = p_k(n-1) + p_l(n-1) - p(n-1), \tag{15}$$

where $p(n)$ denotes the total number of partitions of n; and $p_k(n)$, the number of partitions of n that have at most k terms each. Simple computation, based on the properties of harmonic polynomials presented in Section 3, demonstrates that this formula is also valid for the number $c(\mathscr{H}_n^{n-1}, Y_\lambda)$. This observation also suggested the statement of our basic hypothesis.

Let us now state the basic result of the present study.

Theorem 1. *Suppose that we are given a partition*

$$\lambda = 1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$$

of the number n; we denote by G_1 the subgroup $S_{\alpha_1} \subset S_n$ acting on the first α_i indices, and by G_2 the subgroup

$$(S_2)^{\alpha_2} \times \cdots \times (S_n)^{\alpha_n}$$

acting on the remaining indices.

Then

$$\dim {}^{G_2}\Pi_n(\mathrm{Vect}\,\mathbf{R}^1)^{G_1} = \dim {}^{G_2}(\mathscr{H}_n^{n-1})^{G_1} \tag{16}$$

under the condition that the partition λ possesses the property

$$\sum_{i=2}^{n} \alpha_i \frac{i(i-3)}{2} \geq -1. \tag{17}$$

Note that the proof of Equation (16) would imply the validity of (8) if the additional condition (17) is omitted.

Now let us clarify the meaning of condition (17). Simple computations demonstrate that the right side of Equation (16) coincides with the coefficient of q^{n-1} in the expression

$$q^{\sum_{i=2}^{n} \alpha_i \frac{i(i-1)}{2}} \cdot \frac{\prod_{k=1}^{n} (1-q^k)}{\prod_{i=2}^{n} \left(\prod_{l=1}^{i} (1-q^l)\right)^{\alpha_i} \prod_{m=1}^{\alpha_q} (1-q^m)},$$

and consequently also coincides with the coefficient of q^r in the expression

$$\frac{\prod_{k=\alpha_1+1}^{n} (1-q^k)}{\prod_{i=2}^{n} \left(\prod_{l=1}^{i} (1-q^l)\right)^{\alpha_i}}, \tag{18}$$

where

$$r = n - 1 - \sum_{i=2}^{n} \alpha_i \frac{i(i-1)}{2}. \tag{19}$$

Condition (17) asserts that if the equality

$$n = \sum_{i=1}^{n} i\alpha_i$$

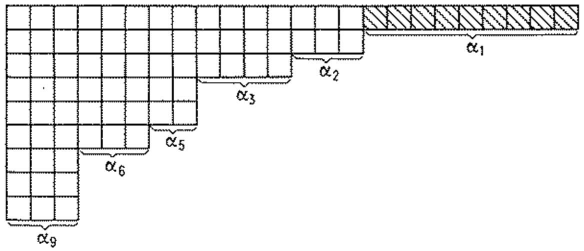
is taken into account, then $r \leq \alpha_1$. Therefore, in computing the coefficient of q^r, the numerator in (18) may be replaced by 1. After this step, the desired coefficient may be obtained by means of a simple combinatorial interpretation. That is to say, it is equal to the number of partitions of r into nonnegative terms, with α_2 groups of two terms each, α_3 groups of three terms each, and so on.

The result may also be written in the following way. If condition (17) is satisfied, the equality

$$\dim {}^{G_2}(\mathscr{H}_n^{n-1})^{G_1} = p_2^{*\alpha_2} * \cdots * p_n^{*\alpha_n}(r) \tag{20}$$

is satisfied, where $*$ denotes the convolution product.

Let us now consider the left side of Equation (16). We enter n vector fields in the cell of Young's diagram that contains α_i columns of length i, $i = 1, 2, \ldots, n$. Then the left-hand side of Equation (16) coincides with the number of linearly independent expressions that may be obtained by means of a permutation of these fields in such a way that each field occurs exactly once in the commutator and such that these expressions are antisymmetric with respect to vector fields that are in the same column of Young's diagram of length equal to or greater than 2 and symmetric with respect to the set of vector fields that occupy columns of unit length (shaded portion of the diagram). The rest of the diagram will be called the *fundamental part*.



(21)

The principle of polarization demonstrates that, without loss of generality, it may be assumed that all the fields that occupy the shaded part of the diagram coincide and, if appropriately chosen, have the form d/dt.

Every expression that is formed in this way has the form $\Phi\, d/dt$, where Φ is a multilinear differential expression that is antisymmetric with respect to the set of functions that specify vector fields in the same column of the fundamental part of Young's diagram.

Suppose that the number of columns in the fundamental part of the diagram (21) is equal to l, where

$$q_1 \geq q_2 \geq \cdots \geq q_l \geq 2$$

are the lengths of the columns and $\Phi_1 = (\varphi_{i1}\, d/dt, \ldots, \varphi_{iq_i}\, d/dt)$ is the set of vector fields that occupy the ith column.

Then vector fields of the form

$$F_{(k_1)}(\Phi_1) \cdots F_{(k_l)}(\Phi_l)\, d/dt \tag{22}$$

will constitute a basis for all possible expressions d/dt, where $(k_i) = (k_{i1}, \ldots, k_{iq_i})$ is a set of integers such that

$$0 \leq k_{i1} < k_{i2} < \cdots < k_{iq_i}, \qquad F_{(k_i)}(\Phi_i) = \det\|\varphi_{ia}^{(k_ib)}\|.$$

The number of distinct fields of the form (22) may easily be computed. We set $l_{ij} = k_{ij} - j + 1$, $1 \leq j \leq q_i$; then $0 \leq l_{ij} \leq \cdots \leq l_{iq_i}$ and

$$\sum_{i=1}^{l} \left(\sum_{j=1}^{q_i} l_{ij} \right) = n - 1 - \sum_{i=1}^{l} \frac{q_i(q_i - 1)}{2} = r.$$

Therefore, the desired number is the number of partitions of r in each cell of the fundamental part of Young's diagram, so that segments of a partition are ordered from top to bottom within each column.

Thus, the dimension of the space generated by vector fields of the form (22) coincides with the right-hand side of Equation (20).

To complete the proof of Theorem 1, it remains to prove that every vector field (22) may be obtained by means of a permutation of the vector fields that occupy the cells of Young's diagram (21).

Lemma. *Suppose that*

$$\xi_i = \varphi_i\, d/dt, \qquad 1 \leq i \leq q, \qquad \eta = d/dt,$$
$$0 \leq k_1 < k_2 < \cdots < k_q.$$

Then the vector field $F_{k_1, \ldots, k_2}(\xi_1, \ldots, \xi_q)\, d/dt$ may be represented as a linear combination of the commutators of the vector fields $\xi_1, \ldots, \xi_q, \eta$. Moreover, if $q \geq 3$, in the algebra $\mathrm{Vect}(\mathbf{R}^1)$, the operator for multiplication by the function

$$F_{k_1, \ldots, k_q}(\xi_1, \ldots, \xi_q) = \det\|\varphi_i^{(k_j)}\|$$

may be obtained by means of a composition of the commutator operators and the vector fields $\xi_1, \ldots, \xi_q, \eta$.

Proof. Both assertions of the lemma may be proved similarly; consequently we present the proof of only the first assertion. For this purpose, we will use induction. If $q = 1, 2$, the assertion may easily be verified directly. Let us take $q > 2$. By the inductive hypothesis, there exists a representation of the fields

$$F_{k_1, \ldots, \hat{k}_i, \ldots, k_q}(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_q)\, d/dt$$

in the form of a linear combination of commutators.

Let us form the new commutators:

$$q \cdot \mathrm{Alt}_\xi \left[(\mathrm{ad}\, \eta)^{k_i} \xi_i, F_{k_1, \ldots, \hat{k}_i, \ldots, k_q}(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_q) \frac{d}{dt} \right]$$

$$= \sum_{j \neq i} F_{k_1, \ldots, k_j + 1, \ldots, k_q}(\xi_1, \ldots, \xi_q)\, d/dt - F_{k_1, \ldots, k_i + 1, \ldots, k_q}(\xi_1, \ldots, \xi_q)\, d/dt,$$

where $i = 1, 2, \ldots, q$. Solving this linear system, we find that all the fields $F_{k_1,\ldots,k_i+1,\ldots,k_q}(\xi_1, \ldots, \xi_q)\, d/dt$ may also be represented in the form of a linear combination of commutators of the fields $\xi_1, \ldots, \xi_1, \eta$. The assertion is proved.

Suppose that the lengths of the columns of Young's diagram satisfy the conditions $q_1 \geq \cdots \geq q_{m-1} \geq 3$ and $q_m = \cdots = q_l = 2$. (Here, obviously, $l - m + 1 = \alpha_2$.)

By means of the formula

$$\langle k \rangle = \sum_{i=1}^{m-1} \left(\sum_{j=1}^{q_i} k_{ij} \right),$$

we may form a weight relative to the vector field (22) corresponding to our choice of $q_1, \ldots, q_l$. Using the lemma, we form commutators that yield the vector fields

$$F_{(k_m)}(\Phi_m)\, d/dt, \ldots, F_{(k_l)}(\Phi_l)\, d/dt.$$

Then we represent the operators for multiplication by the functions $F_{(k_1)}(\Phi_1), \ldots, F_{(k_{m-1})}(\Phi_{m-1})$ in the form of a composition of commutator operators. The number of occurrences of the operator ad η in these compositions is then given by

$$\sum_{i=1}^{m-1} (|k_i| - q_i) \geq \sum_{i=1}^{m-1} \left(\frac{q_i(q_i - 1)}{2} - q_i \right) = \sum_{i=1}^{m-1} \frac{q_i(q_i - 3)}{2}.$$

We form new operators by replacing the operator ad η by $\operatorname{ad}(F_{(k_i)}(\Phi_i)\, d/dt)$, $m \leq i \leq l - 1$ at arbitrary positions. For such a substitution to be possible, it is sufficient that the following inequality is satisfied:

$$\sum_{i=1}^{m-1} \frac{q_i(q_i - 3)}{2} \geq l - m = -\sum_{i=m}^{l-1} \frac{q_i(q_i - 3)}{2};$$

this is equivalent to

$$\sum_{i=1}^{l} \frac{q_i(q_i - 3)}{2} \geq -1,$$

or, using the original notation,

$$\sum_{i=1}^{n} \alpha_i \frac{i(i-3)}{2} \geq -1,$$

which coincides with condition (17).

Finally, let us apply the operators that we have found to a linear combination of commutators, yielding the vector field

$$F_{(k_l)}(\Phi_l)\, d/dt.$$

We then obtain the following expression:

$$F_{(k_1)}(\Phi_1) \cdots F_{(k_l)}(\Phi_l) \frac{d}{dt} + \sum_{(k')} c_{(k')} F_{(k'_1)}(\Phi_1) \cdots \frac{d}{dt}, \tag{23}$$

where the summation on the right-hand side extends over each set (k') such that the weight of the set is less than $\langle k \rangle$.

In fact, the second term in (23) is obtained if, when we eliminate the commutators, some of the derivatives of the functions

$$F_{(k_m)}(\Phi_m), \ldots, F_{(k_{l-1})}(\Phi_{l-1})$$

will be lost, although, obviously, the weight of a field is decreased by the number of such derivatives. The proof is completed by applying induction on the weight.

Analogously, with slight simplifications we may consider the case $q_1 \geq \cdots \geq q_l \geq 3$. The theorem is proved.

Corollary 1. *The S_n-modules of $\Pi_n(\mathrm{Vect}\, \mathbf{R}^1)$ and $\mathscr{H}_n^{n-1}$ contain the hook representations $\pi_{p1}q, p + q = n$ with identical multiplicity.*

Proof. Suppose that a_q and b_q are the multiplicities of the representation $\pi_{p1}q$, $q = 0, 1, \ldots, n$ in $\Pi_n(\mathrm{Vect}\, \mathbf{R}^1)$ and $\mathscr{H}_n^{n-1}$, respectively.

We apply Theorem 1 for the case $G_1 = S_p$ and $G_2 = S_q$. It is well known that

$$\mathrm{Ind}_{S_p \times S_q}^{S_n} (1 \times sgn) = \pi_{p1}q + \pi_{p+1,1}q - 1;$$

consequently, if the corresponding wreath numbers coincide, this will mean that the equalities

$$a_0 = b_0, \quad a_0 + a_1 = b_0 + b_1, \ldots, \quad a_{n-1} + a_n = b_{n-1} + b_n, \quad a_n = b_n,$$

are satisfied, whence the assertion of the corollary follows.

Corollary 2. *The S_n-modules of $\Pi_n(\mathrm{Vect}\, \mathbf{R}^1)$ and $\mathscr{H}_n^{n-1}$ coincide when $n < 7$.*

In fact, if $n < 7$ it follows from Theorem 1 and the Kirillov–Kontsevich formula (cf. [4], [9]) that if the set of representations chosen is large enough, the wreath numbers of these modules will be identical.

6. Growth of a finitely generated subalgebra of the Lie algebra Vect($\mathbb{R}^1$)

Suppose that A_m is the Lie algebra generated by m general vector fields on the real line $\xi_1, \ldots, \xi_m$. This algebra is polygraded by the degrees of the fields $\xi_1, \ldots, \xi_m$ occurring in the commutators:

$$\mathfrak{A}_m = \bigoplus_{(k)} \mathfrak{A}_m^{k_1, \ldots, k_m}.$$

Supposing that hypothesis (8) is valid, let us compute the dimensions of the spaces $\mathfrak{A}_m^{k_1, \ldots, k_m}$. We set $k_1 + \cdots + k_m = n$. We have

$$C[x_1, \ldots, x_n] = C[x_1, \ldots, x_n]^{S_n} \otimes \mathcal{H}_n \tag{24}$$

(cf. Section 3), whence the Poincaré series of the spaces $\mathcal{H}_n^{S_{k_1} \times \cdots \times S_{k_m}}$ is easily found. It has the form

$$\frac{\prod_{k=1}^{n} (1 - q^k)}{\prod_{i=1}^{m} \left(\prod_{l=1}^{k_i} (1 - q^l) \right)}. \tag{25}$$

Consequently, the desired dimension is the coefficient of q^{n-1} in (25) (cf. Section 5). This number may be written in the form

$$\dim \mathfrak{A}_m^{k_1, \ldots, k_m} = r * p_{k_1} * p_{k_m}(n-1), \tag{26}$$

where the series $r(n)$ is given by the equality

$$\sum_{n \geq 0} r(n) q^n = \prod_{l \geq 1} (1 - q^l) = \sum_{m \in Z} (-1)^m q^{\frac{3m^2 + m}{2}}. \tag{27}$$

The rightmost side of the latter relation is the well-known Euler identity (cf. [2]).

We wish to compute the growth of the algebra A_m, i.e., the growth of the series $a_m^{(n)} = \dim \mathfrak{A}_m^{(n)}$, where

$$\mathfrak{A}_m^{(n)} = \bigoplus_{|k| = n} \mathfrak{A}_m^{k_1, \ldots, k_m}.$$

We set

$$f(q_1, \ldots, q_m) = \sum_{(k)} \dim \mathfrak{A}_m^{k_1, \ldots, k_m} q_1^{k_1} \cdots q_m^{k_m},$$

where the summation extends over all possible sets of nonnegative numbers $(k) = (k_1, \ldots, k_m)$.

Using simple combinatorial arguments, we find that

$$f(q_1, \ldots, q_m) = \sum_{k=1}^{m} \frac{q_k^m}{\prod_{i \neq k} (q_k - q_i)} \prod_{l \geq 0} \frac{1}{\prod_{i=1}^{m} (1 - q_i q_k^l)}, \tag{28}$$

where the term $1/(1 - q_k q_k^m)$ in the numerator of the second product has been omitted. The generating function we wish to find for the series

$$a_m^{(n)}, f_m(q) = \sum_{n \geq 0} a_m^{(n)} q^n$$

may be obtained if we set $q_1 = q_2 = \cdots = q_m = q$ in (28) and if the value of the indeterminate form is found by means of L'Hôpital's rule. Unfortunately, the final form of $f_m(q)$ is quite cumbersome, and therefore we present the formula only for $m = 2$ (cf. also [5]):

$$f_2(q) = q^2 \mathcal{P}'(q) + 2q\mathcal{P}(q)(1 - \mathcal{D}(q)),$$

where

$$\mathcal{P}(q) = \prod_{l \geq 1} (1 - q^l)^{-1}, \qquad \mathcal{D}(q) = \sum_{n \geq 1} d(n) q^n, d(n)$$

is the number of divisors of n.

Nevertheless, it is possible to obtain an asymptotic formula for the numbers $a_m^{(n)}$ by using, for example, Theorem 6.2 from [2]. It has the form

$$a_m^{(n)} \sim c n^{3/4(m-2)} \exp(\pi \sqrt{2/3 n(m-1)}). \tag{29}$$

Thus, if hypothesis (8) is true, all the algebras have intermediate growth (cf. [5]). For $m = 2$, the asymptotic formula (29) is then proved; in this case, $c = 1/4\sqrt{3}$.

7. Growth of the series $b(n) = \dim \Pi_n(\text{Vect } \mathbb{R}^1)$

In this section we also suppose that hypothesis (8) is valid. In (26), we set $m = n$ and $k_1 = \cdots = k_n = 1$; then

$$b(n) = \dim \mathfrak{A}_n^{1, \ldots, 1} = r * p_1 * \cdots * p_1(n-1)$$
$$= \sum_{k=0}^{n-1} r(k) \binom{2n-2-k}{n-1}.$$

From (27) it follows that

$$r(k) = \begin{cases} (-1)^m & \text{if } k = (3m^2 + m)/2, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, for any number $N \geq 0$, the following inequalities are satisfied:

$$\sum_{k=0}^{k_1} r(k) \binom{2n-2-k}{n-1} \leq b(n) \leq \sum_{k=0}^{k_2} r(k) \binom{2n-2-k}{n-1}, \tag{30}$$

where

$$k_1 = \frac{3(2N+1)^2 + 2N + 1}{2} = 6N^2 + 7N + 2,$$

$$k_2 = \frac{3(2N)^2 + 2N}{2} = 6N^2 + N.$$

If N is fixed, the asymptotics of the left-hand and right-hand sides of (30) as $n \to \infty$ may easily be calculated. In fact,

$$\binom{2n-2-k}{n-1} \sim \frac{4^n}{2^{k+2}\sqrt{\pi n}}.$$

Consequently, $b(n)$ has the asymptotic expression

$$b(n) \sim c\frac{4^n}{\sqrt{n}}, \tag{31}$$

where the constant c satisfies the conditions

$$\sum_{k=0}^{k_1} r(k)\frac{1}{2^{k+2}\sqrt{\pi}} \le c \le \sum_{k=0}^{k_2} r(k)\frac{1}{2^{k+2}\sqrt{\pi}}.$$

Taking the limit in these expressions as $N \to \infty$ yields

$$c = \frac{1}{4\sqrt{\pi}}\sum_{k=0}^{\infty} r(k)2^{-k} = \frac{1}{4\sqrt{\pi}}\prod_{l\ge 1}(1-2^{-l}). \tag{32}$$

The growth of the series $b(n) = \dim \Pi_n(\text{Vect}\,\mathbf{R}^2)$ is also known as the growth of the manifold of Lie algebras generated by the algebra Vect $\mathbf{R}^1$ [**1**]. Thus, the growth of this manifold is exponential under the assumption that hypothesis (8) is valid.

References

[1] Yu. A. Bahturin, Identities in Lie Algebras (in Russian), Nauka, Moscow, 1985.
[2] G. E. Andrews, Theory of Partitions, Addison-Wesley, 1976.
[3] A. A. Kirillov, *On identities in Lie algebras of Hamiltonian vector fields in the plane* (in Russian), Keldysh Inst. Prikl. Mat. preprint no. 121, 1983.
[4] A. A. Kirillov and M. L. Kontsevich, *Growth of the Lie algebra generated by two vector fields on the line*, Moscow Univ. Math. Bull. no. **4** (1983), 14–20. (Originally published in Vestnik Moskov. Univ. Ser. Mat. Meh. **4** (1983), 15–20.)
[5] A. A. Kirillov, M. L. Kontsevich, and A. I. Molev, *Algebras of intermediate growth* (in Russian), Keldysh Inst. Prikl. Mat. preprint no. 39, 1983.
[6] A. A. Kirillov, V. Yu. Ovsienko, and O. D. Udalova, *Identities in the Lie algebra of vector fields on the real line*, Sel. Math. Sov, in press. (Originally published as Keldysh Inst. Prikl. Mat. preprint no. 135, 1985.)

[7] A. A. Klyachko, *Lie elements in a tensor algebra*. Siberian Math. J. **15**:6 (1974), 914–921. (Originally published in Sibirsk. Mat. Zh. **15**:6 (1974), 1296–1304.)
[8] A. I. Mal'tsev, *On algebras defined by identities* (in Russian), Mat. Sb. **26**(68):1 (1950), 19–33.
[9] A. I. Molev, *A proof of the Kirillov–Kontsevich formulas* (in Russian), Uspehi Mat. Nauk **39** (1984), no. 1 (235), 145–146.
[10] A. I. Molev and L. M. Tsalenko, *Representations of symmetric groups in a free Lie (super) algebra and in a space of harmonic polynomials*, Functional Anal. Appl. **20**:2 (1986), 150–152. (Originally published in Funkts. Anal. i Prilozh. **20**:2 (1986), 76–77.)
[11] A. I. Molev, *On the algebraic structure of the Lie algebra of vector fields on the line*, Math. USSR Sbornik **62**:1(1989), 83–94. (Originally published in Mat. Sb. **134**(176):1 (1987), 82–92.)