Center at the critical level for centralizers in type A

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with the commutation relations

$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\,\delta_{r,-s}\langle X, Y\rangle\,\mathbf{1},$$

where $X[r] = X\,t^r$ for any $X \in \mathfrak{a}$ and $r \in \mathbb{Z}$.

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Note $V(\mathfrak{a}) \cong \mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$ as a vector space.

Hence, $\mathfrak{z}(\widehat{\mathfrak{a}})$ is a subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$.

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Any element of $\mathfrak{z}(\widehat{\mathfrak{a}})$ is called a Segal–Sugawara vector.

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Theorem [Feigin–Frenkel 1992, Frenkel 2007].

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We call $S_1, \dots, S_n$ a complete set of Segal–Sugawara vectors.

Explicit constructions of such sets and a new proof of

the theorem for the classical types A, B, C, D:

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For quantum vertex algebras in types A, B, C, D:

[Jing–Kožić–M.–Yang 2018,   Butorac–Jing–Kožić 2019].

Example: $\mathfrak{a} = \mathfrak{gl}_n$. Defining relations for $\mathrm{U}(\widehat{\mathfrak{gl}}_n)$:

$$E_{ij}[r]\,E_{kl}[s] - E_{kl}[s]\,E_{ij}[r]$$

$$= \delta_{kj}\,E_{il}[r+s] - \delta_{il}\,E_{kj}[r+s] + r\,\delta_{r,-s}\left(\delta_{ij}\delta_{kl} - n\,\delta_{kj}\delta_{il}\right)\mathbf{1}.$$

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For a variable x introduce the $n \times n$ matrix

$$\mathcal{E} = \begin{bmatrix} x + T + E_{11}[-1] & E_{12}[-1] & \dots & E_{1n}[-1] \\ E_{21}[-1] & x + T + E_{22}[-1] & \dots & E_{2n}[-1] \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1}[-1] & E_{n2}[-1] & \dots & x + T + E_{n\,n}[-1] \end{bmatrix}.$$

The column-determinant of $\mathcal{E}$ is a polynomial

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$$\Delta : \phi_k \mapsto -(k-1)(n-k+1)\, \phi_{k-1}$$

for $k = 1, \ldots, n$.

For $n = 2$ the column-determinant $\operatorname{cdet}\mathcal{E}$ equals

$$\big(x + T + E_{11}[-1]\big)\big(x + T + E_{22}[-1]\big) - E_{21}[-1]E_{12}[-1]$$

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with

$$\phi_1 = E_{11}[-1] + E_{22}[-1],$$

$$\phi_2 = E_{11}[-1]\, E_{22}[-1] - E_{21}[-1]\, E_{12}[-1] + E_{22}[-2].$$

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- Affine Harish-Chandra isomorphism, classical $\mathcal{W}$-algebras:

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which yields an $\mathfrak{a}[t]$-module structure on the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{a}[t^{-1}]\big) \cong \mathrm{S}\big(\mathfrak{a}[t, t^{-1}]/\mathfrak{a}[t]\big)$.

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Theorem [Raïs–Tauvel 1992, Beilinson–Drinfeld 1997].

If $P_1, \dots, P_n$ are algebraically independent generators of $\mathrm{S}(\mathfrak{a})^{\mathfrak{a}}$, then the elements $P_{1,(r)}, \dots, P_{n,(r)}$ with $r \geqslant 0$ are algebraically independent generators of $\mathrm{S}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)^{\mathfrak{a}[t]}$.

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Another proof in type A: [Brown–Brundan 2009].

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Suppose that $e \in \mathfrak{g} = \mathfrak{gl}_N$ is a nilpotent matrix with Jordan blocks of sizes $\lambda_1, \ldots, \lambda_n$, where $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ and $\lambda_1 + \cdots + \lambda_n = N$.

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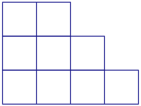
The associated pyramid is a left-justified array of rows of unit boxes; for the blocks $2, 3, 4$ and $N = 9$ the pyramid is



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For any $1 \leqslant i, j \leqslant n$ and $\lambda_j - \min(\lambda_i, \lambda_j) \leqslant r < \lambda_j$ set

$$E_{ij}^{(r)} = \sum_{\substack{\mathrm{row}(a)=i,\ \mathrm{row}(b)=j \\ \mathrm{col}(b)-\mathrm{col}(a)=r}} e_{ab},$$

summed over $a, b \in \{1, \dots, N\}$.

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summed over $a, b \in \{1, \ldots, N\}$.

The elements $E_{ij}^{(r)}$ form a basis of the Lie algebra $\mathfrak{a} = \mathfrak{g}^e$.

Commutation relations for the Lie algebra $\mathfrak{a}$:

$$\big[E_{ij}^{(r)}, E_{kl}^{(s)}\big] = \delta_{kj}\, E_{i\,l}^{(r+s)} - \delta_{i\,l}\, E_{kj}^{(r+s)},$$

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Explicit generators for an arbitrary nilpotent $e \in \mathfrak{g}$:

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$$\big\langle E_{ii}^{(0)}, E_{jj}^{(0)} \big\rangle = \min(\lambda_i, \lambda_j) - \delta_{ij}\big(\lambda_1 + \dots + \lambda_{i-1} + (n - i + 1)\lambda_i\big),$$

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and if $\lambda_i = \lambda_j$ for some $i \neq j$ then

$$\big\langle E_{ij}^{(0)}, E_{ji}^{(0)}\big\rangle = -\big(\lambda_1 + \cdots + \lambda_{i-1} + (n-i+1)\lambda_i\big).$$

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Theorem [Arakawa–Premet 2017]. There exists a complete set of Segal–Sugawara vectors $S_1, \dots, S_N \in \mathfrak{z}(\widehat{\mathfrak{a}})$ so that

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In the case $e = 0$ this is the Feigin–Frenkel theorem.

[AP 2017]: explicit formulas for the S_k in the minimal nilpotent case $\lambda_1 = \dots = \lambda_{n-1} = 1, \ \ \lambda_n = 2$.

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Consider the $n \times n$ matrix $\mathcal{E}$ given by

$$\begin{bmatrix} x + \lambda_1 T + E_{11}(u) & E_{12}(u) & \dots & E_{1n}(u) \\ E_{21}(u) & x + \lambda_2 T + E_{22}(u) & \dots & E_{2n}(u) \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1}(u) & E_{n2}(u) & \dots & x + \lambda_n T + E_{nn}(u) \end{bmatrix}.$$

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Theorem. The coefficients $\phi_k^{(a)}$ with $k = 1, \dots, n$ and

$$\lambda_{n-k+2} + \dots + \lambda_n < a + k \leqslant \lambda_{n-k+1} + \dots + \lambda_n,$$

form a complete set of Segal–Sugawara vectors for $\mathfrak{a}$.

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$$\phi_1(u) = E_{11}(u) + E_{22}(u),$$

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with $b = \lambda_2 - 1, \dots, \lambda_1 + \lambda_2 - 2$.

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For any nonzero $z \in \mathbb{C}$ consider the evaluation homomorphism

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Combine them into the $n \times n$ matrix

$$x + \mathcal{E}(u) = \begin{bmatrix} x + \mathcal{E}_{11}(u) & \mathcal{E}_{12}(u) & \dots & \mathcal{E}_{1n}(u) \\ \mathcal{E}_{21}(u) & x + \mathcal{E}_{22}(u) & \dots & \mathcal{E}_{2n}(u) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E}_{n1}(u) & \mathcal{E}_{n2}(u) & \dots & x + \mathcal{E}_{nn}(u) \end{bmatrix}.$$

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[Brown–Brundan 2009], Takiff case: [M. 1997], [Capelli 1890].

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A general solution: [Arakawa–Premet 2017] following the approach of [Rybnikov 2006].

For $\chi \in \mathfrak{a}^*$ and any nonzero $z \in \mathbb{C}$ consider the homomorphism

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Since $\mathfrak{z}(\widehat{\mathfrak{a}})$ is a commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$, its image is a commutative subalgebra $\mathcal{A}_\chi$ of $\mathrm{U}(\mathfrak{a})$.

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It does not depend on z.

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Moreover, if $\chi \in \mathfrak{a}^*$ is regular, then this family is algebraically independent and $\mathrm{gr}\,\mathcal{A}_\chi = \overline{\mathcal{A}}_\chi$.