Invariants in enveloping algebras and vacuum modules

Alexander Molev

University of Sydney

Invariants of a linear operator

Invariants of a linear operator

Let V be a vector space over $\mathbb{C}$ and let

$$A : V \to V$$

be a linear map.

Invariants of a linear operator

Let V be a vector space over $\mathbb{C}$ and let

$$A : V \to V$$

be a linear map. Choose a basis of V to write A as a matrix.

Invariants of a linear operator

Let V be a vector space over $\mathbb{C}$ and let

$$A : V \to V$$

be a linear map. Choose a basis of V to write A as a matrix. By changing the basis, we get the matrix transformed by

$$A \mapsto TAT^{-1}.$$

Invariants of a linear operator

Let V be a vector space over $\mathbb{C}$ and let

$$A : V \to V$$

be a linear map. Choose a basis of V to write A as a matrix. By changing the basis, we get the matrix transformed by

$$A \mapsto TAT^{-1}.$$

Question: What polynomials in the entries of A remain unchanged?

Invariants of a linear operator

Let V be a vector space over $\mathbb{C}$ and let

$$A : V \to V$$

be a linear map. Choose a basis of V to write A as a matrix. By changing the basis, we get the matrix transformed by

$$A \mapsto TAT^{-1}.$$

Question: What polynomials in the entries of A remain unchanged? Answer: The coefficients of $\det(uI + A)$.

Invariants in symmetric algebra

Invariants in symmetric algebra

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

Invariants in symmetric algebra

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

The adjoint action of $\mathfrak{g}$ on itself extends to the symmetric algebra $\mathrm{S}(\mathfrak{g})$ by

$$Y \cdot X_1 \dots X_k = \sum_{i=1}^{k} X_1 \dots [Y, X_i] \dots X_k, \qquad X_i \in \mathfrak{g}.$$

Invariants in symmetric algebra

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

The adjoint action of $\mathfrak{g}$ on itself extends to the symmetric algebra $\mathrm{S}(\mathfrak{g})$ by

$$Y \cdot X_1 \dots X_k = \sum_{i=1}^{k} X_1 \dots [Y, X_i] \dots X_k, \qquad X_i \in \mathfrak{g}.$$

The subalgebra of invariants is

$$\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} = \{P \in \mathrm{S}(\mathfrak{g}) \mid Y \cdot P = 0 \quad \text{for all} \quad Y \in \mathfrak{g}\}.$$

Let $n = \operatorname{rank} \mathfrak{g}$. Then $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[P_1, \dots, P_n]$, for certain algebraically independent invariants $P_1, \dots, P_n$ whose degrees $d_1, \dots, d_n$ are the exponents of $\mathfrak{g}$ increased by 1.

Let $n = \operatorname{rank} \mathfrak{g}$. Then $S(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[P_1, \dots, P_n]$, for certain algebraically independent invariants $P_1, \dots, P_n$ whose degrees $d_1, \dots, d_n$ are the exponents of $\mathfrak{g}$ increased by 1.

We have the Chevalley isomorphism

$$\varsigma : S(\mathfrak{g})^{\mathfrak{g}} \to S(\mathfrak{h})^W,$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and W is its Weyl group.

Let $n = \operatorname{rank} \mathfrak{g}$. Then $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[P_1, \dots, P_n]$, for certain algebraically independent invariants $P_1, \dots, P_n$ whose degrees $d_1, \dots, d_n$ are the exponents of $\mathfrak{g}$ increased by 1.

We have the Chevalley isomorphism

$$\varsigma : \mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{S}(\mathfrak{h})^{W},$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and W is its Weyl group.

Here we use a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and ς is the projection $\mathrm{S}(\mathfrak{g}) \to \mathrm{S}(\mathfrak{h})$ whose kernel is $\mathrm{S}(\mathfrak{g})(\mathfrak{n}_- \cup \mathfrak{n}_+)$.

Example: $\mathfrak{g} = \mathfrak{gl}_N$. Set

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}$$

Example: $\mathfrak{g} = \mathfrak{gl}_N$. Set

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}$$

and write

$$\det(u + E) = u^N + C_1 u^{N-1} + \dots + C_N.$$

Example: $\mathfrak{g} = \mathfrak{gl}_N$. Set

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}$$

and write

$$\det(u + E) = u^N + C_1 u^{N-1} + \dots + C_N.$$

Then $\quad \mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N} = \mathbb{C}[C_1, \dots, C_N]$

Example: $\mathfrak{g} = \mathfrak{gl}_N$. Set

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}$$

and write

$$\det(u + E) = u^N + C_1\, u^{N-1} + \dots + C_N.$$

Then $\quad \mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N} = \mathbb{C}[C_1, \dots, C_N] \quad$ and

$$\varsigma : \det(u + E) \mapsto (u + \lambda_1) \dots (u + \lambda_N), \qquad \lambda_i = E_{ii}.$$

We have

$$T_k = \operatorname{tr} E^k \in \mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N}$$

for all $k \geqslant 0$,

$$\mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N} = \mathbb{C}[T_1, \dots, T_N]$$

We have

$$T_k = \operatorname{tr} E^k \in \mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N}$$

for all $k \geqslant 0$,

$$\mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N} = \mathbb{C}[T_1, \dots, T_N]$$

and

$$\varsigma : T_k \mapsto \lambda_1^k + \dots + \lambda_N^k.$$

We have

$$T_k = \operatorname{tr} E^k \in \mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N}$$

for all $k \geqslant 0$,

$$\mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N} = \mathbb{C}[T_1, \dots, T_N]$$

and

$$\varsigma : T_k \mapsto \lambda_1^k + \dots + \lambda_N^k.$$

The invariants C_k and T_k are related by the Newton formulas.

Center of universal enveloping algebra

Center of universal enveloping algebra

The adjoint action of $\mathfrak{g}$ on itself extends to the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ by

$$Y \cdot X_1 \ldots X_k = \sum_{i=1}^{k} X_1 \ldots [Y, X_i] \ldots X_k.$$

Center of universal enveloping algebra

The adjoint action of $\mathfrak{g}$ on itself extends to the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ by

$$Y \cdot X_1 \dots X_k = \sum_{i=1}^{k} X_1 \dots [Y, X_i] \dots X_k.$$

The subalgebra of invariants is the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$,

$$\mathrm{Z}(\mathfrak{g}) = \{P \in \mathrm{U}(\mathfrak{g}) \mid Y \cdot P = [Y, P] = 0 \quad \text{for all} \quad Y \in \mathfrak{g}\}.$$

Its elements are called Casimir elements.

We have

$$\mathrm{Z}(\mathfrak{g}) = \mathbb{C}[P_1, \ldots, P_n],$$

for certain algebraically independent invariants $P_1, \ldots, P_n$

whose degrees $d_1, \ldots, d_n$ are the exponents of $\mathfrak{g}$ increased by 1.

We have

$$Z(\mathfrak{g}) = \mathbb{C}[P_1, \dots, P_n],$$

for certain algebraically independent invariants $P_1, \dots, P_n$

whose degrees $d_1, \dots, d_n$ are the exponents of $\mathfrak{g}$ increased by 1.

We have the Harish-Chandra isomorphism

$$\chi : Z(\mathfrak{g}) \to U(\mathfrak{h})^{W_{\text{sh}}}, \qquad \text{with a shifted action of } W.$$

We have

$$Z(\mathfrak{g}) = \mathbb{C}[P_1, \dots, P_n],$$

for certain algebraically independent invariants $P_1, \dots, P_n$

whose degrees $d_1, \dots, d_n$ are the exponents of $\mathfrak{g}$ increased by 1.

We have the Harish-Chandra isomorphism

$$\chi : Z(\mathfrak{g}) \to U(\mathfrak{h})^{W_{\mathsf{sh}}}, \qquad \text{with a shifted action of } W.$$

We use the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus \Big(U(\mathfrak{g})\mathfrak{n}_+ + \mathfrak{n}_- U(\mathfrak{g}) \Big)$$

and χ is the projection $U(\mathfrak{g}) \to U(\mathfrak{h})$.

Example. For $\mathfrak{g} = \mathfrak{gl}_N$ set

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}.$$

Example. For $\mathfrak{g} = \mathfrak{gl}_N$ set

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}.$$

The traces $\operatorname{tr} E^k$ are Casimir elements:

Example. For $\mathfrak{g} = \mathfrak{gl}_N$ set

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}.$$

The traces $\operatorname{tr} E^k$ are Casimir elements:

$$\operatorname{tr} E = \sum_{i=1}^{N} E_{ii}, \qquad \operatorname{tr} E^2 = \sum_{i,j=1}^{N} E_{ij} E_{ji}$$

$$\operatorname{tr} E^3 = \sum_{i,j,k=1}^{N} E_{ij} E_{jk} E_{ki}, \qquad \text{etc.}$$

Example. For $\mathfrak{g} = \mathfrak{gl}_N$ set

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}.$$

The traces $\operatorname{tr} E^k$ are Casimir elements:

$$\operatorname{tr} E = \sum_{i=1}^{N} E_{ii}, \qquad \operatorname{tr} E^2 = \sum_{i,j=1}^{N} E_{ij} E_{ji}$$

$$\operatorname{tr} E^3 = \sum_{i,j,k=1}^{N} E_{ij} E_{jk} E_{ki}, \qquad \text{etc.}$$

Any Casimir element is a unique polynomial in $\operatorname{tr} E^k$, $1 \leqslant k \leqslant N$.

For the Harish-Chandra images we have

$$\chi : \sum_{i=1}^{N} E_{ii} \mapsto \sum_{i=1}^{N} l_i + \binom{N}{2},$$

For the Harish-Chandra images we have

$$\chi : \sum_{i=1}^{N} E_{ii} \mapsto \sum_{i=1}^{N} l_i + \binom{N}{2},$$

$$\chi : \sum_{i,j=1}^{N} E_{ij} E_{ji} \mapsto \sum_{i=1}^{N} l_i^2 + (N-1) \sum_{i=1}^{N} l_i + \binom{N}{3},$$

where $l_i = E_{ii} - i + 1$.

For the Harish-Chandra images we have

$$\chi : \sum_{i=1}^{N} E_{ii} \mapsto \sum_{i=1}^{N} l_i + \binom{N}{2},$$

$$\chi : \sum_{i,j=1}^{N} E_{ij} E_{ji} \mapsto \sum_{i=1}^{N} l_i^2 + (N-1) \sum_{i=1}^{N} l_i + \binom{N}{3},$$

where $l_i = E_{ii} - i + 1$.

In general,

$$1 + \sum_{m=0}^{\infty} \frac{(-1)^m \chi(\operatorname{tr} E^m)}{(u - N + 1)^{m+1}} = \prod_{i=1}^{N} \frac{u + l_i + 1}{u + l_i}.$$

Many more constructions of Casimir elements for the Lie algebras $\mathfrak{gl}_N$, $\mathfrak{o}_N$ and $\mathfrak{sp}_{2n}$ are known.

Many more constructions of Casimir elements for the Lie algebras $\mathfrak{gl}_N$, $\mathfrak{o}_N$ and $\mathfrak{sp}_{2n}$ are known.

In particular, there is a linear basis of $\mathrm{Z}(\mathfrak{gl}_N)$ formed by the quantum immanats $\mathbb{S}_\lambda$ with λ running over partitions with at most N parts (Okounkov–Olshanski, 1996, 1998).

Many more constructions of Casimir elements for the Lie algebras $\mathfrak{gl}_N$, $\mathfrak{o}_N$ and $\mathfrak{sp}_{2n}$ are known.

In particular, there is a linear basis of $\mathrm{Z}(\mathfrak{gl}_N)$ formed by the quantum immanats $\mathbb{S}_\lambda$ with λ running over partitions with at most N parts (Okounkov–Olshanski, 1996, 1998).

The Harish-Chandra images $\chi(\mathbb{S}_\lambda)$ are the shifted Schur polynomials.

Affine Kac–Moody algebras

Affine Kac–Moody algebras

Define an invariant bilinear form on a simple Lie algebra $\mathfrak{g}$,

$$\langle X, Y \rangle = \frac{1}{2h^\vee} \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y),$$

where $h^\vee$ is the dual Coxeter number.

Affine Kac–Moody algebras

Define an invariant bilinear form on a simple Lie algebra $\mathfrak{g}$,

$$\langle X, Y\rangle = \frac{1}{2h^\vee}\,\mathrm{tr}\,(\mathrm{ad}\,X\,\mathrm{ad}\,Y),$$

where $h^\vee$ is the dual Coxeter number.

For the classical types, $\langle X, Y\rangle = \mathrm{const}\cdot\mathrm{tr}\,XY$,

$$h^\vee = \begin{cases} N & \text{for} \quad \mathfrak{g} = \mathfrak{sl}_N, \qquad \mathrm{const} = 1 \\ N-2 & \text{for} \quad \mathfrak{g} = \mathfrak{o}_N, \qquad \mathrm{const} = \frac{1}{2} \\ n+1 & \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n}, \qquad \mathrm{const} = 1. \end{cases}$$

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C} K$$

with the commutation relations

$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\, \delta_{r,-s} \langle X, Y \rangle\, K,$$

where $X[r] = X t^r$ for any $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$.

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

with the commutation relations

$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\,\delta_{r,-s}\langle X, Y\rangle\, K,$$

where $X[r] = X\,t^r$ for any $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$.

Question: What are Casimir elements for $\widehat{\mathfrak{g}}$?

Given $\kappa \in \mathbb{C}$, the universal enveloping algebra $\mathrm{U}_\kappa(\widehat{\mathfrak{g}})$ at the level κ is the quotient of $\mathrm{U}(\widehat{\mathfrak{g}})$ by the ideal generated by $K - \kappa$.

Given $\kappa \in \mathbb{C}$, the universal enveloping algebra $\mathrm{U}_\kappa(\widehat{\mathfrak{g}})$ at the level κ is the quotient of $\mathrm{U}(\widehat{\mathfrak{g}})$ by the ideal generated by $K - \kappa$.

A necessary condition for the existence of Casimir elements:

K is at the critical level, $\kappa = -h^\vee$.

Given $\kappa \in \mathbb{C}$, the universal enveloping algebra $\mathrm{U}_\kappa(\widehat{\mathfrak{g}})$ at the level κ is the quotient of $\mathrm{U}(\widehat{\mathfrak{g}})$ by the ideal generated by $K - \kappa$.

A necessary condition for the existence of Casimir elements:

K is at the critical level, $\kappa = -h^\vee$.

Still, $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ is too small to contain Casimir elements:

the center of $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ is trivial.

Given $\kappa \in \mathbb{C}$, the universal enveloping algebra $\mathrm{U}_\kappa(\widehat{\mathfrak{g}})$ at the level κ is the quotient of $\mathrm{U}(\widehat{\mathfrak{g}})$ by the ideal generated by $K - \kappa$.

A necessary condition for the existence of Casimir elements: K is at the critical level, $\kappa = -h^\vee$.

Still, $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ is too small to contain Casimir elements: the center of $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ is trivial.

By [Kac, 1974], the canonical quadratic Casimir element belongs to an extension of $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})$.

Example: $\mathfrak{g} = \mathfrak{gl}_N$. Defining relations for $\mathrm{U}(\widehat{\mathfrak{gl}}_N)$:

$$E_{ij}[r]\, E_{kl}[s] - E_{kl}[s]\, E_{ij}[r]$$

$$= \delta_{kj}\, E_{il}[r+s] - \delta_{il}\, E_{kj}[r+s] + r\,\delta_{r,-s}\left(\delta_{kj}\,\delta_{il} - \frac{\delta_{ij}\,\delta_{kl}}{N}\right) K.$$

Example: $\mathfrak{g} = \mathfrak{gl}_N$. Defining relations for $\mathrm{U}(\widehat{\mathfrak{gl}}_N)$:

$$E_{ij}[r]\,E_{kl}[s] - E_{kl}[s]\,E_{ij}[r]$$

$$= \delta_{kj}\,E_{i\,l}[r+s] - \delta_{i\,l}\,E_{kj}[r+s] + r\,\delta_{r,-s}\left(\delta_{kj}\,\delta_{i\,l} - \frac{\delta_{ij}\,\delta_{kl}}{N}\right)K.$$

The critical level is $K = -N$.

Example: $\mathfrak{g} = \mathfrak{gl}_N$. Defining relations for $\mathrm{U}(\widehat{\mathfrak{gl}}_N)$:

$$E_{ij}[r]\,E_{kl}[s] - E_{kl}[s]\,E_{ij}[r]$$

$$= \delta_{kj}\,E_{i\,l}[r+s] - \delta_{i\,l}\,E_{kj}[r+s] + r\,\delta_{r,-s}\left(\delta_{kj}\,\delta_{i\,l} - \frac{\delta_{ij}\,\delta_{kl}}{N}\right)K.$$

The critical level is $K = -N$.

For all $r \in \mathbb{Z}$ the sums

$$\sum_{i=1}^{N} E_{ii}[r]$$

are Casimir elements.

For $r \in \mathbb{Z}$ set

$$C_r = \sum_{i,j=1}^{N} \Big(\sum_{s<0} E_{ij}[s]\, E_{ji}[r-s] + \sum_{s\geqslant 0} E_{ji}[r-s]\, E_{ij}[s] \Big).$$

For $r \in \mathbb{Z}$ set

$$C_r = \sum_{i,j=1}^{N} \Big(\sum_{s<0} E_{ij}[s]\, E_{ji}[r-s] + \sum_{s\geqslant 0} E_{ji}[r-s]\, E_{ij}[s] \Big).$$

All C_r are Casimir elements at the critical level.

For $r \in \mathbb{Z}$ set

$$C_r = \sum_{i,j=1}^{N} \Big(\sum_{s<0} E_{ij}[s]\, E_{ji}[r-s] + \sum_{s\geqslant 0} E_{ji}[r-s]\, E_{ij}[s] \Big).$$

All C_r are Casimir elements at the critical level.

They belong to the completed universal enveloping algebra

$\widetilde{\mathrm{U}}_{-N}(\widehat{\mathfrak{gl}}_N)$ defined as the inverse limit

$$\widetilde{\mathrm{U}}_{-N}(\widehat{\mathfrak{gl}}_N) = \lim_{\longleftarrow} \mathrm{U}_{-N}(\widehat{\mathfrak{gl}}_N)/\mathrm{I}_m, \qquad m \to \infty,$$

where I_m is the left ideal of $\mathrm{U}_{-N}(\widehat{\mathfrak{gl}}_N)$ generated by $t^m \mathfrak{gl}_N[t]$.

Introduce the (formal) Laurent series

$$E_{ij}(z) = \sum_{r\in\mathbb{Z}} E_{ij}[r]\, z^{-r-1}$$

Introduce the (formal) Laurent series

$$E_{ij}(z) = \sum_{r\in\mathbb{Z}} E_{ij}[r]\, z^{-r-1}$$

and use the notation

$$E_{ij}(z)_+ = \sum_{r<0} E_{ij}[r]\, z^{-r-1}, \qquad E_{ij}(z)_- = \sum_{r\geqslant 0} E_{ij}[r]\, z^{-r-1}.$$

Introduce the (formal) Laurent series

$$E_{ij}(z) = \sum_{r\in\mathbb{Z}} E_{ij}[r]\, z^{-r-1}$$

and use the notation

$$E_{ij}(z)_{+} = \sum_{r<0} E_{ij}[r]\, z^{-r-1}, \qquad E_{ij}(z)_{-} = \sum_{r\geqslant 0} E_{ij}[r]\, z^{-r-1}.$$

Given two Laurent series $a(z)$ and $b(z)$,

their normally ordered product is defined by

$$:a(z)\,b(z): \; = a(z)_{+}\, b(z) + b(z)\, a(z)_{-}.$$

Note

$$\sum_{r\in\mathbb{Z}} C_r\, z^{-r-2} = \sum_{i,j=1}^{N} \Big(E_{ij}(z)_+ E_{ji}(z) + E_{ji}(z) E_{ij}(z)_-\Big).$$

Note

$$\sum_{r\in\mathbb{Z}} C_r\, z^{-r-2} = \sum_{i,j=1}^{N} \Big(E_{ij}(z)_+ E_{ji}(z) + E_{ji}(z) E_{ij}(z)_-\Big).$$

Hence, all coefficients of the series

$$\mathrm{tr} : E(z)^2 : \; = \sum_{i,j=1}^{N} : E_{ij}(z) E_{ji}(z) :$$

are Casimir elements.

Similarly, all coefficients of the series

$$\mathrm{tr} : E(z)^3 : \; = \sum_{i,j,k=1}^{N} : E_{ij}(z)\, E_{jk}(z)\, E_{ki}(z) :$$

are Casimir elements, where the normal ordering is applied

from right to left.

Similarly, all coefficients of the series

$$\mathrm{tr} : E(z)^3 : \; = \sum_{i,j,k=1}^{N} : E_{ij}(z)\, E_{jk}(z)\, E_{ki}(z) :$$

are Casimir elements, where the normal ordering is applied

from right to left.

However, the claim does not extend to $\mathrm{tr} : E(z)^4 :$!

Similarly, all coefficients of the series

$$\mathrm{tr} : E(z)^3 : \; = \sum_{i,j,k=1}^{N} : E_{ij}(z)\, E_{jk}(z)\, E_{ki}(z) :$$

are Casimir elements, where the normal ordering is applied from right to left.

However, the claim does not extend to $\mathrm{tr} : E(z)^4 :$!

Correction term: all coefficients of the series

$$\mathrm{tr} : E(z)^4 : \; - \mathrm{tr} : \big(\partial_z E(z)\big)^2 :$$

are Casimir elements.

Invariants of the vacuum module

Invariants of the vacuum module

The vacuum module at the critical level is the $\widehat{\mathfrak{g}}$-module

$$V(\mathfrak{g}) = \mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})/\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})\,\mathfrak{g}[t].$$

Invariants of the vacuum module

The vacuum module at the critical level is the $\widehat{\mathfrak{g}}$-module

$$V(\mathfrak{g}) = \mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})/\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})\,\mathfrak{g}[t].$$

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the algebra of $\mathfrak{g}[t]$-invariants

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\,v = 0\}.$$

Invariants of the vacuum module

The vacuum module at the critical level is the $\widehat{\mathfrak{g}}$-module

$$V(\mathfrak{g}) = \mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})/\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})\,\mathfrak{g}[t].$$

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the algebra of $\mathfrak{g}[t]$-invariants

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\,v = 0\}.$$

Note $\quad V(\mathfrak{g}) \cong \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \quad$ as a vector space.

Invariants of the vacuum module

The vacuum module at the critical level is the $\widehat{\mathfrak{g}}$-module

$$V(\mathfrak{g}) = \mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})/\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})\,\mathfrak{g}[t].$$

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the algebra of $\mathfrak{g}[t]$-invariants

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\,v = 0\}.$$

Note $V(\mathfrak{g}) \cong \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ as a vector space.

Hence, $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

Properties:

- The algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is commutative.

Properties:

- The algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is commutative.
- The subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ is invariant with respect to the translation operator T defined as the derivation $T = -d/dt$.

Properties:

- The algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is commutative.
- The subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ is invariant with respect to the translation operator T defined as the derivation $T = -d/dt$.

Any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a Segal–Sugawara vector.

Theorem (Feigin–Frenkel, 1992, Frenkel, 2007).

There exist Segal–Sugawara vectors $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

$n = \operatorname{rank}\mathfrak{g}$, such that

Theorem (Feigin–Frenkel, 1992, Frenkel, 2007).

There exist Segal–Sugawara vectors $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

$n = \operatorname{rank} \mathfrak{g}$, such that

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}\,[T^k S_l \mid l = 1, \dots, n, \ \ k \geqslant 0].$$

Theorem (Feigin–Frenkel, 1992, Frenkel, 2007).

There exist Segal–Sugawara vectors $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

$n = \operatorname{rank}\mathfrak{g}$, such that

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}\,[T^k S_l \mid l = 1, \dots, n, \ \ k \geqslant 0].$$

We call $S_1, \dots, S_n$ a complete set of Segal–Sugawara vectors.

Theorem (Feigin–Frenkel, 1992, Frenkel, 2007).

There exist Segal–Sugawara vectors $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

$n = \operatorname{rank}\mathfrak{g}$, such that

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}\,[T^k S_l \mid l = 1, \dots, n, \ \ k \geqslant 0].$$

We call $S_1, \dots, S_n$ a complete set of Segal–Sugawara vectors.

Explicit constructions of such sets and a new proof of

the theorem for the classical types A, B, C, D:

[Chervov–Talalaev, 2006, Chervov–M., 2009, M. 2013].

Example: $\mathfrak{g} = \mathfrak{gl}_N$.

Example: $\mathfrak{g} = \mathfrak{gl}_N$.

Set $\tau = -d/dt$ and consider the $N \times N$ matrix

$$\tau + E[-1] = \begin{bmatrix} \tau + E_{11}[-1] & E_{12}[-1] & \dots & E_{1N}[-1] \\ E_{21}[-1] & \tau + E_{22}[-1] & \dots & E_{2N}[-1] \\ \vdots & \vdots & \ddots & \vdots \\ E_{N1}[-1] & E_{N2}[-1] & \dots & \tau + E_{NN}[-1] \end{bmatrix}.$$

The coefficients $S_1, \dots, S_N$ of the polynomial

$$\mathrm{cdet}\big(\tau + E[-1]\big) = \tau^N + S_1\,\tau^{N-1} + \dots + S_{N-1}\,\tau + S_N$$

form a complete set of Segal–Sugawara vectors.

The coefficients $S_1, \dots, S_N$ of the polynomial

$$\mathrm{cdet}\big(\tau + E[-1]\big) = \tau^N + S_1\,\tau^{N-1} + \dots + S_{N-1}\,\tau + S_N$$

form a complete set of Segal–Sugawara vectors.

For $N = 2$

$$\mathrm{cdet}\big(\tau + E[-1]\big) = \big(\tau + E_{11}[-1]\big)\big(\tau + E_{22}[-1]\big) - E_{21}[-1]E_{12}[-1]$$

$$= \tau^2 + S_1\,\tau + S_2$$

The coefficients $S_1, \dots, S_N$ of the polynomial

$$\mathrm{cdet}\big(\tau + E[-1]\big) = \tau^N + S_1\,\tau^{N-1} + \dots + S_{N-1}\,\tau + S_N$$

form a complete set of Segal–Sugawara vectors.

For $N = 2$

$$\begin{aligned}\mathrm{cdet}\big(\tau + E[-1]\big) &= \big(\tau + E_{11}[-1]\big)\big(\tau + E_{22}[-1]\big) - E_{21}[-1]E_{12}[-1] \\ &= \tau^2 + S_1\,\tau + S_2\end{aligned}$$

with

$$S_1 = E_{11}[-1] + E_{22}[-1],$$

$$S_2 = E_{11}[-1]\,E_{22}[-1] - E_{21}[-1]\,E_{12}[-1] + E_{22}[-2].$$

To get another family of Segal–Sugawara vectors, expand

$$\operatorname{tr}\big(\tau + E[-1]\big)^m = U_{m\,0}\,\tau^m + U_{m\,1}\,\tau^{m-1} + \cdots + U_{m\,m}$$

To get another family of Segal–Sugawara vectors, expand

$$\operatorname{tr}\big(\tau + E[-1]\big)^m = U_{m\,0}\,\tau^m + U_{m\,1}\,\tau^{m-1} + \cdots + U_{m\,m}$$

All coefficients $U_{m\,i}$ belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$.

To get another family of Segal–Sugawara vectors, expand

$$\mathrm{tr}\,\big(\tau + E[-1]\big)^m = U_{m\,0}\,\tau^m + U_{m\,1}\,\tau^{m-1} + \dots + U_{m\,m}$$

All coefficients $U_{m\,i}$ belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$.

The elements $U_{1\,1}, \dots, U_{N\,N}$ form

a complete set of Segal–Sugawara vectors.

To get another family of Segal–Sugawara vectors, expand

$$\operatorname{tr}\big(\tau + E[-1]\big)^m = U_{m\,0}\,\tau^m + U_{m\,1}\,\tau^{m-1} + \dots + U_{m\,m}$$

All coefficients $U_{m\,i}$ belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$.

The elements $U_{1\,1}, \dots, U_{N\,N}$ form

a complete set of Segal–Sugawara vectors.

The following are Segal–Sugawara vectors for $\mathfrak{gl}_N$:

$$\operatorname{tr} E[-1], \qquad \operatorname{tr} E[-1]^2, \qquad \operatorname{tr} E[-1]^3, \qquad \operatorname{tr} E[-1]^4 - \operatorname{tr} E[-2]^2.$$

The corresponding central elements in $\widetilde{\mathrm{U}}_{-N}(\widehat{\mathfrak{gl}}_N)$ are recovered by the state-field correspondence map

$$Y : V(\mathfrak{gl}_N) \to \mathrm{End}\, V(\mathfrak{gl}_N)[[z, z^{-1}]]$$

The corresponding central elements in $\widetilde{\mathrm{U}}_{-N}(\widehat{\mathfrak{gl}}_N)$ are recovered by the state-field correspondence map

$$Y : V(\mathfrak{gl}_N) \to \operatorname{End} V(\mathfrak{gl}_N)[[z, z^{-1}]]$$

applied to Segal–Sugawara vectors, i.e., elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$.

The corresponding central elements in $\widetilde{\mathrm{U}}_{-N}(\widehat{\mathfrak{gl}}_N)$ are recovered by the state-field correspondence map

$$Y : V(\mathfrak{gl}_N) \to \operatorname{End} V(\mathfrak{gl}_N)[[z, z^{-1}]]$$

applied to Segal–Sugawara vectors, i.e., elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$.

By definition,

$$Y : E_{ij}[-1] \mapsto E_{ij}(z) = \sum_{r\in\mathbb{Z}} E_{ij}[r]\, z^{-r-1}.$$

Also,

$$Y : E_{ij}[-r-1] \mapsto \frac{1}{r!}\,\partial_z^{\,r} E_{ij}(z), \qquad r \geqslant 0,$$

Also,

$$Y : E_{ij}[-r-1] \mapsto \frac{1}{r!}\,\partial_z^r E_{ij}(z), \qquad r \geqslant 0,$$

and

$$Y : E_{ij}[-1]\,E_{kl}[-1] \mapsto\ :E_{ij}(z)\,E_{kl}(z):$$

Also,

$$Y : E_{ij}[-r-1] \mapsto \frac{1}{r!}\,\partial_z^r\, E_{ij}(z), \qquad r \geqslant 0,$$

and

$$Y : E_{ij}[-1]\, E_{kl}[-1] \mapsto\ :E_{ij}(z)\, E_{kl}(z):$$

We have

$$Y : \operatorname{tr} E[-1] \mapsto \operatorname{tr} E(z)$$

$$Y : \operatorname{tr} E[-1]^2 \mapsto \operatorname{tr} :E(z)^2:$$

$$Y : \operatorname{tr} E[-1]^3 \mapsto \operatorname{tr} :E(z)^3:$$

$$Y : \operatorname{tr} E[-1]^4 - \operatorname{tr} E[-2]^2 \mapsto \operatorname{tr} :E(z)^4: \ - \operatorname{tr} : \big(\partial_z E(z)\big)^2 :$$

Write

$$\operatorname{tr} : \bigl(\partial_z + E(z)\bigr)^m : \; = U_{m0}(z)\,\partial_z^m + \cdots + U_{mm}(z).$$

Write

$$\mathrm{tr} : \big(\partial_z + E(z)\big)^m : \; = U_{m0}(z)\, \partial_z^m + \cdots + U_{mm}(z).$$

Theorem. The coefficients of the Laurent series

$$U_{11}(z), \ldots, U_{NN}(z)$$

are topological generators of the center of $\widetilde{\mathrm{U}}_{-N}(\widehat{\mathfrak{gl}}_N)$.

Proving the Feigin–Frenkel theorem for the classical types:

Proving the Feigin–Frenkel theorem for the classical types:

- Produce Segal–Sugawara vectors $S_1, \dots, S_n$ explicitly.

Proving the Feigin–Frenkel theorem for the classical types:

- Produce Segal–Sugawara vectors $S_1, \dots, S_n$ explicitly.
- Show that all elements $T^k S_l$ with $l = 1, \dots, n$ and $k \geqslant 0$ are algebraically independent and generate $\mathfrak{z}(\widehat{\mathfrak{g}})$.

Proving the Feigin–Frenkel theorem for the classical types:

- Produce Segal–Sugawara vectors $S_1, \dots, S_n$ explicitly.
- Show that all elements $T^k S_l$ with $l = 1, \dots, n$ and $k \geqslant 0$ are algebraically independent and generate $\mathfrak{z}(\widehat{\mathfrak{g}})$.

Use the classical limit:

$$\operatorname{gr} \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \cong \mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$$

Proving the Feigin–Frenkel theorem for the classical types:

- Produce Segal–Sugawara vectors $S_1, \dots, S_n$ explicitly.
- Show that all elements $T^k S_l$ with $l = 1, \dots, n$ and $k \geqslant 0$ are algebraically independent and generate $\mathfrak{z}(\widehat{\mathfrak{g}})$.

Use the classical limit:

$$\operatorname{gr} \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \cong \mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$$

which yields a $\mathfrak{g}[t]$-module structure on the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \cong \mathrm{S}\big(\mathfrak{g}[t, t^{-1}]/\mathfrak{g}[t]\big)$.

Let $X_1, \ldots, X_d$ be a basis of $\mathfrak{g}$ and let $P = P(X_1, \ldots, X_d)$ be a $\mathfrak{g}$-invariant in the symmetric algebra $\mathrm{S}(\mathfrak{g})$.

Let $X_1, \dots, X_d$ be a basis of $\mathfrak{g}$ and let $P = P(X_1, \dots, X_d)$ be a $\mathfrak{g}$-invariant in the symmetric algebra $\mathrm{S}(\mathfrak{g})$. Then each element

$$P_{(r)} = T^r P\big(X_1[-1], \dots, X_d[-1]\big), \qquad r \geqslant 0,$$

is a $\mathfrak{g}[t]$-invariant in the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

Let $X_1, \dots, X_d$ be a basis of $\mathfrak{g}$ and let $P = P(X_1, \dots, X_d)$ be a $\mathfrak{g}$-invariant in the symmetric algebra $\mathrm{S}(\mathfrak{g})$. Then each element

$$P_{(r)} = T^r P\big(X_1[-1], \dots, X_d[-1]\big), \qquad r \geqslant 0,$$

is a $\mathfrak{g}[t]$-invariant in the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

Theorem (Raïs–Tauvel, 1992, Beilinson–Drinfeld, 1997).

If $P_1, \dots, P_n$ are algebraically independent generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then the elements $P_{1,(r)}, \dots, P_{n,(r)}$ with $r \geqslant 0$ are algebraically independent generators of $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{g}[t]}$.

Take a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

Take a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

and consider the (affine) Harish-Chandra homomorphism

$$\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{h}} \to \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big),$$

Take a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

and consider the (affine) Harish-Chandra homomorphism

$$\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{h}} \to \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big),$$

the projection modulo the left ideal generated by $t^{-1}\mathfrak{n}_+[t^{-1}]$.

Take a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

and consider the (affine) Harish-Chandra homomorphism

$$\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{h}} \to \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big),$$

the projection modulo the left ideal generated by $t^{-1}\mathfrak{n}_+[t^{-1}]$.

The restriction to $\mathfrak{z}(\widehat{\mathfrak{g}})$ yields the Harish-Chandra isomorphism

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \to \mathcal{W}({}^L\mathfrak{g}),$$

Take a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

and consider the (affine) Harish-Chandra homomorphism

$$\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{h}} \to \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big),$$

the projection modulo the left ideal generated by $t^{-1}\mathfrak{n}_+[t^{-1}]$.

The restriction to $\mathfrak{z}(\widehat{\mathfrak{g}})$ yields the Harish-Chandra isomorphism

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \to \mathcal{W}({}^L\mathfrak{g}),$$

where $\mathcal{W}({}^L\mathfrak{g})$ is the classical $\mathcal{W}$-algebra associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$ [Feigin and Frenkel, 1992].

Classical $\mathcal{W}$-algebras

Classical $\mathcal{W}$-algebras

Let $\mu_1, \dots \mu_n$ be a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Classical $\mathcal{W}$-algebras

Let $\mu_1, \dots \mu_n$ be a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Set $\mu_i[r] = \mu_i \, t^r$ and identify

$$\mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big) = \mathbb{C}\left[\mu_1[r], \dots, \mu_n[r] \mid r < 0\right] =: \mathcal{P}_n.$$

Classical $\mathcal{W}$-algebras

Let $\mu_1, \dots \mu_n$ be a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Set $\mu_i[r] = \mu_i\, t^r$ and identify

$$\mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big) = \mathbb{C}\left[\mu_1[r], \dots, \mu_n[r] \mid r < 0\right] =: \mathcal{P}_n.$$

The classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ is defined by

$$\mathcal{W}(\mathfrak{g}) = \{P \in \mathcal{P}_n \mid V_i\, P = 0, \quad i = 1, \dots, n\},$$

Classical $\mathcal{W}$-algebras

Let $\mu_1, \dots \mu_n$ be a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Set $\mu_i[r] = \mu_i\, t^r$ and identify

$$\mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big) = \mathbb{C}\left[\mu_1[r], \dots, \mu_n[r] \mid r < 0\right] =: \mathcal{P}_n.$$

The classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ is defined by

$$\mathcal{W}(\mathfrak{g}) = \{P \in \mathcal{P}_n \mid V_i\, P = 0, \quad i = 1, \dots, n\},$$

the V_i are the screening operators.

Example. For $\mathcal{W}(\mathfrak{gl}_N)$ the operators $V_1, \dots, V_{N-1}$ are

Example. For $\mathcal{W}(\mathfrak{gl}_N)$ the operators $V_1, \dots, V_{N-1}$ are

$$V_i = \sum_{r=0}^{\infty} V_{i\,(r)} \left(\frac{\partial}{\partial \mu_i[-r-1]} - \frac{\partial}{\partial \mu_{i+1}[-r-1]} \right),$$

Example. For $\mathcal{W}(\mathfrak{gl}_N)$ the operators $V_1, \dots, V_{N-1}$ are

$$V_i = \sum_{r=0}^{\infty} V_{i\,(r)} \left(\frac{\partial}{\partial \mu_i[-r-1]} - \frac{\partial}{\partial \mu_{i+1}[-r-1]} \right),$$

$$\sum_{r=0}^{\infty} V_{i\,(r)}\, z^r = \exp \sum_{m=1}^{\infty} \frac{\mu_i[-m] - \mu_{i+1}[-m]}{m}\, z^m.$$

Define the elements $\mathcal{E}_1, \dots, \mathcal{E}_N$ by the Miura transformation

$$\big(\tau + \mu_N[-1]\big) \dots \big(\tau + \mu_1[-1]\big) = \tau^N + \mathcal{E}_1 \tau^{N-1} + \dots + \mathcal{E}_N.$$

Define the elements $\mathcal{E}_1, \dots, \mathcal{E}_N$ by the Miura transformation

$$\big(\tau + \mu_N[-1]\big) \dots \big(\tau + \mu_1[-1]\big) = \tau^N + \mathcal{E}_1\,\tau^{N-1} + \dots + \mathcal{E}_N.$$

Explicitly,

$$\mathcal{E}_m = e_m\big(T + \mu_1[-1], \dots, T + \mu_N[-1]\big)$$

is the noncommutative elementary symmetric function,

$$e_m(x_1, \dots, x_p) = \sum_{i_1 > \dots > i_m} x_{i_1} \dots x_{i_m}.$$

Then

$$\mathcal{W}(\mathfrak{gl}_N) = \mathbb{C}[T^k\mathcal{E}_1, \ldots, T^k\mathcal{E}_N \mid k \geqslant 0].$$

Then

$$\mathcal{W}(\mathfrak{gl}_N) = \mathbb{C}[T^k\mathcal{E}_1, \ldots, T^k\mathcal{E}_N \mid k \geqslant 0].$$

Also,

$$\mathcal{W}(\mathfrak{gl}_N) = \mathbb{C}[T^k\mathcal{H}_1, \ldots, T^k\mathcal{H}_N \mid k \geqslant 0],$$

Then

$$\mathcal{W}(\mathfrak{gl}_N) = \mathbb{C}[T^k\mathcal{E}_1, \dots, T^k\mathcal{E}_N \mid k \geqslant 0].$$

Also,

$$\mathcal{W}(\mathfrak{gl}_N) = \mathbb{C}[T^k\mathcal{H}_1, \dots, T^k\mathcal{H}_N \mid k \geqslant 0],$$

where

$$\mathcal{H}_m = h_m\big(T + \mu_1[-1], \dots, T + \mu_N[-1]\big)$$

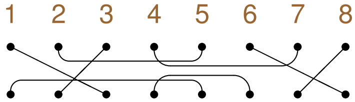
is the noncommutative complete symmetric function,

$$h_m(x_1, \dots, x_p) = \sum_{i_1 \leqslant \dots \leqslant i_m} x_{i_1} \dots x_{i_m}.$$

Brauer algebra $\mathcal{B}_m(\omega)$

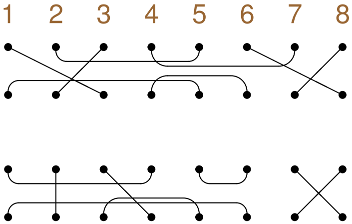
Brauer algebra $\mathcal{B}_m(\omega)$

Multiplication of m-diagrams ($m = 8$):



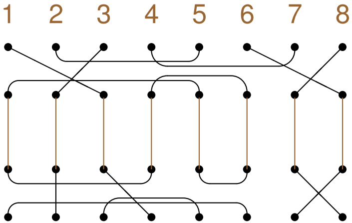
Brauer algebra $\mathcal{B}_m(\omega)$

Multiplication of m-diagrams ($m = 8$):



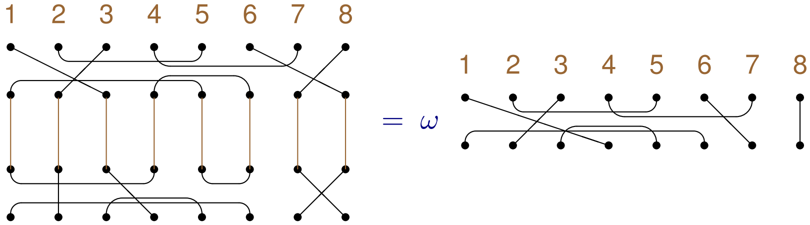
Brauer algebra $\mathcal{B}_m(\omega)$

Multiplication of m-diagrams ($m = 8$):

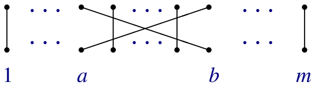


Brauer algebra $\mathcal{B}_m(\omega)$

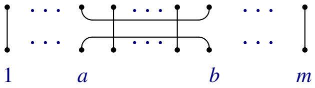
Multiplication of m-diagrams ($m = 8$):



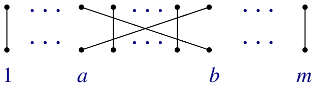
For $1 \leqslant a < b \leqslant m$ denote by s_{ab} and ϵ_{ab} the diagrams



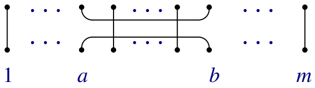
and



For $1 \leqslant a < b \leqslant m$ denote by s_{ab} and ϵ_{ab} the diagrams



and



The symmetrizer in the Brauer algebra $\mathcal{B}_m(\omega)$

is the idempotent $s^{(m)}$ such that

$$s_{ab}\, s^{(m)} = s^{(m)}\, s_{ab} = s^{(m)} \qquad \text{and} \qquad \epsilon_{ab}\, s^{(m)} = s^{(m)}\, \epsilon_{ab} = 0.$$

Action in tensors

Action in tensors

In the case $\mathfrak{g} = \mathfrak{o}_N$ set $\omega = N$. The generators of $\mathcal{B}_m(N)$ act in the tensor space

$$\underbrace{\mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m}$$

by the rule

$$s_{ab} \mapsto P_{ab}, \qquad \epsilon_{ab} \mapsto Q_{ab}, \qquad 1 \leqslant a < b \leqslant m,$$

Action in tensors

In the case $\mathfrak{g} = \mathfrak{o}_N$ set $\omega = N$. The generators of $\mathcal{B}_m(N)$ act in the tensor space

$$\underbrace{\mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m}$$

by the rule

$$s_{ab} \mapsto P_{ab}, \qquad \epsilon_{ab} \mapsto Q_{ab}, \qquad 1 \leqslant a < b \leqslant m,$$

where $i' = N - i + 1$ and

$$Q_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)}.$$

In the case $\mathfrak{g} = \mathfrak{sp}_N$ with $N = 2n$ set $\omega = -N$. The generators of $\mathcal{B}_m(-N)$ act in the tensor space $(\mathbb{C}^N)^{\otimes m}$ by

$$s_{ab} \mapsto -P_{ab}, \qquad \epsilon_{ab} \mapsto -Q_{ab}, \qquad 1 \leqslant a < b \leqslant m,$$

In the case $\mathfrak{g} = \mathfrak{sp}_N$ with $N = 2n$ set $\omega = -N$. The generators of $\mathcal{B}_m(-N)$ act in the tensor space $(\mathbb{C}^N)^{\otimes m}$ by

$$s_{ab} \mapsto -P_{ab}, \qquad \epsilon_{ab} \mapsto -Q_{ab}, \qquad 1 \leqslant a < b \leqslant m,$$

with $\varepsilon_i = -\varepsilon_{n+i} = 1$ for $i = 1, \dots, n$ and

$$Q_{ab} = \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j \, 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)}.$$

In the case $\mathfrak{g} = \mathfrak{sp}_N$ with $N = 2n$ set $\omega = -N$. The generators of $\mathcal{B}_m(-N)$ act in the tensor space $(\mathbb{C}^N)^{\otimes m}$ by

$$s_{ab} \mapsto -P_{ab}, \qquad \epsilon_{ab} \mapsto -Q_{ab}, \qquad 1 \leqslant a < b \leqslant m,$$

with $\varepsilon_i = -\varepsilon_{n+i} = 1$ for $i = 1, \dots, n$ and

$$Q_{ab} = \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j \, 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)}.$$

In both cases denote by $S^{(m)}$ the image of the symmetrizer $s^{(m)}$ under the action in tensors,

$$S^{(m)} \in \underbrace{\operatorname{End}\mathbb{C}^N \otimes \dots \otimes \operatorname{End}\mathbb{C}^N}_{m}.$$

Explicitly,

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \left(1 + \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{N/2 + b - a - 1} \right),$$

Explicitly,

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \left(1 + \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{N/2 + b - a - 1} \right),$$

and

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \left(1 - \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{n - b + a + 1} \right).$$

Explicitly,

$$S^{(m)} = \frac{1}{m!} \prod_{1\leqslant a<b\leqslant m} \Big(1 + \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{N/2+b-a-1}\Big),$$

and

$$S^{(m)} = \frac{1}{m!} \prod_{1\leqslant a<b\leqslant m} \Big(1 - \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{n-b+a+1}\Big).$$

Set

$$\gamma_m(\omega) = \frac{\omega+m-2}{\omega+2m-2}, \qquad \omega = \begin{cases} N & \text{for} \quad \mathfrak{g} = \mathfrak{o}_N \\ -2n & \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n}. \end{cases}$$

Types B, C and D

Types B, C and D

Let $\mathfrak{g} = \mathfrak{o}_N$, $\mathfrak{sp}_N$ with $N = 2n$ or $N = 2n + 1$.

Types B, C and D

Let $\mathfrak{g} = \mathfrak{o}_N$, $\mathfrak{sp}_N$ with $N = 2n$ or $N = 2n + 1$.

Set

$$F_{ij} = E_{ij} - E_{j'i'} \qquad \text{or} \qquad F_{ij} = E_{ij} - \varepsilon_i \, \varepsilon_j \, E_{j'i'}$$

Types B, C and D

Let $\mathfrak{g} = \mathfrak{o}_N, \ \mathfrak{sp}_N$ with $N = 2n$ or $N = 2n + 1$.

Set

$$F_{ij} = E_{ij} - E_{j'i'} \qquad \text{or} \qquad F_{ij} = E_{ij} - \varepsilon_i \, \varepsilon_j \, E_{j'i'}$$

and

$$F_{ij}[r] = F_{ij} \, t^r \in \widehat{\mathfrak{g}}.$$

Types B, C and D

Let $\mathfrak{g} = \mathfrak{o}_N, \quad \mathfrak{sp}_N$ with $N = 2n$ or $N = 2n + 1$.

Set

$$F_{ij} = E_{ij} - E_{j'i'} \qquad \text{or} \qquad F_{ij} = E_{ij} - \varepsilon_i \, \varepsilon_j \, E_{j'i'}$$

and

$$F_{ij}[r] = F_{ij} \, t^r \in \widehat{\mathfrak{g}}.$$

Combine into a matrix

$$F[r] = \sum_{i,j=1}^{N} e_{ij} \otimes F_{ij}[r] \in \operatorname{End} \mathbb{C}^N \otimes \mathrm{U}_{-h^\vee}\big(\widehat{\mathfrak{g}}\big).$$

Theorem. All coefficients of the polynomial in $\tau = -d/dt$

$$\gamma_m(\omega)\,\mathrm{tr}\,S^{(m)}\big(\tau + F[-1]_1\big)\ldots\big(\tau + F[-1]_m\big)$$

$$= \phi_{m0}\,\tau^m + \phi_{m1}\,\tau^{m-1} + \cdots + \phi_{mm}$$

Theorem. All coefficients of the polynomial in $\tau = -d/dt$

$$\gamma_m(\omega)\,\mathrm{tr}\, S^{(m)}\big(\tau + F[-1]_1\big)\dots\big(\tau + F[-1]_m\big)$$

$$= \phi_{m\,0}\,\tau^m + \phi_{m\,1}\,\tau^{m-1} + \dots + \phi_{m\,m}$$

belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$.

Theorem. All coefficients of the polynomial in $\tau = -d/dt$

$$\gamma_m(\omega)\,\mathrm{tr}\,S^{(m)}\big(\tau + F[-1]_1\big)\dots\big(\tau + F[-1]_m\big)$$
$$= \phi_{m\,0}\,\tau^m + \phi_{m\,1}\,\tau^{m-1} + \dots + \phi_{m\,m}$$

belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$.

Moreover, in the case $\mathfrak{g} = \mathfrak{o}_{2n}$, the Pfaffian

$$\mathrm{Pf}\,F[-1] = \frac{1}{2^n n!}\sum_{\sigma\in\mathfrak{S}_{2n}} \mathrm{sgn}\,\sigma\cdot F_{\sigma(1)\,\sigma(2)'}[-1]\dots F_{\sigma(2n-1)\,\sigma(2n)'}[-1]$$

belongs to $\mathfrak{z}(\widehat{\mathfrak{o}}_{2n})$ [M. 2013].

Corollary. The elements $\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n\,2n}$ form a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n+1}$ and $\mathfrak{sp}_{2n}$.

Corollary. The elements $\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n\,2n}$ form a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n+1}$ and $\mathfrak{sp}_{2n}$.

The elements $\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n-2\,2n-2}, \mathrm{Pf}\, F[-1]$ form a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n}$.

Examples. Complete sets of Segal–Sugawara vectors:

for $\mathfrak{o}_3:\qquad \operatorname{tr} F[-1]^2$

for $\mathfrak{o}_4:\qquad \operatorname{tr} F[-1]^2, \quad \operatorname{Pf} F[-1]$

for $\mathfrak{o}_5:\qquad \operatorname{tr} F[-1]^2, \quad \operatorname{tr} F[-1]^4 - \frac{1}{2} \operatorname{tr} F[-2]^2$

for $\mathfrak{o}_6:\qquad \operatorname{tr} F[-1]^2, \quad \operatorname{tr} F[-1]^4, \quad \operatorname{Pf} F[-1].$

Examples. Complete sets of Segal–Sugawara vectors:

$$\text{for} \quad \mathfrak{o}_3: \qquad \operatorname{tr} F[-1]^2$$

$$\text{for} \quad \mathfrak{o}_4: \qquad \operatorname{tr} F[-1]^2, \quad \operatorname{Pf} F[-1]$$

$$\text{for} \quad \mathfrak{o}_5: \qquad \operatorname{tr} F[-1]^2, \quad \operatorname{tr} F[-1]^4 - \frac{1}{2} \operatorname{tr} F[-2]^2$$

$$\text{for} \quad \mathfrak{o}_6: \qquad \operatorname{tr} F[-1]^2, \quad \operatorname{tr} F[-1]^4, \quad \operatorname{Pf} F[-1].$$

$$\text{for} \quad \mathfrak{sp}_2: \qquad \operatorname{tr} F[-1]^2$$

$$\text{for} \quad \mathfrak{sp}_4: \qquad \operatorname{tr} F[-1]^2, \quad \operatorname{tr} F[-1]^4 - 5 \operatorname{tr} F[-2]^2.$$