Quantum immanants, Bethe subalgebras and Sugawara operators

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- The dual series $\mathbb{T}_\mu^+(u)$ are invariants of the quantum vacuum module [Jing, Kožić, M. and Yang 2018].

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- Coefficients of the power series $\mathbb{T}_\mu(u)$ generate a Bethe subalgebra of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ [Nazarov 1998].
- The dual series $\mathbb{T}^+_\mu(u)$ are invariants of the quantum vacuum module [Jing, Kožić, M. and Yang 2018].
- Taking quasi-classical limits we get Sugawara operators – Casimir elements for $\widehat{\mathfrak{gl}}_N$ at the critical level.

Young diagrams and tableaux

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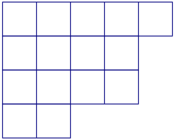
The figure illustrates the diagram of the partition $(5, 4, 4, 2)$ of 15, its length is 4:



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The following is a standard tableau of shape $(4, 4, 1)$:

1	3	4	5
2	6	7	9
8			

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The vector space V_μ admits an orthonormal Young basis parameterized by the set of standard μ-tableaux $\mathcal{U}$.

The group algebra $\mathbb{C}[\mathfrak{S}_m]$ is isomorphic to the direct sum

$$\mathbb{C}[\mathfrak{S}_m] \cong \bigoplus_{\mu \vdash m} \mathrm{Mat}_{f_\mu}(\mathbb{C}),$$

$f_\mu = \dim V_\mu$ is the number of standard tableaux of shape μ.

The diagonal matrix units $e_{\mathcal{U}} = e_{\mathcal{U}\mathcal{U}} \in \mathrm{Mat}_{f_\mu}(\mathbb{C})$ are primitive idempotents of $\mathbb{C}[\mathfrak{S}_m]$. We have $\mathbb{C}[\mathfrak{S}_m]\, e_{\mathcal{U}} \cong V_\mu$ so that explicit formulas for $e_{\mathcal{U}} \in \mathbb{C}[\mathfrak{S}_m]$ provide realizations of V_μ.

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The Jucys–Murphy elements $x_1, \dots, x_m \in \mathbb{C}[\mathfrak{S}_m]$ are defined by

$$x_a = (1\,a) + \dots + (a-1\ a) \quad \text{for} \quad a = 2, \dots, m$$

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and $x_1 = 0$. We have

$$x_a\, e_{\mathcal{U}} = e_{\mathcal{U}}\, x_a = c_a(\mathcal{U})\, e_{\mathcal{U}}, \qquad a = 1, \dots, m,$$

$c_a(\mathcal{U}) = j - i$ is the content of the box $(i, j) \in \mu$ occupied by a.

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Jucys–Murphy formula [Jucys 1971, Murphy 1981]:

$$e_{\mathcal{U}} = e_{\mathcal{V}} \, \frac{(x_m - a_1) \dots (x_m - a_l)}{(c - a_1) \dots (c - a_l)}$$

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where $a_1, \dots, a_l$ are the contents of all addable boxes of ν except for α, while c is the content of the latter.

Example. Take $\mu = (2^2)$ and let $\mathcal{U}$ be

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Hence

$$e_{\mathcal{U}} = e_{\mathcal{V}} \frac{(x_4 - 2)(x_4 + 2)}{(-2)\,2}, \qquad x_4 = (1\,4) + (2\,4) + (3\,4).$$

Fusion procedure

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Take m variables $u_1, \ldots, u_m$ and consider the rational function

$$\phi(u_1, \ldots, u_m) = \prod_{1 \leqslant a < b \leqslant m} \Big(1 - \frac{(a\,b)}{u_a - u_b}\Big),$$

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Suppose that $\mu \vdash m$ and let $\mathcal{U}$ be a standard μ-tableau. Set $c_a = c_a(\mathcal{U})$ for $a = 1, \dots, m$. We have [Jucys 1966]:

$$\phi(u_1, \dots, u_m)\big|_{u_1 = c_1}\big|_{u_2 = c_2} \dots \big|_{u_m = c_m} = \frac{m!}{f_\mu}\, e_{\mathcal{U}}.$$

Schur–Weyl duality

The symmetric group $\mathfrak{S}_m$ acts by permuting the tensor factors in the tensor product space

$$(\mathbb{C}^N)^{\otimes m} = \underbrace{\mathbb{C}^N \otimes \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m}.$$

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If $\ell(\mu) \leqslant N$ then $\mathcal{E}_{\mathcal{U}}(\mathbb{C}^N)^{\otimes m} \cong L(\mu)$ is an irreducible $\mathfrak{gl}_N$-module

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$$(\mathbb{C}^N)^{\otimes m} \cong \bigoplus_{\mu \vdash m,\ \ell(\mu) \leqslant N} V_\mu \otimes L(\mu).$$

Introduce the matrix

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij} \in \operatorname{End} \mathbb{C}^N \otimes \mathrm{U}(\mathfrak{gl}_N).$$

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defined by

$$E_a = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes E_{ij}.$$

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The element $\mathbb{S}_\mu$ does not depend on $\mathcal{U}$.

Theorem [Okounkov 1996, Okounkov and Olshanski 1997].

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The eigenvalue of $\mathbb{S}_\mu$ in the highest weight module $L(\lambda)$ with $\lambda = (\lambda_1, \dots, \lambda_N)$ (the Harish-Chandra image) is the factorial Schur polynomial,

$$s^*_\mu(\lambda) = \sum_{\mathrm{sh}(\mathcal{T})=\mu} \prod_{\alpha\in\mu} \big(\lambda_{\mathcal{T}(\alpha)} + c(\alpha)\big),$$

summed over semistandard tableaux $\mathcal{T}$ of shape μ with entries in $\{1, \dots, N\}$.

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are obtained as coefficients of the Capelli determinant

$$C(u) = \mathrm{cdet} \begin{bmatrix} u + E_{11} & E_{12} & \dots & E_{1N} \\ \vdots & \vdots & & \vdots \\ E_{N1} & E_{N2} & \dots & u + E_{NN} - N + 1 \end{bmatrix}.$$

Bethe subalgebras in Yangian

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The Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is a unital associative algebra with generators $t_{ij}^{(r)}$, where $1 \leqslant i,j \leqslant N$ and $r = 1,2,\ldots$ and the defining relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)},$$

where $r,s = 0,1,\ldots$ and $t_{ij}^{(0)} = \delta_{ij}$.

In terms of the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)}\, u^{-r} \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$$

the defining relations are written in the form

$$(u - v)\,[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u).$$

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Set

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$$

and use the notation $T_a(u)$ with $a = 1, \dots, m$ for formal series in u^{-1} with coefficients in the tensor product algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \dots \otimes \mathrm{End}\,\mathbb{C}^N}_{m} \otimes \mathrm{Y}(\mathfrak{gl}_N).$$

The defining relations for the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ can be written in the matrix form as

$$R(u-v)\,T_1(u)\,T_2(v) = T_2(v)\,T_1(u)\,R(u-v),$$

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where

$$R(u) = 1 - P\,u^{-1}$$

is the Yang R-matrix,

$$P : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N$$

is the permutation operator.

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$$\mathbb{T}_\mu(u) = \mathrm{tr}_{1,\dots,m}\, \mathcal{E}_{\mathcal{U}}\, T_1(u + c_1) \dots T_m(u + c_m).$$

It does not depend on $\mathcal{U}$.

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Using the evaluation homomorphism

$$\mathrm{ev} : \mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N), \qquad T(u) \mapsto 1 + E u^{-1},$$

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we get

$$\mathbb{S}_\mu = (u + c_1) \dots (u + c_m)\, \mathrm{ev}\big(\mathbb{T}_\mu(u)\big)\Big|_{u=0}.$$

Consider the left ideal I of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ generated by all coefficients of $t_{ij}(u)$ with $1 \leqslant i < j \leqslant N$.

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Yangian version of the Harish-Chandra homomorphism:

$$\mathrm{Y}(\mathfrak{gl}_N)_0 \to \mathbb{C}[\lambda_i^{(r)} \,|\, i = 1, \dots, N,\ r \geqslant 1], \qquad t_{ii}^{(r)} \mapsto \lambda_i^{(r)}.$$

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Combine the elements $\lambda_i^{(r)}$ into the formal series

$$\lambda_i(u) = 1 + \sum_{r=1}^{\infty} \lambda_i^{(r)} u^{-r}, \qquad i = 1, \dots, N,$$

so that $t_{ii}(u) \mapsto \lambda_i(u)$.

Theorem [cf. Okounkov 1996, Nazarov 1998].

The coefficients of all series $\mathbb{T}_\mu(u)$ pairwise commute.

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The Harish-Chandra image of $\mathbb{T}_\mu(u)$ coincides with the Yangian character of the evaluation module $L(\mu)$:

$$\mathbb{T}_\mu(u) \mapsto \sum_{\mathrm{sh}(\mathcal{T})=\mu} \prod_{\alpha\in\mu} \lambda_{\mathcal{T}(\alpha)}\big(u + c(\alpha)\big),$$

summed over semistandard tableau $\mathcal{T}$ of shape μ with entries in $\{1,\dots,N\}$.

Introduce the rational function in variables $u_1, \dots, u_m$ by

$$R(u_1, \dots, u_m) = \prod_{1 \leqslant a < b \leqslant m} \Big(1 - \frac{P_{ab}}{u_a - u_b} \Big).$$

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By the fusion procedure,

$$R(u_1, \dots, u_m)\big|_{u_1=c_1}\big|_{u_2=c_2} \dots \big|_{u_m=c_m} = \frac{m!}{f_\mu}\, \mathcal{E}_{\mathcal{U}}.$$

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A key point in the proof is the identity

$$R(u_1, \dots, u_m)\, T_1(u_1) \dots T_m(u_m) = T_m(u_m) \dots T_1(u_1) R(u_1, \dots, u_m),$$

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$$R(u_1, \dots, u_m) = \prod_{1 \leqslant a < b \leqslant m} \Big(1 - \frac{P_{ab}}{u_a - u_b}\Big).$$

By the fusion procedure,

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and its consequence implied by the fusion procedure:

$$\mathcal{E}_{\mathcal{U}}\,T_1(u + c_1) \dots T_m(u + c_m) = T_m(u + c_m) \dots T_1(u + c_1)\,\mathcal{E}_{\mathcal{U}}.$$

Quantum vacuum modules

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The double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ is generated by the central element C and elements $t_{ij}^{(r)}$ and $t_{ij}^{(-r)}$, where $1 \leqslant i, j \leqslant N$ and $r = 1, 2, \ldots$.

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where the coefficients of powers of u, v belong to

$$\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{DY}(\mathfrak{gl}_N)$$

and

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \qquad \text{and} \qquad T^+(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}^+(u).$$

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$$g(u) = 1 + \sum_{i=1}^{\infty} g_i\, u^{-i}, \qquad g_i \in \mathbb{C},$$

is uniquely determined by the relation

$$g(u+N) = g(u)\,(1 - u^{-2}).$$

The (quantum) vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$ at the level $c \in \mathbb{C}$

over the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ is defined as the quotient

$$\mathcal{V}_c(\mathfrak{gl}_N) = \mathrm{DY}(\mathfrak{gl}_N)/\mathrm{DY}(\mathfrak{gl}_N)\langle C - c,\ t_{ij}^{(r)} \,|\, r \geqslant 1\rangle.$$

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On the vacuum vector $1 \in \mathcal{V}_c(\mathfrak{gl}_N)$ we have

$$C\,1 = c\,1 \qquad \text{and} \qquad t_{ij}^{(r)}\,1 = 0 \quad \text{for} \quad r \geqslant 1.$$

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As a vector space, the vacuum module is isomorphic to the dual Yangian $\mathrm{Y}^+(\mathfrak{gl}_N)$, which is the subalgebra of $\mathrm{DY}(\mathfrak{gl}_N)$ generated by the elements $t_{ij}^{(-r)}$.

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Introduce the subspace of invariants by

$$\mathfrak{z}(\widehat{\mathcal{V}}) = \{v \in \widehat{\mathcal{V}} \mid t_{ij}(u)\,v = \delta_{ij}\,v\},$$

so that any element of $\mathfrak{z}(\widehat{\mathcal{V}})$ is annihilated by all $t_{ij}^{(r)}$ with $r \geqslant 1$.

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Proposition. $\mathfrak{z}(\widehat{\mathcal{V}})$ is a subalgebra of the completed dual Yangian $\mathrm{Y}^+(\mathfrak{gl}_N)$.

Construction of invariants

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For a standard tableau $\mathcal{U}$ of shape $\mu \vdash m$ with $\ell(\mu) \leqslant N$,

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This is a power series in u independent of $\mathcal{U}$, whose coefficients

are elements of the completed vacuum module $\widehat{\mathcal{V}}$.

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Introduce the series

$$\Phi_m(u) = \sum_{k=0}^{m} (-1)^k \binom{N-k}{m-k} \mathrm{tr}_{1,\dots,k} A^{(k)} \, T_1^+(u) \dots T_k^+(u-k+1),$$

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Segal–Sugawara vectors from the invariants

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Here $X[r] = X\,t^r$ for $X \in \mathfrak{gl}_N$ and any $r \in \mathbb{Z}$.

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Proposition. The assignments

$$E_{ij}[r-1] \mapsto \bar{t}_{ij}^{(r)}, \qquad E_{ij}[-r] \mapsto \bar{t}_{ij}^{(-r)} \qquad \text{and} \qquad K \mapsto \overline{C}$$

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with $r \geqslant 1$ define an algebra isomorphism

$$\mathrm{U}(\widehat{\mathfrak{gl}}_N) \to \operatorname{gr} \mathrm{DY}(\mathfrak{gl}_N).$$

By the proposition, $\operatorname{gr} \mathrm{Y}^{+}(\mathfrak{gl}_N) \cong \mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)$ so that $\widehat{\mathcal{V}}$ is a quantization of the vacuum module at the critical level over $\widehat{\mathfrak{gl}}_N$:

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Any element of $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$ is called a Segal–Sugawara vector.

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$$\mathrm{tr}_{1,\dots,m} A^{(m)} \big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_m\big)$$
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Corollary.

All elements $\phi_{m\,a}$ are Segal–Sugawara vectors.

Example. $m = N$.

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Consider the $N \times N$ matrix $\tau + E[-1]$ given by

$$\tau + E[-1] = \begin{bmatrix} \tau + E_{11}[-1] & E_{12}[-1] & \dots & E_{1N}[-1] \\ E_{21}[-1] & \tau + E_{22}[-1] & \dots & E_{2N}[-1] \\ \vdots & \vdots & \ddots & \vdots \\ E_{N1}[-1] & E_{N2}[-1] & \dots & \tau + E_{NN}[-1] \end{bmatrix}.$$

The coefficients $\phi_1, \dots, \phi_N$ of the polynomial

$$\mathrm{cdet}\big(\tau + E[-1]\big) = \tau^N + \phi_1\,\tau^{N-1} + \dots + \phi_{N-1}\,\tau + \phi_N$$

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That is, the elements $(\partial_t)^r\phi_a$ with $r \geqslant 0$ and $a = 1, \dots, N$ are algebraically independent generators of the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$. [Chervov–Talalaev 2006, Chervov–M. 2009].