Lecture 4

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$$t_{ij}(u)\,\zeta = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N,$$

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for some formal series

$$\lambda_i(u) = 1 + \lambda_i^{(1)} u^{-1} + \lambda_i^{(2)} u^{-2} + \dots, \qquad \lambda_i^{(r)} \in \mathbb{C}.$$

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In terms of the Drinfeld presentation, the conditions read

$$e_i(u)\,\zeta = 0 \qquad \text{for} \quad 1 \leqslant i \leqslant N-1, \qquad \text{and}$$

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$$h_i(u) = \begin{vmatrix} t_{1\,1}(u) & \dots & t_{1\,i-1}(u) & t_{1\,i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1\,1}(u) & \dots & t_{i-1\,i-1}(u) & t_{i-1\,i}(u) \\ t_{i\,1}(u) & \dots & t_{i\,i-1}(u) & \boxed{t_{i\,i}(u)} \end{vmatrix}.$$

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The Verma module $M(\lambda(u))$ is the quotient of $\mathrm{Y}(\mathfrak{gl}_N)$ by the left ideal generated by all the coefficients of the series $t_{ij}(u)$ for $1 \leqslant i < j \leqslant N$ and $t_{ii}(u) - \lambda_i(u)$ for $1 \leqslant i \leqslant N$.

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The Verma module $M(\lambda(u))$ is a universal highest weight representation of $\mathrm{Y}(\mathfrak{gl}_N)$ with the highest weight $\lambda(u)$ and the highest vector $1_{\lambda(u)}$ which is the image of the element $1 \in \mathrm{Y}(\mathfrak{gl}_N)$ in the quotient.

The PBW theorem implies that given any order on the set of generators $t_{ji}^{(r)}$ with $1 \leqslant i < j \leqslant N$ and $r \geqslant 1$, the elements

$$t_{j_1 i_1}^{(r_1)} \dots t_{j_m i_m}^{(r_m)} 1_{\lambda(u)}, \qquad m \geqslant 0,$$

with ordered products, form a basis of $M(\lambda(u))$.

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Proposition. Suppose that L is a highest weight representation of $\mathrm{Y}(\mathfrak{gl}_N)$ with the highest weight $\lambda(u) = \big(\lambda_1(u), \dots, \lambda_N(u)\big)$.

Then each coefficient of the quantum determinant $\operatorname{qdet} T(u)$ acts on L as multiplication by a scalar determined by

$$\operatorname{qdet} T(u)|_L = \lambda_1(u) \dots \lambda_N(u - N + 1).$$

Proof. This is clear from the formula

$$\operatorname{qdet} T(u) = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{p(1)\,1}(u) \dots t_{p(N)\,N}(u - N + 1).$$

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Identify the elements $E_{ij} \in \mathfrak{gl}_N$ with their images $t_{ij}^{(1)}$ in $\mathrm{Y}(\mathfrak{gl}_N)$ under the embedding $\mathrm{U}(\mathfrak{gl}_N) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_N)$

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$$[E_{ij}, t_{kl}(u)] = \delta_{kj}\, t_{il}(u) - \delta_{il}\, t_{kj}(u).$$

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$$M(\lambda(u))_\mu = \{\eta \in M(\lambda(u)) \mid E_{ii}\,\eta = \mu_i\,\eta, \quad i = 1, \dots, N\}.$$

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We will identify μ with the element $\mu_1\varepsilon_1 + \cdots + \mu_N\varepsilon_N \in \mathfrak{h}^*$, with $\varepsilon_i = E_{ii}^*$ for the Cartan subalgebra $\mathfrak{h} = \langle E_{11}, \dots, E_{NN}\rangle$.

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If α and β are two weights, then α precedes β if $\beta - \alpha$ is a $\mathbb{Z}_+$-linear combination of the N-tuples $\varepsilon_i - \varepsilon_j$ with $i < j$.

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Theorem. Every finite-dimensional irreducible representation L of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is isomorphic to a unique irreducible highest weight representation $L(\lambda(u))$.

Proof. Introduce the following subspace of L,

$$L^0 = \{\xi \in L \mid t_{ij}(u)\,\xi = 0, \qquad 1 \leqslant i < j \leqslant N\}.$$

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This set is finite and hence contains a maximal weight μ with respect to the partial ordering.

The corresponding weight vector ξ belongs to L^0, because the weight of $t_{ij}(u)\,\xi$ is $\mu + \varepsilon_i - \varepsilon_j$. By the maximality of μ, we must have $t_{ij}(u)\,\xi = 0$ for $i < j$.

Next, the subspace L^0 is invariant with respect to the action of all elements $t_{kk}^{(r)}$.

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Moreover, the elements $t_{kk}^{(r)}$ with $k = 1, \ldots, N$ and $r \geqslant 1$ act on L^0 as pairwise commuting operators.

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Moreover, the elements $t_{kk}^{(r)}$ with $k = 1, \ldots, N$ and $r \geqslant 1$ act on L^0 as pairwise commuting operators.

Hence, any simultaneous eigenvector $\zeta \in L^0$ for these operators is the highest vector. $\square$

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So, $L(\lambda)$ is generated by a nonzero vector ζ such that

$$E_{ij}\,\zeta = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

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The representation $L(\lambda)$ is finite-dimensional if and only if $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$ for all $i = 1, \dots, N-1$.

The evaluation homomorphism

$$\mathrm{ev} : t_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1}.$$

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Note that $L(\lambda)$ is a highest weight representation of the Yangian with the highest vector ζ, and the components of the highest weight are given by

$$\lambda_i(u) = 1 + \lambda_i u^{-1}, \qquad i = 1, \dots, N.$$

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Proposition. The cyclic span $\mathrm{Y}(\mathfrak{gl}_N)\,\zeta$ is a highest weight representation with the highest vector ζ and the highest weight $\big(\lambda_1(u), \ldots, \lambda_N(u)\big)$,

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$$\lambda_i(u) = (1 + \lambda_i^{(1)} u^{-1})\,(1 + \lambda_i^{(2)} u^{-1}) \ldots (1 + \lambda_i^{(k)} u^{-1}).$$

Proof. We have

$$t_{ij}(u)\big(\eta_1 \otimes \ldots \otimes \eta_k\big) = \sum_{a_1,\ldots,a_{k-1}} t_{ia_1}(u)\,\eta_1 \otimes t_{a_1a_2}(u)\,\eta_2 \otimes \ldots \otimes t_{a_{k-1}j}(u)\,\eta_k\,,$$

summed over $a_1, \ldots, a_{k-1} \in \{1, \ldots, N\}$.

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If $i < j$ and for every $m = 1, \ldots, k$ we have $\eta_m = \zeta_m$, then each summand is zero because it contains a factor of the form $t_{kl}(u)\,\zeta_m$ with $k < l$, which is zero.

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Similarly, if $i = j$, then the only nonzero summand corresponds to the case where each index a_m equals i. □

Representations of $\mathrm{Y}(\mathfrak{gl}_2)$

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Consider the irreducible highest weight representation $L(\lambda(u))$ of $\mathrm{Y}(\mathfrak{gl}_2)$ with an arbitrary highest weight $\lambda(u) = \big(\lambda_1(u), \lambda_2(u)\big)$.

Proposition. If $\dim L(\lambda(u)) < \infty$, then there exists a formal series

$$f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \dots, \qquad f_r \in \mathbb{C},$$

such that $f(u)\lambda_1(u)$ and $f(u)\lambda_2(u)$ are polynomials in u^{-1}.

Proof. By twisting the action of $\mathrm{Y}(\mathfrak{gl}_2)$ on $L(\lambda(u))$ by the automorphism $T(u) \mapsto f(u)\, T(u)$ with $f(u) = \lambda_2(u)^{-1}$, we obtain a module over $\mathrm{Y}(\mathfrak{gl}_2)$ which is isomorphic to the irreducible highest weight representation $L(\nu(u), 1)$ with $\nu(u) = \lambda_1(u)/\lambda_2(u)$.

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Let ζ denote the highest vector of $L(\nu(u), 1)$. Since this representation is finite-dimensional, there exist coefficients $c_i \in \mathbb{C}$ with $c_m \neq 0$ such that

$$\xi := \sum_{i=1}^{m} c_i\, t_{21}^{(i)}\, 1_{\lambda(u)} = 0.$$

Then we have $t_{12}^{(r)}\xi = 0$ for all $r \geqslant 1$.

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By the defining relations,

$$t_{12}^{(r)}\, t_{21}^{(i)}\, 1_{\lambda(u)} = \sum_{a=1}^{\min\{r,i\}} \Big(t_{22}^{(a-1)} t_{11}^{(r+i-a)} - t_{22}^{(r+i-a)} t_{11}^{(a-1)} \Big) 1_{\lambda(u)}$$

$$= \nu^{(r+i-1)}\, 1_{\lambda(u)}.$$

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Hence, for all $r \geqslant 1$ we have the relations

$$\sum_{i=1}^{m} c_i\, \nu^{(r+i-1)} = 0.$$

They imply that for some coefficients $b_i \in \mathbb{C}$ we have

$$\nu(u)\,(c_1 + c_2\, u + \dots + c_m\, u^{m-1}) = b_1 + b_2\, u + \dots + b_m\, u^{m-1}.$$

Thus, $\nu(u)$ is a rational function in u^{-1}, so that taking $f(u)$ to be its denominator, we find that both $f(u)\,\nu(u)$ and $f(u)\,1$ are polynomials in u^{-1}. $\square$

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By the proposition, it suffices to understand the representations with the highest weights, whose components $\lambda_1(u)$ and $\lambda_2(u)$ are polynomials in u^{-1}.

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Write the decompositions

$$\lambda_1(u) = (1 + \alpha_1 u^{-1}) \dots (1 + \alpha_k u^{-1}),$$

$$\lambda_2(u) = (1 + \beta_1 u^{-1}) \dots (1 + \beta_k u^{-1}),$$

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where the constants α_i and β_i are complex numbers.

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$$E_{11}\,\zeta = \alpha\,\zeta, \qquad E_{22}\,\zeta = \beta\,\zeta, \qquad E_{12}\,\zeta = 0.$$

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$$E_{11}\,\zeta = \alpha\,\zeta, \qquad E_{22}\,\zeta = \beta\,\zeta, \qquad E_{12}\,\zeta = 0.$$

If $\alpha - \beta \in \mathbb{Z}_+$, then the vectors $(E_{21})^r\zeta$ with $r = 0, 1, \dots, \alpha - \beta$ form a basis of $L(\alpha, \beta)$ so that $\dim L(\alpha, \beta) = \alpha - \beta + 1$.

For any $\alpha, \beta \in \mathbb{C}$ consider the irreducible highest weight representation $L(\alpha, \beta)$ of the Lie algebra $\mathfrak{gl}_2$ and equip it with a $\mathrm{Y}(\mathfrak{gl}_2)$-module structure.

Let ζ denote the highest vector of $L(\alpha, \beta)$. Then

$$E_{11}\,\zeta = \alpha\,\zeta, \qquad E_{22}\,\zeta = \beta\,\zeta, \qquad E_{12}\,\zeta = 0.$$

If $\alpha - \beta \in \mathbb{Z}_+$, then the vectors $(E_{21})^r\zeta$ with $r = 0, 1, \dots, \alpha - \beta$ form a basis of $L(\alpha, \beta)$ so that $\dim L(\alpha, \beta) = \alpha - \beta + 1$.

If $\alpha - \beta \notin \mathbb{Z}_+$, then a basis of $L(\alpha, \beta)$ is formed by the vectors $(E_{21})^r\zeta$, where r runs over all nonnegative integers.

Given the expansions

$$\lambda_1(u) = (1 + \alpha_1 u^{-1}) \dots (1 + \alpha_k u^{-1}),$$

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renumber the coefficients, if necessary to satisfy the following condition for every $i = 1, \dots, k-1$:

if the multiset

$$\{\alpha_p - \beta_q \mid i \leqslant p, q \leqslant k\}$$

contains nonnegative integers, then $\alpha_i - \beta_i$ is minimal amongst them.

Proposition. If the condition holds, then the representation $L(\lambda_1(u), \lambda_2(u))$ of $\mathrm{Y}(\mathfrak{gl}_2)$ is isomorphic to the tensor product module

$$L := L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_k, \beta_k).$$

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Proof. Let ζ_i be the highest vector of $L(\alpha_i, \beta_i)$ for $i = 1, \ldots, k$.

By the proposition above, the cyclic span $\mathrm{Y}(\mathfrak{gl}_2)\,\zeta$ of the vector $\zeta = \zeta_1 \otimes \ldots \otimes \zeta_k$ is a highest weight module with the highest weight $(\lambda_1(u), \lambda_2(u))$.

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By the proposition above, the cyclic span $\mathrm{Y}(\mathfrak{gl}_2)\,\zeta$ of the vector $\zeta = \zeta_1 \otimes \ldots \otimes \zeta_k$ is a highest weight module with the highest weight $(\lambda_1(u), \lambda_2(u))$. Therefore, the proposition will follow if we prove that the tensor product module L is irreducible.

Step 1. Show that any vector $\xi \in L$ satisfying $t_{12}(u)\xi = 0$ is proportional to ζ.

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$$\xi = \sum_{r=0}^{p} (E_{21})^r \zeta_1 \otimes \xi_r, \qquad \text{where} \quad \xi_r \in L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_k, \beta_k)$$

and $p \leqslant \alpha_1 - \beta_1$ if this difference is in $\mathbb{Z}_+$.

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We have

$$\sum_{r=0}^{p} \Big(t_{11}(u)(E_{21})^r \zeta_1 \otimes t_{12}(u)\xi_r + t_{12}(u)(E_{21})^r \zeta_1 \otimes t_{22}(u)\xi_r \Big) = 0.$$

Taking the coefficient of $(E_{21})^p \zeta_1$ gives

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Therefore,

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To complete Step 1, we show that p is zero.

Suppose that $p \geqslant 1$. Then taking the coefficient of $(E_{21})^{p-1}\zeta_1$ in $t_{12}(u)\xi = 0$ we derive

$$(1+(\alpha_1-p+1)u^{-1})\, t_{12}(u)\, \xi_{p-1} + u^{-1}\, p\, (\alpha_1-\beta_1-p+1)\, t_{22}(u)\, \xi_p = 0.$$

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Multiply by u^k and set $u = -\alpha_1 + p - 1$ we obtain the relation

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$$p\,(\alpha_1-\beta_1-p+1)(\alpha_1-\beta_2-p+1)\ldots(\alpha_1-\beta_k-p+1) = 0.$$

But this is impossible due to the conditions on the parameters α_i and β_i. Thus, $p = 0$.

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The above argument shows that M contains the vector ζ. It remains to prove that the cyclic span $K = \mathrm{Y}(\mathfrak{gl}_2)\zeta$ coincides with L.

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Use the dual $\mathrm{Y}(\mathfrak{gl}_2)$-module L^* which is defined by

$$(y\,\omega)(\eta) = \omega(\varkappa(y)\,\eta) \quad \text{for} \quad y \in \mathrm{Y}(\mathfrak{gl}_2) \quad \text{and} \quad \omega \in L^*,\ \eta \in L,$$

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for the anti-automorphism

$$\varkappa:\ t_{ij}(u) \mapsto t_{3-i,3-j}(-u).$$

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However, this contradicts the claim verified in Step 1. □

Theorem. The irreducible highest weight representation $L(\lambda_1(u), \lambda_2(u))$ of $\mathrm{Y}(\mathfrak{gl}_2)$ is finite-dimensional if and only if there exists a monic polynomial $P(u)$ in u such that

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Proof. By the propositions, if $\dim L(\lambda_1(u), \lambda_2(u)) < \infty$, then

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and consider the tensor product module

$$L = L(\gamma_1+1,\gamma_1)\otimes L(\gamma_2+1,\gamma_2)\otimes\dots\otimes L(\gamma_p+1,\gamma_p). \qquad \square$$

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Every such representation is isomorphic to the restriction of a $\mathrm{Y}(\mathfrak{gl}_2)$-module of the form

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where each difference $\alpha_i - \beta_i$ is a positive integer.

Proof. Use the decomposition

$$\mathrm{Y}(\mathfrak{gl}_2) = \mathrm{ZY}(\mathfrak{gl}_2) \otimes \mathrm{Y}(\mathfrak{sl}_2).$$

$\square$

Representations of $\mathrm{Y}(\mathfrak{gl}_N)$

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Every tuple $(P_1(u), \dots, P_{N-1}(u))$ arises in this way.

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Proof. For $i = 1, \dots, N-1$ let Y_i be the subalgebra of $\mathrm{Y}(\mathfrak{gl}_N)$ generated by the coefficients of the series $t_{kl}(u)$ with $k, l \in \{i, i+1\}$.

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The cyclic span $\mathrm{Y}_i\,\zeta$ of the highest vector ζ of $L(\lambda(u))$ is a highest weight representation of $\mathrm{Y}(\mathfrak{gl}_2)$ with the highest weight $(\lambda_i(u), \lambda_{i+1}(u))$.

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The cyclic span $\mathrm{Y}_i\,\zeta$ of the highest vector ζ of $L(\lambda(u))$ is a highest weight representation of $\mathrm{Y}(\mathfrak{gl}_2)$ with the highest weight $(\lambda_i(u), \lambda_{i+1}(u))$. Apply the previous theorem for $\mathrm{Y}(\mathfrak{gl}_2)$.

For the converse claim, note that if $L(\nu(u))$ and $L(\mu(u))$ are the irreducible highest weight modules with the highest weights

$$\nu(u) = \big(\nu_1(u), \dots, \nu_N(u)\big) \quad \text{and} \quad \mu(u) = \big(\mu_1(u), \dots, \mu_N(u)\big),$$

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then the cyclic span $\mathrm{Y}(\mathfrak{gl}_N)(\zeta \otimes \zeta')$ is a highest weight module with the highest weight $\big(\nu_1(u)\,\mu_1(u), \dots, \nu_N(u)\,\mu_N(u)\big)$.

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The cyclic span corresponds to the set of Drinfeld polynomials $\big(P_1(u)\,Q_1(u), \dots, P_{N-1}(u)\,Q_{N-1}(u)\big)$, where the $P_i(u)$ and $Q_i(u)$ are the Drinfeld polynomials for $L(\nu(u))$ and $L(\mu(u))$, respectively.

Therefore, we only need to establish the sufficiency of the conditions for the fundamental representations of $\mathrm{Y}(\mathfrak{gl}_N)$ associated with the tuples of Drinfeld polynomials

$$(1,\ldots,1,u+a,1,\ldots,1), \qquad a \in \mathbb{C}.$$

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Such a tuple is associated with the evaluation module $L(\lambda)$, where $\lambda = (a + 1, \dots, a + 1, a, \dots, a)$,

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Such a tuple is associated with the evaluation module $L(\lambda)$, where $\lambda=(a+1,\dots,a+1,a,\dots,a)$, since

$$\lambda_j(u)=\begin{cases}1+(a+1)\,u^{-1} & \text{for} \quad j=1,\dots,i,\\ 1+a\,u^{-1} & \text{for} \quad j=i+1,\dots,N. \end{cases} \qquad \square$$

Recall the Drinfeld presentation of $\mathrm{Y}(\mathfrak{sl}_N)$: the generators are κ_{ir} and $\xi_{ir}^{\pm}$ with $i = 1, \dots, N-1$ and $r \geqslant 0$, subject to the defining relations:

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$$[\kappa_{i\,r}, \kappa_{j\,s}] = 0,$$

$$[\xi_{i\,r}^{+}, \xi_{j\,s}^{-}] = \delta_{ij}\,\kappa_{i\,r+s},$$

$$[\kappa_{i\,0}, \xi_{j\,s}^{\pm}] = \pm\,(\alpha_i, \alpha_j)\,\xi_{j\,s}^{\pm},$$

$$[\kappa_{i\,r+1}, \xi_{j\,s}^{\pm}] - [\kappa_{i\,r}, \xi_{j\,s+1}^{\pm}] = \pm\,\frac{(\alpha_i, \alpha_j)}{2}\big(\kappa_{i\,r}\,\xi_{j\,s}^{\pm} + \xi_{j\,s}^{\pm}\,\kappa_{i\,r}\big),$$

$$[\xi_{i\,r+1}^{\pm}, \xi_{j\,s}^{\pm}] - [\xi_{i\,r}^{\pm}, \xi_{j\,s+1}^{\pm}] = \pm\,\frac{(\alpha_i, \alpha_j)}{2}\big(\xi_{i\,r}^{\pm}\,\xi_{j\,s}^{\pm} + \xi_{j\,s}^{\pm}\,\xi_{i\,r}^{\pm}\big),$$

$$\sum_{p \in \mathfrak{S}_m} [\xi_{i\,r_{p(1)}}^{\pm}, [\xi_{i\,r_{p(2)}}^{\pm}, \dots [\xi_{i\,r_{p(m)}}^{\pm}, \xi_{j\,s}^{\pm}] \dots]] = 0,$$

with $i \neq j$ and $m = 1 - c_{ij}$ in the last relation.

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$$\xi_{i\,r}^{+}\,\zeta = 0 \quad \text{for all} \quad i = 1, \dots, N-1 \quad \text{and} \quad r \geqslant 0.$$

Moreover, this vector satisfies

$$\Big(1 + \sum_{r=0}^{\infty} \kappa_{ir}\, u^{-r-1}\Big)\,\zeta = \frac{Q_i(u+1)}{Q_i(u)}\,\zeta \quad \text{for} \quad i = 1, \dots, N-1,$$

Corollary. Every finite-dimensional irreducible representation of the Yangian $\mathrm{Y}(\mathfrak{sl}_N)$ contains a unique, up to a constant factor, vector $\zeta \neq 0$ such that

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Now use the automorphism of $\mathrm{Y}(\mathfrak{sl}_N)$ defined by

$$\xi_i^+(u) \mapsto \xi_i^-(-u), \quad \xi_i^-(u) \mapsto \xi_i^+(-u), \quad \kappa_i(u) \mapsto \kappa_i(-u). \qquad \square$$

Representations of $\mathrm{Y}(\mathfrak{a})$

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Recall that the Yangian $\mathrm{Y}(\mathfrak{a})$ is generated by elements κ_{ir} and $\xi_{ir}^{\pm}$ with $i = 1, \dots, n$ and $r \geqslant 0$, subject to the relations:

$$[\kappa_{i\,r}, \kappa_{j\,s}] = 0,$$

$$[\xi_{i\,r}^{+}, \xi_{j\,s}^{-}] = \delta_{ij}\, \kappa_{i\,r+s},$$

$$[\kappa_{i\,0}, \xi_{j\,s}^{\pm}] = \pm\, (\alpha_i, \alpha_j)\, \xi_{j\,s}^{\pm},$$

$$[\kappa_{i\,r+1}, \xi_{j\,s}^{\pm}] - [\kappa_{i\,r}, \xi_{j\,s+1}^{\pm}] = \pm\, \frac{(\alpha_i, \alpha_j)}{2} \big(\kappa_{i\,r}\, \xi_{j\,s}^{\pm} + \xi_{j\,s}^{\pm}\, \kappa_{i\,r}\big),$$

$$[\xi_{i\,r+1}^{\pm}, \xi_{j\,s}^{\pm}] - [\xi_{i\,r}^{\pm}, \xi_{j\,s+1}^{\pm}] = \pm\, \frac{(\alpha_i, \alpha_j)}{2} \big(\xi_{i\,r}^{\pm}\, \xi_{j\,s}^{\pm} + \xi_{j\,s}^{\pm}\, \xi_{i\,r}^{\pm}\big),$$

$$\sum_{p \in \mathfrak{S}_m} [\xi_{i\,r_{p(1)}}^{\pm}, [\xi_{i\,r_{p(2)}}^{\pm}, \dots [\xi_{i\,r_{p(m)}}^{\pm}, \xi_{j\,s}^{\pm}] \dots]] = 0,$$

with $i \neq j$ and $m = 1 - c_{ij}$ in the last relation.

Note that the subalgebra of $\mathrm{Y}(\mathfrak{a})$ generated by the elements κ_{ir} and $\xi_{ir}^{\pm}$ with a fixed $i \in \{1, \dots, n\}$ and $r \geqslant 0$, is isomorphic to the Yangian $\mathrm{Y}(\mathfrak{sl}_2)$.

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Namely, the coefficients of the series

$$\kappa_i(d_i u), \qquad \xi_i^{+}(d_i u) \qquad \text{and} \qquad d_i^{-1}\xi_i^{-}(d_i u)$$

with $d_i = (\alpha_i, \alpha_i)/2$ satisfy the $\mathrm{Y}(\mathfrak{sl}_2)$ defining relations.

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Yangian characters

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Denote by $\mathcal{P}_N$ the abelian group whose elements are the tuples $\lambda(u) = \big(\lambda_1(u), \dots, \lambda_N(u)\big)$ where each $\lambda_i(u)$ is a formal series in u^{-1} with constant term 1 with respect to the component-wise multiplication.

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Consider the group ring $\mathbb{Z}[\mathcal{P}_N]$ of the abelian group $\mathcal{P}_N$ whose elements are finite linear combinations of the form $\sum m_{\lambda(u)}[\lambda(u)]$, where $m_{\lambda(u)} \in \mathbb{Z}$.

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For any $\lambda(u) \in \mathcal{P}_N$, the corresponding Gelfand–Tsetlin subspace $V_{\lambda(u)}$ consists of the vectors $v \in V$ with the property that for each $i = 1, \dots, N$ and each $r \geqslant 1$ there exists $p \geqslant 1$ such that $\big(h_i^{(r)} - \lambda_i^{(r)}\big)^p \, v = 0$.

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Then the Gelfand–Tsetlin character of V is the element of $\mathbb{Z}[\mathcal{P}_N]$ defined by

$$\mathrm{ch}\, V = \sum_{\lambda(u) \in \mathcal{P}_N} \big(\dim V_{\lambda(u)}\big)\, [\lambda(u)].$$

Multiplicativity property:

$$\mathrm{ch}\,(V \otimes W) = \mathrm{ch}\,V \cdot \mathrm{ch}\,W$$

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Consider the evaluation module $L(\lambda)$ over $\mathrm{Y}(\mathfrak{gl}_N)$, where $\lambda = (\lambda_1, \dots, \lambda_N)$ is a partition.

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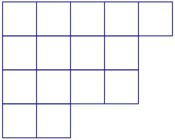
Identify λ with its Young diagram; for $\lambda = (5, 4, 4, 2)$ we have



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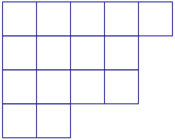

The content of the box $\alpha = (i, j)$ is $c(\alpha) = j - i$.

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By $\mathcal{T}(\alpha)$ we denote the entry of $\mathcal{T}$ in the box $\alpha \in \lambda$.

Theorem. The Gelfand–Tsetlin character of the $\mathrm{Y}(\mathfrak{gl}_N)$-module $L(\lambda)$ is given by

$$\operatorname{ch} L(\lambda) = \sum_{\mathcal{T}} \prod_{\alpha \in \lambda} x_{\mathcal{T}(\alpha),\, c(\alpha)},$$

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$$x_{i,a} = \Big[\Big(1, \dots, \frac{u+a+i}{u+a+i-1}, \dots, 1\Big)\Big], \quad 1 \leqslant i \leqslant N, \quad a \in \mathbb{C}.$$

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Remark. The specialization $x_{i,a} \mapsto x_i$ yields the Schur polynomial to recover the Weyl character formula.

Another specialization $x_{i,a} \mapsto x_i - b_{i+a}$ produces the factorial Schur polynomial associated with the sequence b_i.

Proof. The coefficients of the quantum determinant act on $L(\lambda)$ as scalar operators found from

$$\mathrm{qdet}\, T(u)\big|_{L(\lambda)} = \big(1 + \lambda_1\, u^{-1}\big) \dots \big(1 + \lambda_N\, (u - N + 1)^{-1}\big).$$

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we can write

$$h_1(u)\, h_2(u-1)\dots h_N(u-N+1)|_{L(\lambda)} = \prod_{\alpha\in\lambda} \frac{u+c(\alpha)+1}{u+c(\alpha)}.$$

Use the Gelfand–Tsetlin basis of $L(\lambda)$ parameterized by the semistandard λ-tableaux.

Such a tableau $\mathcal{T}$ can be viewed as the sequence of diagrams

$$\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_N = \lambda,$$

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Example. For $\lambda = (5, 4, 4, 2)$ and the tableau

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we have the sequence

$$\Lambda_1 = (3), \quad \Lambda_2 = (5,2), \quad \Lambda_3 = (5,4,1),$$
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The diagrams Λ_i represent the rows of the corresponding Gelfand–Tsetlin pattern:

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associated with the chain of subalgebras

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \mathfrak{gl}_3 \subset \mathfrak{gl}_4 \subset \mathfrak{gl}_5.$$

For any basis vector $\zeta_{\mathcal{T}} \in L(\lambda)$ and any $1 \leqslant k \leqslant N$ we have

$$h_1(u)\, h_2(u-1) \dots h_k(u-k+1)\, \zeta_{\mathcal{T}} = \prod_{\alpha \in \Lambda_k} \frac{u + c(\alpha) + 1}{u + c(\alpha)}\, \zeta_{\mathcal{T}}.$$

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This implies

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The element of $\mathbb{Z}[\mathcal{P}_N]$ corresponding to the action of $h_k(u)$ is

$$\prod_{\alpha \in \Lambda_k / \Lambda_{k-1}} x_{k,\, c(\alpha)},$$

which yields the character formula. $\square$

References

References

J. Brundan and A. Kleshchev, *Representations of shifted Yangians and finite W-algebras*, Mem. Amer. Math. Soc. **196** (2008), no. 918.

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V. Chari and A. Pressley, *Fundamental representations of Yangians and rational R-matrices*, J. Reine Angew. Math. **417** (1991), 87–128.