A CONSTRUCTION OF LATTICES FOR CERTAIN HYPERBOLIC BUILDINGS

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Abstract. We construct a nonuniform lattice and an infinite family of uniform lattices in the automorphism group of a hyperbolic building with all links a fixed finite building of rank 2 associated to a Chevalley group. We use complexes of groups and basic facts about spherical buildings.

1. Introduction

Let Δ be the finite building of rank 2 associated to a Chevalley group. A (k, Δ)–*building* is a hyperbolic polygonal complex X, with the link at each vertex Δ, and each 2–cell a regular hyperbolic k-gon, $k \geq 5$. Let $\mathrm{Aut}(X)$ be the group of cellular isometries of X. Since X is locally finite, $\mathrm{Aut}(X)$, with the compact-open topology, is locally compact. Let μ be a Haar measure on $G = \mathrm{Aut}(X)$. A discrete subgroup $\Gamma \leq G$ is a *lattice* if $\mu(\Gamma\backslash G)$ is finite, and Γ is *uniform* if $\Gamma\backslash G$ is compact.

Very few lattices in $\mathrm{Aut}(X)$ are known. Some (k, Δ)-buildings are Kac–Moody buildings, in which case they have a nonuniform lattice (Rémy, [5]). Bourdon in [1] and Gaboriau–Paulin in [3] constructed uniform lattices using polygons of groups, and Bourdon, by lifting lattices for classical buildings, constructed uniform and nonuniform lattices for certain $(4, \Delta)$–buildings (Example 1.5.2 of [1]).

Using complexes of groups, for each k divisible by 4, and each Δ, we construct a nonuniform lattice and an infinite family of uniform lattices for the unique locally reflexive (k, Δ)-building X with trivial holonomy (see below for definitions). The construction applies the Levi decomposition and basic facts about spherical buildings. A consequence is that the set of covolumes of lattices for X is nondiscrete.

2. Preliminaries

Let X be a (k, Δ)–building and $G = \mathrm{Aut}(X)$. The following characterisation of lattices in G is the same as Proposition 1.4.2 of [1], except that we consider the action on vertices rather than on 2-cells.

Proposition 2.1. *Suppose $G\backslash X$ is compact. Let $\Gamma \leq G$ act properly discontinuously on X and let $\mathcal{V}$ be a set of representatives of the vertices of $\Gamma\backslash X$. Then Γ is a lattice if and only if the series*

$$\sum_{v \in \mathcal{V}} \frac{1}{|\Gamma_v|} \tag{1}$$

converges, and Γ is uniform if and only if $\Gamma\backslash X$ is compact.

The Haar measure μ on G may be normalised so that $\mu(\Gamma\backslash G)$ equals the series (1) (Serre, [7]).

We next state local conditions for the universal cover of a complex of groups to be a (k,Δ)–building (see [2] for Haefliger's theory of complexes of groups). Each 2–cell in a (k,Δ)–building is isometric to a regular hyperbolic k–gon R with vertex angles $\frac{\pi}{m}$, where $m \geq 3$ is an integer determined by Δ. Let Y be a polygonal complex, such that each 2–cell of Y is isometric to a 2–simplex in the barycentric subdivision R' of R. We then say that a vertex of Y is an *n–vertex*, for $n = 0, 1, 2$, if it is mapped to the barycentre of an n–dimensional cell of R. The following is an easy generalisation of Theorem 0.1 of [3].

Theorem 2.2. *Suppose $G(Y)$ is a complex of groups over Y, such that the local development at each n–vertex of Y is: for $n = 0$, the barycentric subdivision of Δ; for $n = 1$, the complete bipartite graph $K_{2,s}$, with s the valence of a vertex of Δ; and for $n = 2$, the 1–skeleton of R'. Then $G(Y)$ is developable, with universal cover (the barycentric subdivision of) a (k,Δ)–building.*

For a fixed (k,Δ), there may be uncountably many (k,Δ)–buildings (see, for example, Theorem 0.2 of [3]). We now recall conditions, due to Haglund in [4], under which local data does specify the building. For each edge a of X, let $\mathcal{U}(a)$ be the union of the 2–cells of X which meet a. Then X is *locally reflexive* if every $\mathcal{U}(a)$ has a *reflection*, that is, an automorphism of order 2 which exchanges the ends of a, and preserves each 2–cell containing a. Suppose C is a 2–cell of X. Label the edges of C cyclically by $a_1, \ldots, a_k$, and let v be the vertex of C contained in the edges a_1 and a_k. A locally reflexive building X has *trivial holonomy* if for each 2–cell C of X, there is a set of reflections $\sigma_1, \ldots, \sigma_k$ of the subcomplexes $\mathcal{U}(a_1)$, $\ldots, \mathcal{U}(a_k)$, such that the composition $\sigma_k \circ \cdots \circ \sigma_1$ is the identity on the link of v in X. Finally, X is *homogeneous* if $\mathrm{Aut}(X)$ acts transitively on the set of vertices of X.

Theorem 2.3 (Haglund, [4])**.** *Let $k \geq 6$ be even. Then there exists a unique locally reflexive (k,Δ)–building X with trivial holonomy, and X is homogeneous.*

3. Construction of lattices

Let Δ be the spherical building of rank 2 associated to a finite Chevalley group $\mathcal{G}$. Then Δ is a generalised m–gon, that is, a bipartite graph with diameter m and girth $2m$, for $m \in \{3,4,6,8\}$ (see [6]). Let B be the Borel subgroup of $\mathcal{G}$ and let P be a maximal parabolic subgroup of $\mathcal{G}$. Recall that $\mathcal{G}$ acts on Δ by type-preserving automorphisms, hence so does P.

Lemma 3.1. *The quotient graph $P\backslash\Delta$ is a ray of m edges. Moreover, there are subgroups U_P, L_P and $K_P < H_1 < \cdots < H_{m-2} < B$ of P such that the quotient graph of groups for the action of P on Δ is:*

Proof. The group P is the stabiliser in $\mathcal{G}$ of a vertex v of Δ, and B is the stabiliser of an edge containing v. Since $\mathcal{G}$ acts transitively on the set of vertices of each type in Δ, P acts transitively on the sets of vertices of Δ at distances $j = 1, 2, \ldots, m = \mathrm{girth}(\Delta)$ from v. Hence the quotient $P\backslash\Delta$ is a ray of m edges, with L_P, K_P and the H_i the subgroups of P stabilising vertices and edges of Δ as shown.

By Theorem 6.18 of [6], there is a subgroup U_P of P such that $P = U_P \rtimes L_P$. We now show $B = U_P \rtimes K_P$. By definition of U_P (see [6]), we have $U_P < B < P$, thus $U_P \lhd B$. As $K_P < L_P$ and $U_P \cap L_P = 1$, it follows that $U_P \cap K_P = 1$. Vertices at distance m in Δ have the same valence (Exercise 6.3 of [6]), so

$$[L_P : K_P] = [P : B] = [U_P L_P : B]$$

Hence $|B| = |U_P||K_P|$ and so $B = U_P K_P$. We conclude that $B = U_P \rtimes K_P$. □

Consider the complex of groups $G(Y_1)$ in Figure 1. Here $m = 3$ so $H = H_1 = H_{m-2}$, and each 2–cell is isometric to a 2-simplex in the barycentric subdivision of a regular hyperbolic k-gon with vertex angles $\frac{\pi}{3}$. We write D_k for the dihedral group of order k and $\mathbb{Z}_2$ for $\mathbb{Z}/2\mathbb{Z}$. The copy of D_k at each 2–vertex is generated by the two adjacent copies of $\mathbb{Z}_2$. All other maps between local groups are natural inclusions. The construction of $G(Y_1)$ for other values of m is similar: each 2–cell of Y_1 is isometric to a 2-simplex in the barycentric subdivision of a regular hyperbolic k-gon with vertex angles $\frac{\pi}{m}$, and there are m 2–vertices with groups $K_P \times D_k$, $H_i \times D_k$ for $1 \le i \le m-2$, and $(U_P \rtimes K_P) \times D_k$.

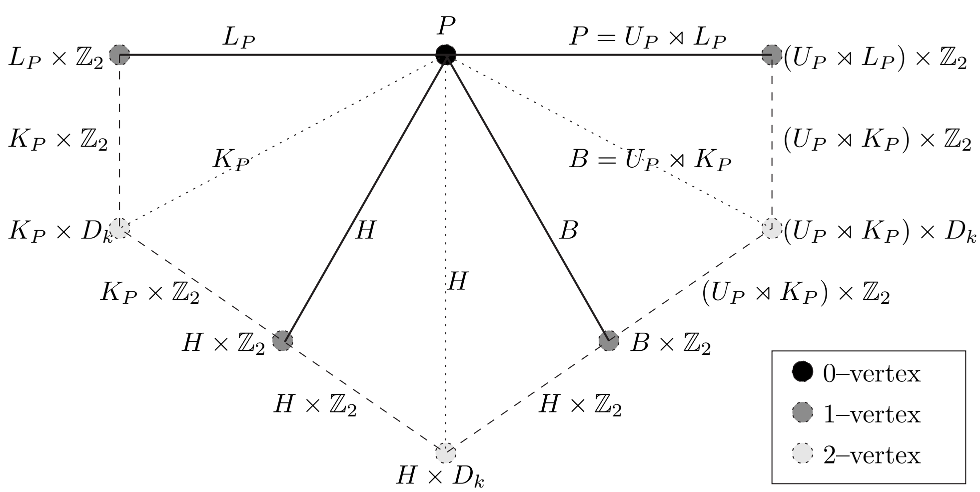

FIGURE 1. Complex of groups $G(Y_1)$

For $n \ge 1$ let U_P^n be the direct product of n copies of U_P. Since U_P is normal in P, for any subgroup Q of P we may form $U_P^n \rtimes Q$, with the action on each copy of U_P by conjugation in P. Hence, in Figure 1, we may replace each copy of $B = U_P \rtimes L_P$ and $P = U_P \rtimes L_P$ by respectively $U_P^n \rtimes L_P$ and $U_P^n \rtimes L_P$, and each L_P, K_P and H by respectively $U_P^{n-1} \rtimes L_P$, $U_P^{n-1} \rtimes K_P$ and $U_P^{n-1} \rtimes H$ (and similarly for other values of m). Call the resulting complex of groups $G(Y_n)$.

Assume k is divisible by 4. As sketched in Figure 2 for the case $m = 3$, we may form a complex of groups $G(Y_\infty)$ by "gluing" together $G(Y_1)$, $G(Y_2)$, and so on. More precisely, for $n \ge 1$, we identify the cells of $G(Y_n)$ and $G(Y_{n+1})$ with local groups $(U_P^n \rtimes L_P) \times \mathbb{Z}_2$, $(U_P^n \rtimes K_P) \times \mathbb{Z}_2$ and $(U_P^n \rtimes K_P) \times D_k$. We then remove the $\mathbb{Z}_2$–factors and replace D_k by $D_{\frac{k}{2}}$ (since k is divisible by 4, $\frac{k}{2}$ is even).

By Lemma 3.1 and Theorem 2.2, the universal cover X of $G(Y_\infty)$ is a (k, Δ)–building. We verify that X is locally reflexive and has trivial holonomy, using the

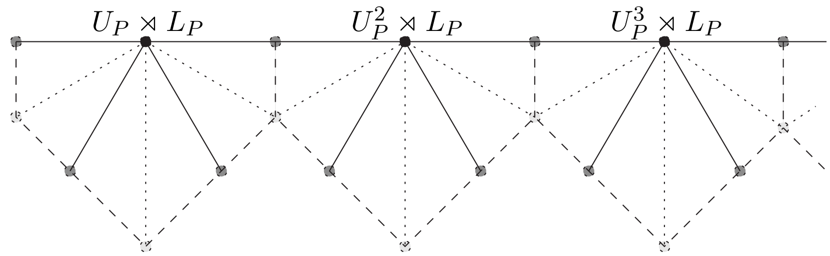

Figure 2. Sketch of $G(Y_\infty)$

direct products with $\mathbb{Z}_2$. Let $\Gamma = \pi_1(G(Y_\infty))$ and let $N = \ker(\Gamma \to G = \mathrm{Aut}(X))$, so that Γ/N may be regarded as a subgroup of G. Then N is contained in each local group of $G(Y_\infty)$, so has bounded order. By abuse of notation, we identify Γ and Γ/N.

Since X is homogeneous, $G\backslash X$ is compact, and since the local groups of $G(Y_\infty)$ are all finite, Γ acts properly discontinuously. Thus, by Proposition 2.1, as the series

$$\sum_{v\in\mathcal{V}} \frac{1}{|\Gamma_v|} = \sum_{n=1}^{\infty} \frac{1}{|U_P^n \rtimes L_P|} = \frac{1}{|L_P|} \sum_{n=1}^{\infty} \frac{1}{|U_P|^n} \tag{2}$$

is convergent, Γ is a nonuniform lattice in G. Moreover, an infinite family of uniform lattices in G is obtained by, for each $n \geq 1$, gluing together $G(Y_1), \ldots, G(Y_n)$. The covolumes of these lattices are the partial sums of the series (2), hence the set of covolumes of lattices in G is nondiscrete.

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