A numerical analysis framework for linear and non-linear elasticity equations

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Joint work with B. P. Lamichhane (U. Newcastle)

Plan

1. **Numerical methods for elasticity equations**

2. **Gradient Schemes for elasticity equations**
   - 4 discrete elements
   - 3 properties
   - Convergence results

3. **Examples of Gradient Schemes for elasticity equations**
   - Displacement-based formulation
   - Stabilised nodal strain formulation
   - Hu-Washizu-based formulation

4. **Conclusion**

Plan

Tasks of the numerical analyst

1. Design numerical methods.
2. Test them in simple and real-world applications (benchmarking).
3. Analyse their convergence and other properties.

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   under assumptions compatible with real-world applications.

Linear and non-linear elasticity

$$\begin{cases} -\mathrm{div}(\boldsymbol{\sigma}(x, \varepsilon(\overline{\mathbf{u}}))) = \mathbf{F}\,, & \text{in } \Omega, \\ \varepsilon(\overline{\mathbf{u}}) = \frac{\nabla\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T}{2}\,, & \text{in } \Omega, \\ \overline{\mathbf{u}} = 0\,, & \text{on } \Gamma_D\,, \\ \boldsymbol{\sigma}(x, \varepsilon(\overline{\mathbf{u}}))\mathbf{n} = \mathbf{g}\,, & \text{on } \Gamma_N, \end{cases}$$

- Example: linear elasticity $\boldsymbol{\sigma}(x, \varepsilon(\overline{\mathbf{u}})) = \mathbb{C}\varepsilon(\overline{\mathbf{u}})$.

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- Example: linear elasticity $\boldsymbol{\sigma}(x, \varepsilon(\overline{\mathbf{u}})) = \mathbb{C}\varepsilon(\overline{\mathbf{u}})$.

Weak formulation:

$$\begin{cases} \text{Find } \overline{\mathbf{u}} \in \mathbf{H}^1_{\Gamma_D}(\Omega) \text{ such that, for any } \mathbf{v} \in \mathbf{H}^1_{\Gamma_D}(\Omega), \\ \displaystyle\int_\Omega \boldsymbol{\sigma}(x, \varepsilon(\overline{\mathbf{u}})(x)) : \varepsilon(\mathbf{v})(x) \mathrm{d}x \quad = \quad \int_\Omega \mathbf{F}(x) \cdot \mathbf{v}(x) \mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + \displaystyle\int_{\Gamma_N} \mathbf{g}(x) \cdot \gamma(\mathbf{v})(x)\, dS(x). \end{cases}$$

where $\mathbf{H}^1_{\Gamma_D}(\Omega) = \{\mathbf{v} \in H^1(\Omega)^d \,:\, \gamma(\mathbf{v}) = 0 \text{ on } \Gamma_D\}$ and γ is the trace operator.

Numerical methods and their convergence analysis

Methods:

- Finite Element (or Mixed FE) based.

- Sometimes with projections or modifications to stabilise in the nearly-incompressible limit (mostly for linear elasticity).

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Convergence analysis

- Based on error estimates, establish optimal orders of convergence.

- Mostly/only done for:
  - Linear elasticity: conforming methods, or non-conforming methods when $\overline{\mathbf{u}} \in H^2$.
  - Non-linear elasticity: conforming methods, under sometimes very strong assumptions on $\overline{\mathbf{u}}$ (e.g. $C^2(\overline{\Omega})$).

References

General theory

- Brenner & Scott, 1994.
- Ciarlet, 1978.

Linear elasticity

- Braess, Carstensen & Reddy, 2004.
- Brenner & Sung, 1992.
- Burman & Hansbo, 2006.
- Lamichhane, Reddy & Wohlmuth, 2006.

Non-linear elasticity

- Braess & Ming, 2005.
- Carstensen & Dolzmann, 2004.
- Gatica & Stephan, 2002.

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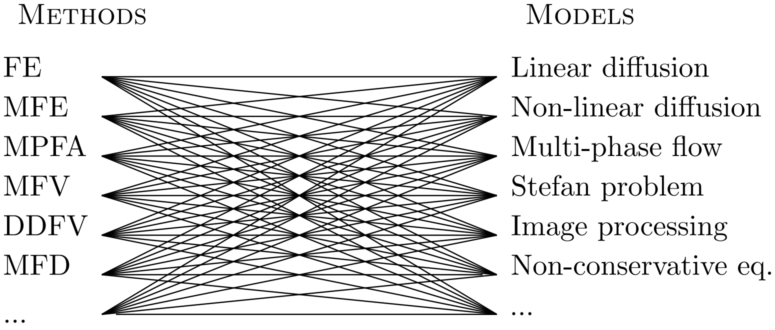
Gradient schemes for diffusion equations

- Developed for diffusion equations (Eymard, Guichard, Herbin, Gallouët, D.: 2012+): linear, non-linear, stationary, transient, non-local...

- Unified convergence analysis of numerous numerical schemes for anisotropic diffusion equations for numerous models.



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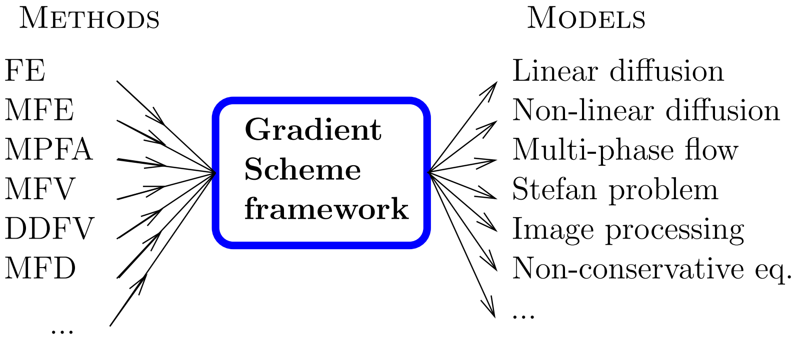

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Weak formulation of the elasticity equations:

$$\left\{ \begin{array}{l} \text{Find } \overline{\mathbf{u}} \in \mathbf{H}^1_{\Gamma_D}(\Omega) \text{ such that, for any } \mathbf{v} \in \mathbf{H}^1_{\Gamma_D}(\Omega), \\ \displaystyle\int_\Omega \boldsymbol{\sigma}(x, \boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)) : \boldsymbol{\varepsilon}(\mathbf{v})(x)\mathrm{d}x \quad = \quad \int_\Omega \mathbf{F}(x) \cdot \mathbf{v}(x)\mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \displaystyle\int_{\Gamma_N} \mathbf{g}(x) \cdot \gamma(\mathbf{v})(x)\, dS(x). \end{array} \right.$$

Gradient Scheme framework for elasticity: 4 discrete elements

A Gradient Discretisation is $\mathcal{D} = (X_\mathcal{D}, \Pi_\mathcal{D}, \mathcal{T}_\mathcal{D}, \nabla_\mathcal{D})$ with

- $\mathbf{X}_{\mathcal{D},\Gamma_D} = \mathbb{R}^{d.o.f.}$ discrete space (with Dirichlet boundary conditions on Γ_D),
- $\Pi_\mathcal{D} : \mathbf{X}_{\mathcal{D},\Gamma_D} \to L^2(\Omega)$ a reconstruction of functions,
- $\mathcal{T}_\mathcal{D} : \mathbf{X}_{\mathcal{D},\Gamma_D} \to L^2(\partial\Omega)$ a discrete trace operator,
- $\nabla_\mathcal{D} : \mathbf{X}_{\mathcal{D},\Gamma_D} \to L^2(\Omega)^d$ a discrete gradient such that $|| \cdot ||_\mathcal{D} = ||\nabla_\mathcal{D} \cdot ||_{L^2(\Omega)^d}$ is a norm on $\mathbf{X}_{\mathcal{D},\Gamma_D}$.

Gradient Scheme for elasticity equations

Continuous equation

$$\left\{\begin{array}{l} \text{Find } \overline{\mathbf{u}} \in \mathbf{H}^1_{\Gamma_D}(\Omega) \text{ such that, for any } \mathbf{v} \in \mathbf{H}^1_{\Gamma_D}(\Omega), \\ \displaystyle\int_\Omega \boldsymbol{\sigma}(x, \boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)) : \boldsymbol{\varepsilon}(\mathbf{v})(x)\mathrm{d}x \quad = \quad \int_\Omega \mathbf{F}(x) \cdot \mathbf{v}(x)\mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + \displaystyle\int_{\Gamma_N} \mathbf{g}(x) \cdot \gamma(\mathbf{v})(x)\, dS(x). \end{array}\right.$$

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Discretisation

$$\left\{\begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{X}_{\mathcal{D},\Gamma_D} \text{ such that, for any } \mathbf{v} \in \mathbf{X}_{\mathcal{D},\Gamma_D}, \\ \displaystyle\int_\Omega \boldsymbol{\sigma}(x, \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{u})(x)) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}x = \int_\Omega \mathbf{F}(x)\cdot\Pi_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}x \\ \qquad\qquad\qquad\qquad + \displaystyle\int_{\Gamma_N} \mathbf{g}(x)\cdot\mathcal{T}_{\mathcal{D}}(\mathbf{v})(x)\, dS(x). \end{array}\right.$$

where $\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{u}) = \frac{\nabla_{\mathcal{D}}\mathbf{u}+(\nabla_{\mathcal{D}}\mathbf{u})^T}{2}$.

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Gradient Scheme framework for elasticity: 3 properties to ensure convergence

A sequence $(\mathcal{D}_m)$ of Gradient discretisations is:

Coercive if there exists C such that, for all m and $\mathbf{v} \in \mathbf{X}_{\mathcal{D}_m,\Gamma_D}$,

$$||\Pi_{\mathcal{D}_m}\mathbf{v}||_{L^2} \leq C||\nabla_{\mathcal{D}_m}\mathbf{v}||_{L^2} ,$$

$$||\mathcal{T}_{\mathcal{D}_m}\mathbf{v}||_{L^2} \leq C||\nabla_{\mathcal{D}_m}\mathbf{v}||_{L^2} ,$$

$$||\nabla_{\mathcal{D}_m}\mathbf{v}||_{L^2} \leq C||\varepsilon_{\mathcal{D}_m}\mathbf{v}||_{L^2}$$

(Poincaré's, trace and Körn's inequalities).

Gradient Scheme framework for elasticity: 3 properties to ensure convergence

A sequence $(\mathcal{D}_m)$ of Gradient discretisations is:

Consistent if, for all $\varphi \in \mathbf{H}^1_{\Gamma_D}(\Omega)$

$$S_{\mathcal{D}_m}(\varphi) := \min_{\mathbf{v}\in\mathbf{X}_{\mathcal{D}_m,\Gamma_D}} \left(||\Pi_{\mathcal{D}_m}\mathbf{v} - \varphi||_{L^2} + ||\nabla_{\mathcal{D}_m}\mathbf{v} - \nabla\varphi||_{L^2}\right)$$

tends to 0 as $m \to \infty$.

(Ultimate density of the range of the discrete reconstruction and trace).

Gradient Scheme framework for elasticity: 3 properties to ensure convergence

A sequence $(\mathcal{D}_m)$ of Gradient discretisations is:

Limit-conforming if, for all $\boldsymbol{\tau} \in (L^2)^{d\times d}$ such that $\mathrm{div}(\boldsymbol{\tau}) \in (L^2)^d$ and $\gamma_{\mathbf{n}}(\boldsymbol{\tau}) \in L^2(\Gamma_N)$,

$$W_{\mathcal{D}_m}(\boldsymbol{\varphi}) := \max_{\substack{\mathbf{v} \in \mathbf{X}_{\mathcal{D}_m,\Gamma_D} \\ \mathbf{v}\neq 0}} \frac{\left| \int_\Omega (\nabla_{\mathcal{D}_m}\mathbf{v}) : \boldsymbol{\tau} + (\Pi_{\mathcal{D}_m}\mathbf{v})\mathrm{div}(\boldsymbol{\tau}) - \int_{\Gamma_N} \gamma_{\mathbf{n}}(\boldsymbol{\tau}) \cdot \mathcal{T}_{\mathcal{D}_m}(\mathbf{v}) \right|}{||\nabla_{\mathcal{D}_m} v||_{L^p}}$$

tends to 0 as $m \to \infty$.

($\lim_m (\nabla_{\mathcal{D}_m})^* \approx -\mathrm{div}$ *and* $\lim_{m\to\infty} \mathcal{T}_{\mathcal{D}_m} \approx \gamma$ *in weak topology).*

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Linear elasticity: $-\mathrm{div}(\mathbb{C}\varepsilon(\overline{\mathbf{u}})) = \mathbf{F}$ in Ω.

Theorem (Error estimates for linear elasticity)

Assume that $\mathbb{C}$ is bounded and coercive, that $\mathbf{F} \in (L^2)^d$ and that $\mathbf{g} \in (L^2)^d$. If $(\mathcal{D}_m)$ is a **coercive** *family of Gradient Discretization then*

$$\|\overline{\mathbf{u}} - \Pi_{\mathcal{D}_m}\mathbf{u}_m\|_{\mathbf{L}^2(\Omega)} + \|\nabla\overline{\mathbf{u}} - \nabla_{\mathcal{D}_m}\mathbf{u}_m\|_{\mathbf{L}^2(\Omega)^d} \leq CW_{\mathcal{D}_m}(\mathbb{C}\varepsilon(\overline{\mathbf{u}})) + CS_{\mathcal{D}_m}(\overline{\mathbf{u}}).$$

In particular, if $(\mathcal{D}_m)$ is **consistent** *and* **limit-conforming***, then $\Pi_{\mathcal{D}_m}\mathbf{u}_m \to \overline{\mathbf{u}}$ and $\nabla_{\mathcal{D}_m}\mathbf{u}_m \to \nabla\overline{\mathbf{u}}$ in L^2.*

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- $\mathbb{C}(x)$ may be discontinuous, no regularity assumption on $\overline{\mathbf{u}}$.

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- $\mathbb{C}(x)$ may be discontinuous, no regularity assumption on $\overline{\mathbf{u}}$.
- Error estimates if $\mathbb{C}$ is Lipschitz and $\overline{\mathbf{u}} \in H^2$:

$$W_{\mathcal{D}_m}(\mathbb{C}\varepsilon(\overline{\mathbf{u}})) + S_{\mathcal{D}_m}(\overline{\mathbf{u}}) = \mathcal{O}(h_m) \qquad (h_m = \text{mesh size}).$$

Non-linear elasticity: $-\mathrm{div}(\boldsymbol{\sigma}(\cdot, \varepsilon(\overline{\mathbf{u}}))) = \mathbf{F}$ in Ω.

Theorem (Convergence for non-linear elasticity)

Assume that $\boldsymbol{\sigma}$ has a linear growth, is coercive and strictly monotone, that $\mathbf{F} \in (L^2)^d$ and that $\mathbf{g} \in (L^2)^d$.

If $(\mathcal{D}_m)$ is a **coercive, consistent** *and* **limit-conforming** *family of Gradient Discretization then, up to a subsequence, $\Pi_{\mathcal{D}_m}\mathbf{u}_m \to \overline{\mathbf{u}}$ and $\nabla_{\mathcal{D}_m}\mathbf{u}_m \to \nabla\overline{\mathbf{u}}$ in L^2.*

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Covered models:

- Damage models $\boldsymbol{\sigma}(x,\boldsymbol{\varepsilon}) = (1-D(\boldsymbol{\varepsilon}))\mathbb{C}(x)\boldsymbol{\varepsilon}$ (Cervera, Chiumenti, Codina 2010).

- non-linear Hencky-von Mises elasticity $\sigma = \lambda(\mathrm{dev}(\boldsymbol{\varepsilon}))\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu(\mathrm{dev}(\boldsymbol{\varepsilon}))\boldsymbol{\varepsilon}$.

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Conforming methods (Galerkin approximation)

Replace $\mathbf{H}^1_{\Gamma_D}$ with $\mathbf{X}_{\mathcal{D},\Gamma_D}$ in the weak continuous formulation!

- $\mathbf{X}_{\mathcal{D},\Gamma_D}$ = finite-dimensional subspace of $\mathbf{H}^1_{\Gamma_D}(\Omega)$,
- $\Pi_{\mathcal{D}} = \mathrm{Id}$, $\mathcal{T}_{\mathcal{D}} = \gamma$ and $\nabla_{\mathcal{D}} = \nabla$.

Example: any low- or high-degree conforming Finite Element method (e.g. $P1$ on triangles or simplices, bilinear functions on quadrilaterals, etc.)

Non-conforming method: Crouzeix-Raviart

Given T a triangulation of Ω,

- $\mathbf{X}_{\mathcal{D},\Gamma_D}$ =space of piecewise linear functions on T, which are continuous at the edge mid-points,
- $\Pi_{\mathcal{D}} = \mathrm{Id}$, $\mathcal{T}_{\mathcal{D}}$ =restriction to $\partial\Omega$ and $\nabla_{\mathcal{D}}$ =broken gradient.

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► May not be coercive (no Körn inequality) if $\Gamma_D \neq \partial\Omega$.
Higher order methods (still Gradient Schemes) then required.

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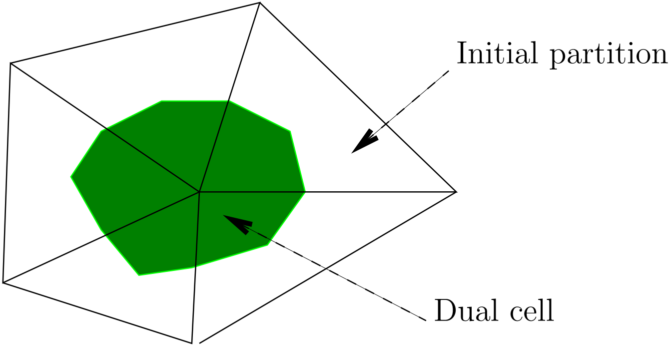
Formulation

$\mathbf{V}_h$ standard Finite Element space on a partition $\mathcal{T}_h$ of Ω.
$\mathcal{T}_h^*$ =dual mesh.



Formulation

In the weak formulation of the FE scheme, replace

$$\int_\Omega \mathbb{C}\varepsilon(\mathbf{u}_h) : \varepsilon(\mathbf{v}_h)$$

with

$$\int_\Omega \Pi_h^* \varepsilon(\mathbf{u}_h) : \mathbb{C}\varepsilon(\mathbf{v}_h) + \int_\Omega \mathbb{D}(\varepsilon(\mathbf{u}_h) - \Pi_h^* \varepsilon(\mathbf{u}_h)) : \varepsilon(\mathbf{v}_h)\, dx$$

where $\mathbb{D}$ is symmetric definite positive and $\Pi_h^* =$orthogonal projection on piecewise constant functions on $\mathcal{T}_h^*$.

(Flanagan & Belytschko 1981, Puso & Solberg 2006, Lamichhane 2009)

Gradient Discretisation

- $\mathbf{X}_{\mathcal{D},\Gamma_D} = \mathbf{V}_h$, $\Pi_{\mathcal{D}} = \mathrm{Id}$, $\mathcal{T}_{\mathcal{D}} = \gamma$,
- For $\mathbf{v} \in \mathbf{X}_{\mathcal{D},\Gamma_D}$,

$$\nabla_{\mathcal{D}}\mathbf{v} = \Pi_h^* \nabla \mathbf{v} + \mathbb{C}^{-1/2}\mathbb{D}^{1/2}(\nabla \mathbf{v} - \Pi_h^* \nabla \mathbf{v})$$

(for $\mathbb{C}$ and $\mathbb{D}$ piecewise constant on $\mathcal{T}_h^*$).

► Orthogonality properties of Π_h^* and $\mathrm{I} - \Pi_h^*$ eliminate the cross products in $\int_\Omega \mathbb{C}\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{u}) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})$.

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▶ Consistency and limit-conformity follow because

$$\nabla_{\mathcal{D}}\mathbf{v} = \nabla\mathbf{v} + \mathcal{L}_h\nabla\mathbf{v}$$

where $\mathcal{L}_h : (L^2)^d \to (L^2)^d$ is self-adjoint, bounded and converges pointwise to 0.

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Basics

- Based on a 3-field formulation ($\mathbf{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ approximated in three different spaces).

- Gives stable numerical scheme in the nearly-incompressible limit.

- Can be reduced to a displacement formulation by static condensation.

(Lamichhane, Reddy & Wohlmuth, 2006).

Reduced displacement formulation of the Hu-Washizu method

$\mathbf{V}_h$ space of bilinear conforming Finite Element on quadrilaterals.
In the weak formulation of the FE scheme, replace

$$\int_\Omega \mathbb{C}\varepsilon(\mathbf{u}_h) : \varepsilon(\mathbf{v}_h)$$

with

$$\int_\Omega \mathbb{C}_h P_{S_h}\varepsilon(\mathbf{u}_h) : P_{S_h}\varepsilon(\mathbf{v}_h)\, dx.$$

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with

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- $S_h =$ a suitable sub-space of $\mathbf{V}_h$ (several possible examples),
- $P_{S_h} =$ orthogonal projection on S_h,
- $\mathbb{C}_h =$ approximation of $\mathbb{C}$ defined by

$$\forall \boldsymbol{\tau} \in \mathbf{V}_h\,,:\; \mathbb{C}_h\boldsymbol{\tau} = \mathbb{C}P_{S_h^c}\boldsymbol{\tau} + \theta P_{S_h^t}\boldsymbol{\tau}$$

where

$$S_h^c = \{\boldsymbol{\tau} \in S_h \;:\; \mathbb{C}\boldsymbol{\tau} \in S_h\}\,, \quad S_h = S_h^c \oplus S_h^t.$$

Gradient Discretisation

- $\mathbf{X}_{\mathcal{D},\Gamma_D} = \mathbf{V}_h$, $\Pi_{\mathcal{D}} = \mathrm{Id}$, $\mathcal{T}_{\mathcal{D}} = \gamma$,
- For $\mathbf{v} \in \mathbf{X}_{\mathcal{D},\Gamma_D}$,

$$\nabla_{\mathcal{D}}\mathbf{v} = P_{S_h^c}\nabla\mathbf{v} + \sqrt{\theta}\mathbb{C}^{-1/2}P_{S_h^t}\nabla\mathbf{v}.$$

► The particular choices of S_h (and orthogonality properties) eliminate the cross products.

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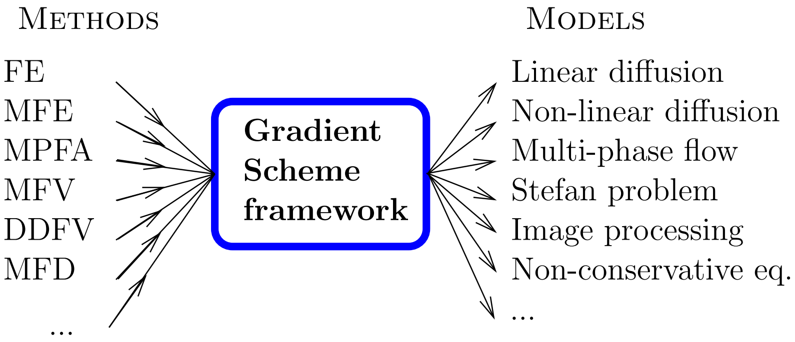
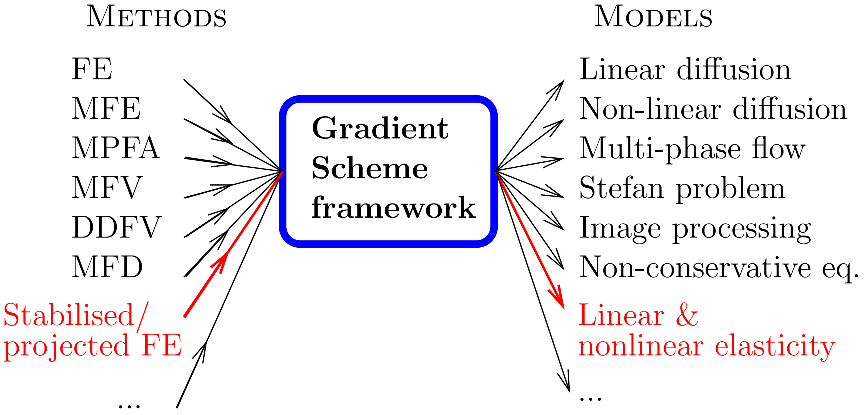


Methods
FE
MFE
MPFA
MFV
DDFV
MFD
...
Gradient
Scheme
framework
Models
Linear diffusion
Non-linear diffusion
Multi-phase flow
Stefan problem
Image processing
Non-conservative eq.
...

Add two branches...



Thanks.