Classifications of Symmetric Normal Form Games

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Normal Form Game

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- A non-empty (finite) set of **strategies** A_i; and
- A **payoff** function $u_i : A \to \mathbb{R}$ where $A = \times_{i \in N} A_i$ is the set of **strategy profiles** or **outcomes**.

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Then:

$$A = \{(a, c, e), (a, c, f), (a, d, e), (a, d, f), (b, c, e), (b, c, f), (b, d, e), (b, d, f)\}$$

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- $N = \{1, 2, 3\}$;
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Then:

$$A = \{(a,c,e),(a,c,f),(a,d,e),(a,d,f),(b,c,e),(b,c,f),(b,d,e),(b,d,f)\}$$

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We can display the payoffs in tables as follows:

	e	f
c	1, 1, 1	2, 2, 3
d	2, 3, 2	5, 4, 4

$(a, ,)$

	e	f
c	3, 2, 2	4, 5, 4
d	4, 4, 5	6, 6, 6

$(b, ,)$

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We can display the payoffs in tables as follows:

$(a, ,)$	e	f
c	1, 1, 1	2, 2, 3
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$(b, ,)$	e	f
c	3, 2, 2	4, 5, 4
d	4, 4, 5	6, 6, 6

The payoff to player 3 for the strategy profile (b, d, e) is $u_3(b, d, e) = 5$.

Player Permutations S_N Acting on Strategy Profiles A

Suppose each player has the same strategy set.

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$$\pi(s_1, \ldots, s_n) = (s_{\pi^{-1}(1)}, \ldots, s_{\pi^{-1}(n)})$$

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Take $\pi = (123) \in S_3$ and $(s_1, s_2, s_3) \in A$.

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$$\text{Eg. } \pi(a, b, a) = (a, a, b)$$

Label-Dependent Notions of Symmetry

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	a	b
a	$1, 1, 1$	$2, 2, 3$
b	$2, 3, 2$	$5, 4, 4$

$(a, ,)$

	a	b
a	$3, 2, 2$	$4, 5, 4$
b	$4, 4, 5$	$6, 6, 6$

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Eg. let $\pi = (123)$, $\pi(a, b, a) = (a, a, b)$ as before, and we see that $u_2(a, b, a) = u_{\pi(2)}(\pi(a, b, a)) = u_3(a, a, b) = 3$.

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- $\langle (123), (12) \rangle = S_3$.

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Standard symmetric 3-player game.

	a	b
a	$1, 1, 1$	$2, 3, 4$
b	$3, 4, 2$	$5, 6, 7$

$(a, ,)$

	a	b
a	$4, 2, 3$	$7, 5, 6$
b	$6, 7, 5$	$8, 8, 8$

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Note: Must have $u_i(a, a, a) = u_j(a, a, a)$ for all $i, j \in N$ etc.

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Example

$$g = \left((123); \begin{pmatrix} a & b \\ d & c \end{pmatrix}, \begin{pmatrix} c & d \\ e & f \end{pmatrix}, \begin{pmatrix} e & f \\ a & b \end{pmatrix}\right)$$

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$$g(2) = 3, g(b, d, e) = (a, c, f)$$

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Note: $\mathrm{bij}(\Gamma) \cong (S_m \,\mathrm{Wr}\, S_n)$.

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- **player** n**-transitive** if for each $\pi \in S_N$ there exists $g \in G$ such that $g(i) = \pi(i)$ for all $i \in N$; and
- **strategy trivial** if for each $g \in G_i$, $g(s_i) = s_i$ for all $s_i \in A_i$ (ie. $\tau_i = \mathrm{id}_{A_i}$).

Automorphism Group

An **automorphism** of Γ is an invariant bijection $g \in \text{bij}(\Gamma)$

$$\text{ie. } u_i(s) = u_{g(i)}(g(s)) \text{ for all } i \in N,\ s \in A$$

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	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

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$$\begin{aligned}\text{Aut}(\Gamma) = \{&(e; (\begin{smallmatrix} H & T \\ H & T \end{smallmatrix}), (\begin{smallmatrix} H & T \\ H & T \end{smallmatrix})), (e; (\begin{smallmatrix} H & T \\ T & H \end{smallmatrix}), (\begin{smallmatrix} H & T \\ T & H \end{smallmatrix})), \\ &((12); (\begin{smallmatrix} H & T \\ H & T \end{smallmatrix}), (\begin{smallmatrix} H & T \\ T & H \end{smallmatrix})), ((12); (\begin{smallmatrix} H & T \\ T & H \end{smallmatrix}), (\begin{smallmatrix} H & T \\ H & T \end{smallmatrix}))\}\end{aligned}$$

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H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

$$\begin{aligned}\text{Aut}(\Gamma) = \{&(e; \left(\begin{smallmatrix} H & T \\ H & T \end{smallmatrix}\right), \left(\begin{smallmatrix} H & T \\ H & T \end{smallmatrix}\right)), (e; \left(\begin{smallmatrix} H & T \\ T & H \end{smallmatrix}\right), \left(\begin{smallmatrix} H & T \\ T & H \end{smallmatrix}\right)), \\ &((12); \left(\begin{smallmatrix} H & T \\ H & T \end{smallmatrix}\right), \left(\begin{smallmatrix} H & T \\ T & H \end{smallmatrix}\right)), ((12); \left(\begin{smallmatrix} H & T \\ T & H \end{smallmatrix}\right), \left(\begin{smallmatrix} H & T \\ H & T \end{smallmatrix}\right))\}\end{aligned}$$

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$\mathrm{Aut}(\Gamma)$ is player transitive, is not strategy trivial and contains no proper transitive subgroups.

Label-Independent Notions of Symmetry

Theorem

The following conditions are equivalent:

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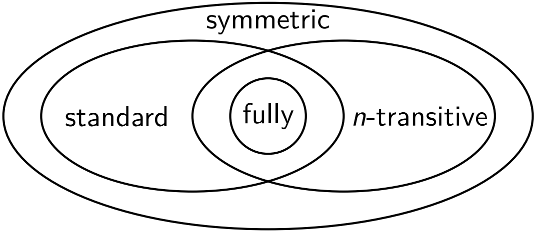

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	c	d
a	α, α	γ, β
b	β, γ	δ, δ

Example: n-transitive standard non-fully symmetric

	e	f
c	$1, 1, 1$	$2, 3, 4$
d	$3, 4, 2$	$4, 3, 2$

$(a, ,)$

	e	f
c	$4, 2, 3$	$2, 4, 3$
d	$3, 2, 4$	$1, 1, 1$

$(b, ,)$

$$\mathrm{Aut}(\Gamma) = \langle ((123); \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ e & f \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ a & b \end{smallmatrix}\right)), ((12); \left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ b & a \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ f & e \end{smallmatrix}\right)) \rangle$$

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- $((12); \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ e & f \end{smallmatrix}\right)) \notin \mathrm{Aut}(\Gamma)$.

Example: only-transitive non-standard symmetric

	g	h
e	1, 1, 2, 2	3, 4, 4, 3
f	4, 3, 3, 4	2, 2, 1, 1

$(a, c, ,)$

	g	h
e	3, 4, 4, 3	1, 1, 2, 2
f	2, 2, 1, 1	4, 3, 3, 4

$(a, d, ,)$

	g	h
e	4, 3, 3, 4	2, 2, 1, 1
f	1, 1, 2, 2	3, 4, 4, 3

$(b, c, ,)$

	g	h
e	2, 2, 1, 1	4, 3, 3, 4
f	3, 4, 4, 3	1, 1, 2, 2

$(b, d, ,)$

$$\begin{aligned}\mathrm{Aut}(\Gamma) \geq \langle &\Big((12)\circ(34); \left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ h & g \end{smallmatrix}\right), \left(\begin{smallmatrix} g & h \\ e & f \end{smallmatrix}\right)\Big), \\ &\Big((13)\circ(24); \left(\begin{smallmatrix} a & b \\ f & e \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ h & g \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ a & b \end{smallmatrix}\right), \left(\begin{smallmatrix} g & h \\ c & d \end{smallmatrix}\right)\Big), \\ &\Big((14)\circ(23); \left(\begin{smallmatrix} a & b \\ h & g \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ f & e \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} g & h \\ a & b \end{smallmatrix}\right)\Big)\rangle\end{aligned}$$

Questions?

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Bonus Example: only-transitive non-standard symmetric

	g	h
e	1, 2, 3, 4	5, 6, 7, 8
f	7, 8, 5, 6	8, 5, 6, 7

$(a, c, ,)$

	g	h
e	4, 1, 2, 3	6, 7, 8, 5
f	3, 4, 1, 2	2, 3, 4, 1

$(a, d, ,)$

	g	h
e	2, 3, 4, 1	3, 4, 1, 2
f	6, 7, 8, 5	4, 1, 2, 3

$(b, c, ,)$

	g	h
e	8, 5, 6, 7	7, 8, 5, 6
f	5, 6, 7, 8	1, 2, 3, 4

$(b, d, ,)$

$$\mathrm{Aut}(\Gamma) \geq \langle ((1234); \left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ e & f \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right), \left(\begin{smallmatrix} g & h \\ a & b \end{smallmatrix}\right)) \rangle$$