# The Triapsis Semigroup

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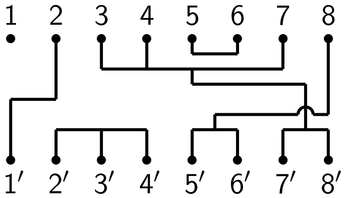

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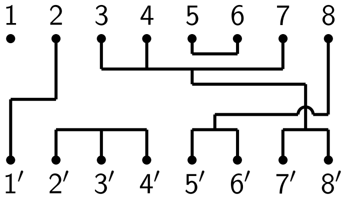


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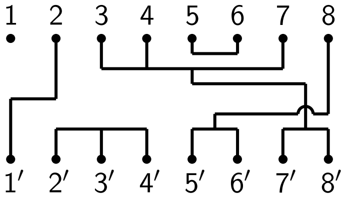


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**transversal components** are edges that connect vertices in both rows.

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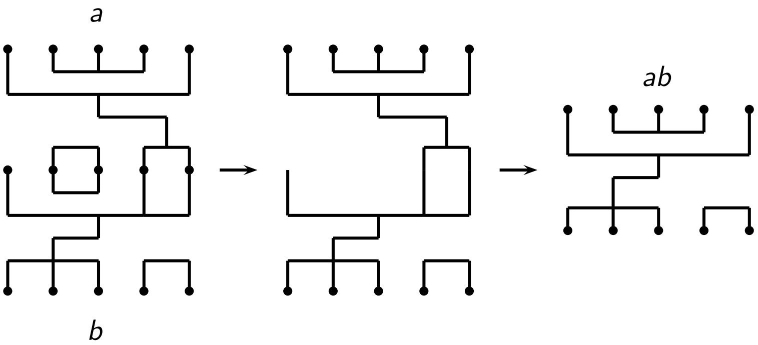

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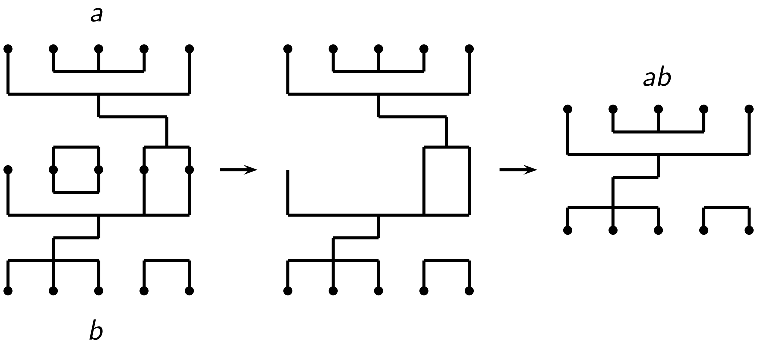


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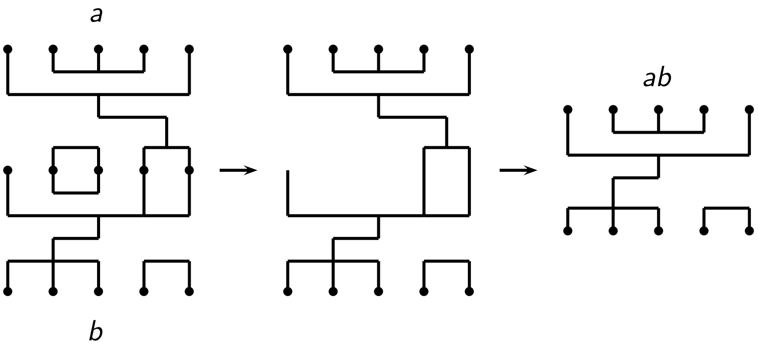


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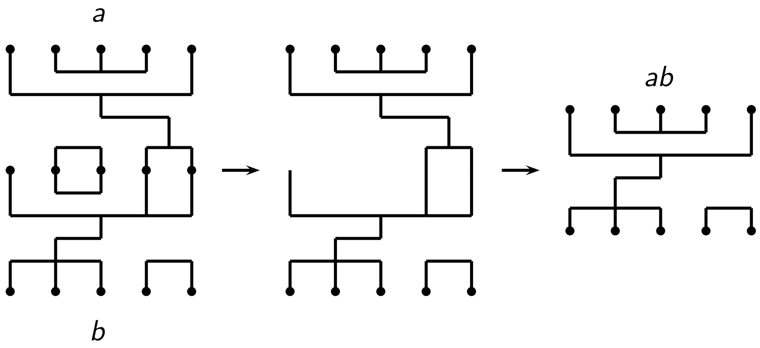


- place $a$ on top of $b$;
- remove the middle dots and stranded loops; and
- clip loose ends and collapse remaining loops.

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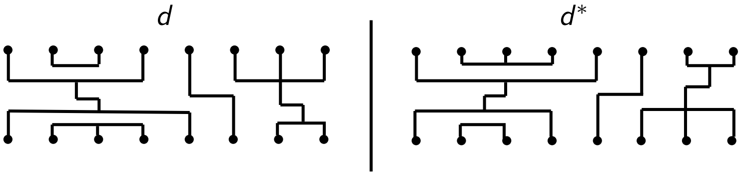

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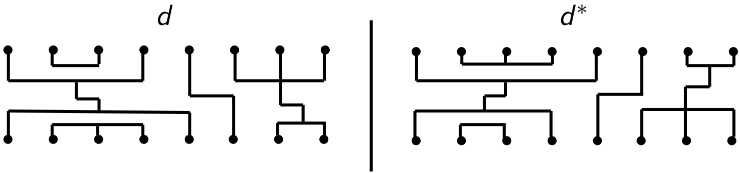


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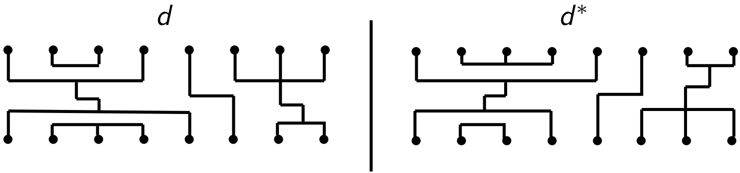


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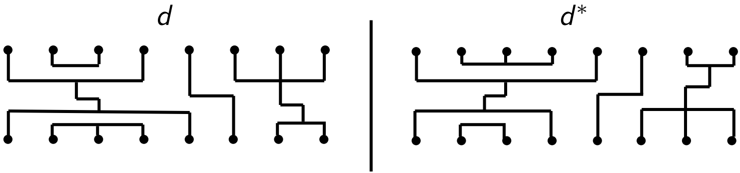


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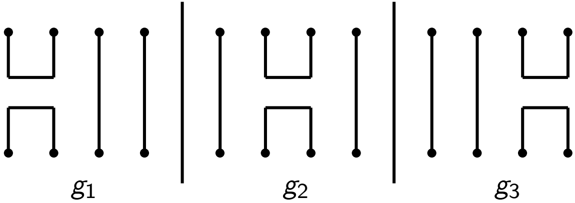
## Example

Three generators of $\mathcal{J}_4$.




We call the **hooks** in the generators **diapses**.

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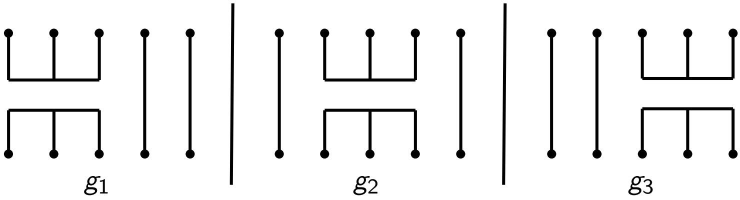

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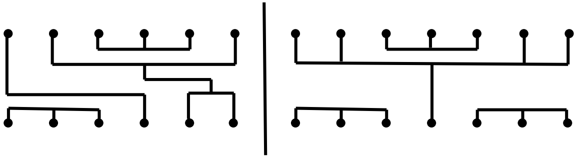

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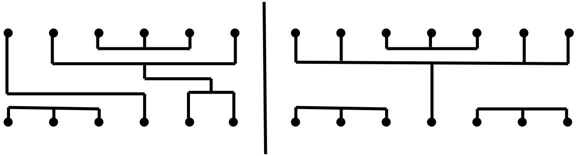

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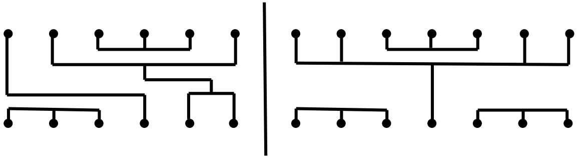

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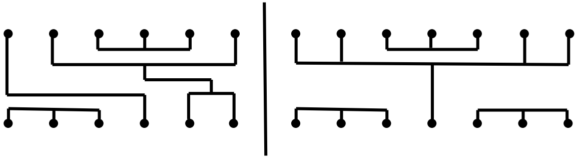

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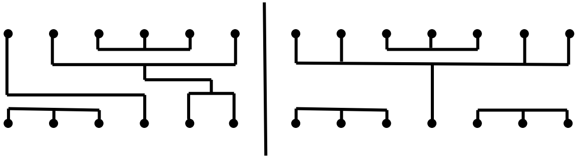

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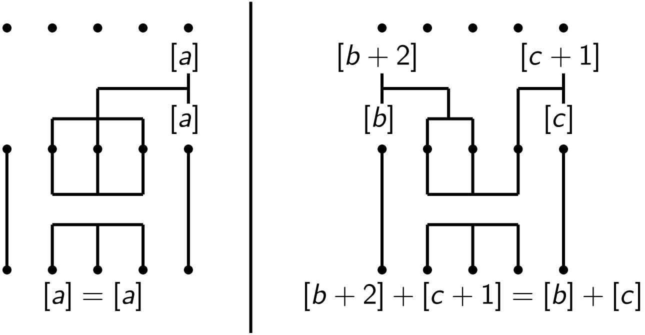

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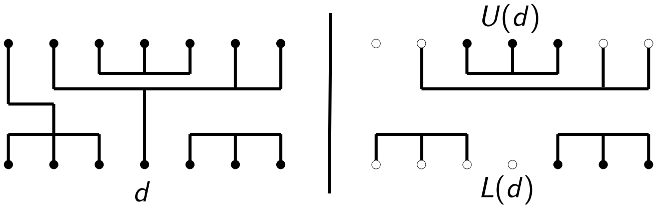


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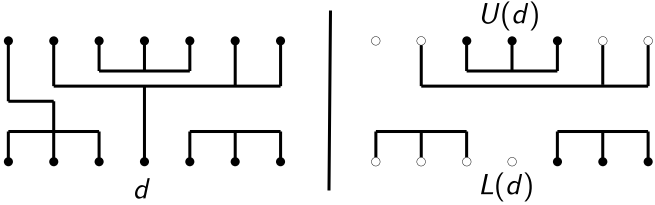


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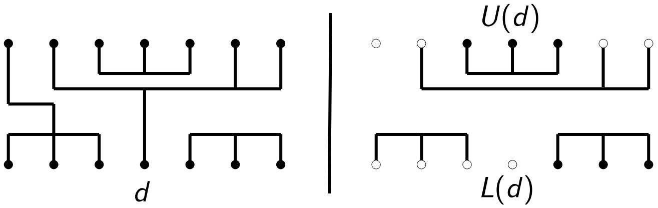


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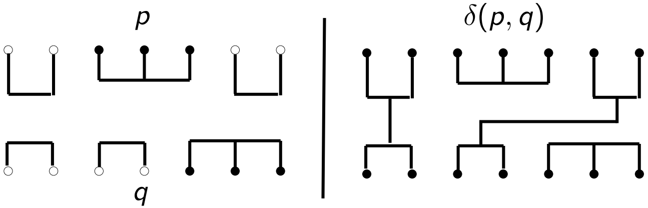

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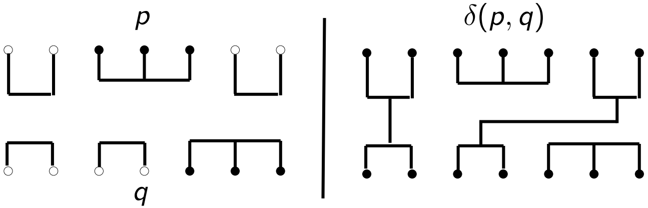


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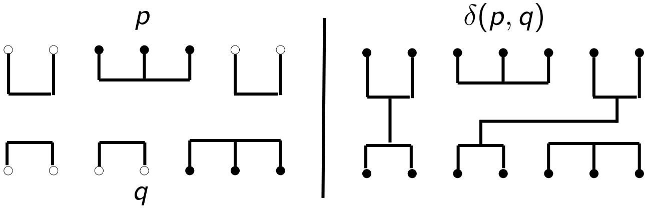


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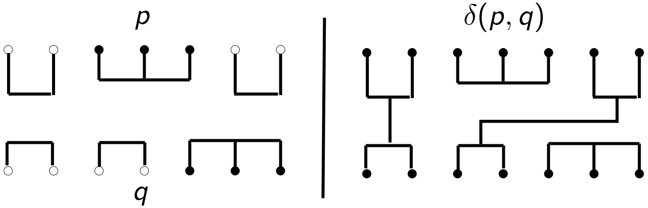

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## Theorem (Howie)

If $T \leq S$ is regular then Green's $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ relations are just their respective restrictions on $S$, ie. $\mathcal{L}^T = \mathcal{L}^S \cap T^2$.

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- $a\mathcal{J}b$ iff $rank(a) = rank(b)$.

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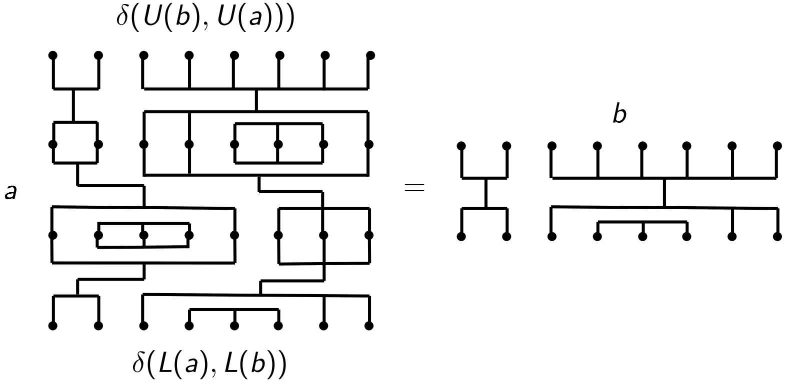

Questions?