A Yang-Baxter equation from sutured Floer homology

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Outline

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  - Jones and Alexander polynomials
  - Khovanov homology
  - Floer homology
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  - Floer homology
  - Categorification
- Indicate how a generalised Yang-Baxter equation is found in sutured Floer homology, further tying this story together.
  - Generalised to "Higher genus"
  - Generalised to "Higher dimension"

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E.g. take $V = \mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ and

$$R = \begin{pmatrix} 1+u & & & \\ & u & 1 & \\ & 1 & u & \\ & & & 1+u \end{pmatrix}$$

w.r.t. basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$.

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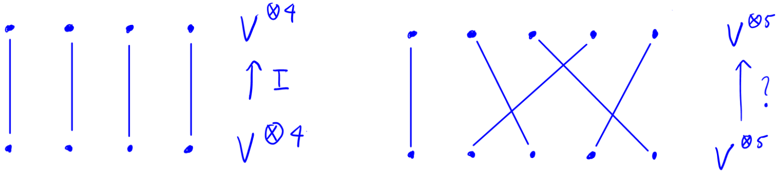
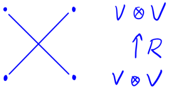
Represent V by a point, $V^{\otimes n}$ by n points, maps $V^{\otimes n} \longrightarrow V^{\otimes n}$ by lines between them.



Origin and meaning of the Yang-Baxter equation

Think of $R : V \otimes V \longrightarrow V \otimes V$ as representing an *interaction*



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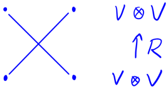

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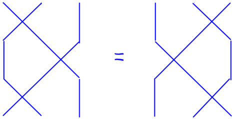

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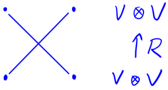

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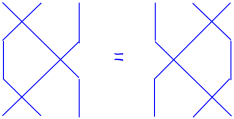

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Evolutions of system are equivalent if isotopic as *braids*.

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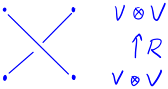

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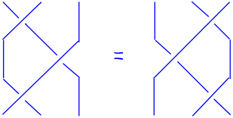

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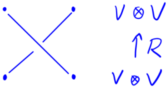

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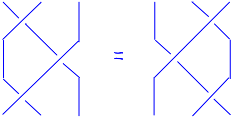

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Evolutions of system are equivalent if isotopic as *braids*.
The group of *braids* on n strands has a presentation

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{rl} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, \ldots, n-1 \end{array} \right\rangle.$$

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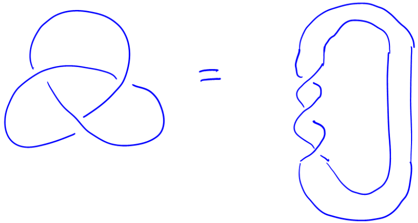
Any knot is the closure of a braid, and it turns out we can obtain *knot invariants* also.



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- $U(\mathfrak{g})$ has a presentation (Serre 1965) over $\mathbb{C}$ with $3n$ generators

$$X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, H_1, H_2, \ldots, H_n$$

and relations

$$[H_i, H_j] = 0, \quad [X_i, Y_j] = \delta_{ij} H_i,$$
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some others...

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A large and interesting source of solutions to the Yang-Baxter equation comes from *quantum groups*.

- Given semisimple $\mathfrak{g}$ and enveloping $U(\mathfrak{g})$, take its *quantum enveloping algebra* or *quantum group* $U_q(\mathfrak{g})$.
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- In quantum groups we find things like *quantum integers* $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+1}$.

A quantum group

A quantum group we're interested in: $U_q\mathfrak{sl}(1|1)$.

$$U_q(\mathfrak{sl}(1|1)) = \mathbb{Q}(q)\left\langle E, F, H^{\pm 1} \;\middle|\; \begin{array}{c} E^2 = F^2 = 0, \\ EH = HE, FH = HF, \\ EF + FE = \frac{H - H^{-1}}{q - q^{-1}} \end{array} \right\rangle$$

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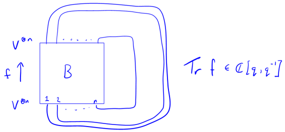

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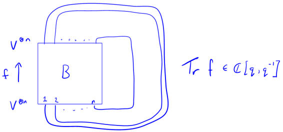

Taking simple $\mathfrak{g}$, V gives well-known knot invariants.

$\mathfrak{g}$	V	Invariant	
$\mathfrak{sl}(2)$	V_2	Jones polynomial	(Witten 1989, Reshetikhin-Turaev 1990)
$\mathfrak{sl}(2)$	V_n	Coloured Jones	(Turaev 1994, Melvin-Morton 1995)
$\mathfrak{sl}(1\|1)$	V_2	Alexander	(Kauffman-Saleur 1991)

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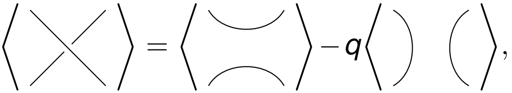

$$\langle \bigcirc L \rangle = (q+q^{-1})\langle L \rangle$$

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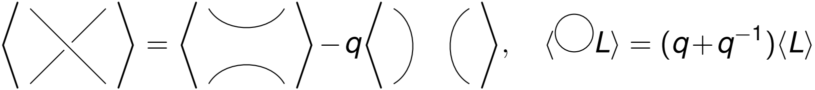

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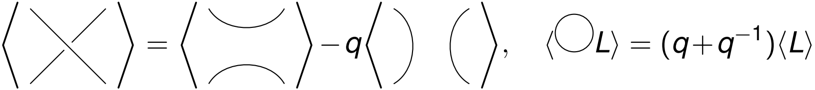

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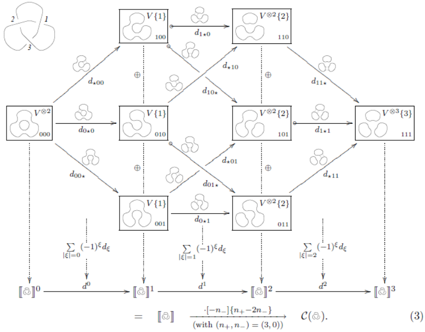
- Khovanov (late 1990s) took this idea to much greater algebraic lengths...

Khovanov homology

Resolve crossings → arrange resolutions into cube → vertices = tensor powers of 2-dim vector space V, edges = homomorphisms based on $U_q(sl(2))$ (1+1)-dimensional TQFT → find differential → Take homology



(Source: Bar–Natan, "On Khovanov's categorification of the Jones polynomial")

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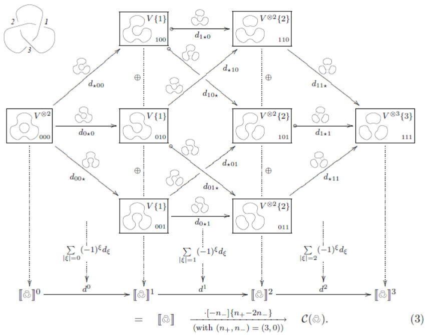

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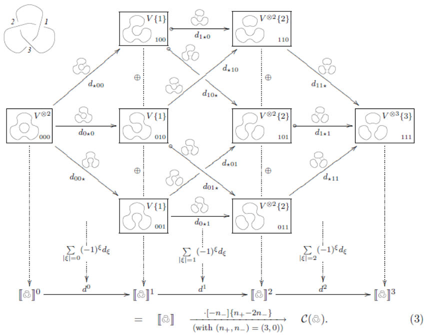

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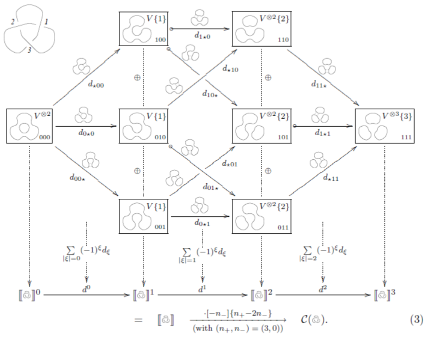

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- Its definition includes $U_q\mathfrak{sl}(2)$.
- Its *Euler characteristic* is $J(K)$:

$$\sum_j t^j \sum_i (-1)^i \dim \mathrm{Kh}_{i,j}(K) = J(K).$$

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Floer homology originated in symplectic geometry.
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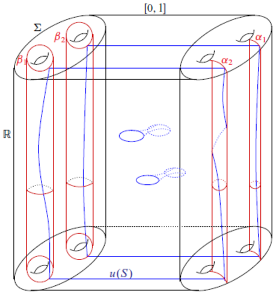

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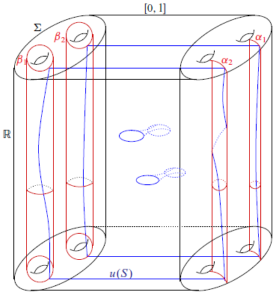

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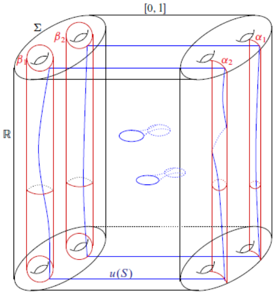
- Take chain complex generated by boundary conditions $\alpha \cap \beta$, ∂ counting holomorphic curves.



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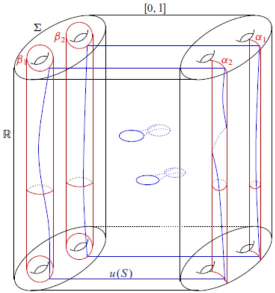

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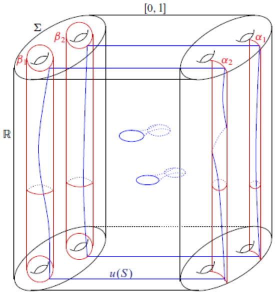

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Long-standing question:

How are Floer homology and $U_q(\mathfrak{sl}(1|1))$ related?

Outline

Sutured Floer homology of product manifolds

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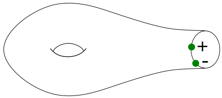

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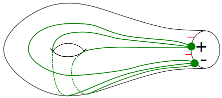

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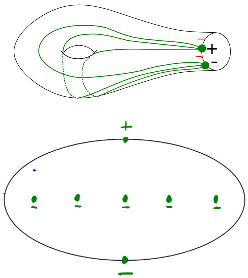

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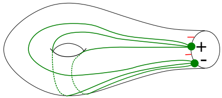

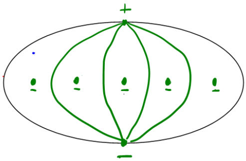

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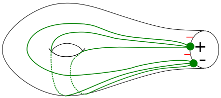

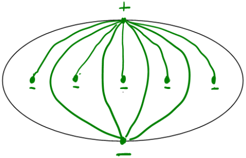

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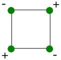
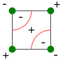
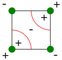
$SFH($  $\times S^1) = \mathbb{V} = \mathbb{Z}\mathbf{0} \oplus \mathbb{Z}\mathbf{1}$,  $= \mathbf{0}$,  $= \mathbf{1}$

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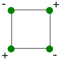
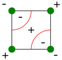
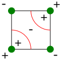
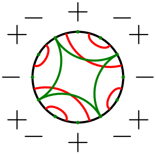
$SFH($  $\times S^1) = \mathbb{V} = \mathbb{Z}\mathbf{0} \oplus \mathbb{Z}\mathbf{1}$,  $= \mathbf{0}$,  $= \mathbf{1}$



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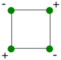
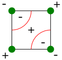
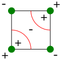
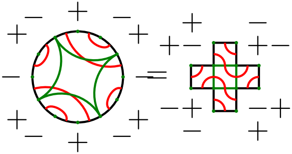
$SFH($  $\times S^1) = \mathbb{V} = \mathbb{Z}\mathbf{0} \oplus \mathbb{Z}\mathbf{1}$,  $= \mathbf{0}$,  $= \mathbf{1}$



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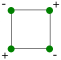
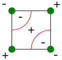
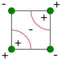
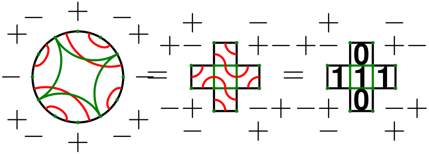
$SFH($  $\times S^1) = \mathbb{V} = \mathbb{Z}\mathbf{0} \oplus \mathbb{Z}\mathbf{1}$,  $= \mathbf{0}$,  $= \mathbf{1}$



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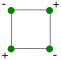
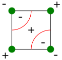
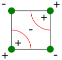
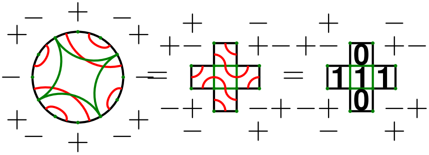
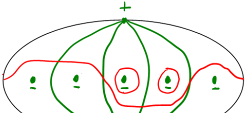
$SFH($  $\times S^1) = \mathbb{V} = \mathbb{Z}\mathbf{0} \oplus \mathbb{Z}\mathbf{1}$,  $= \mathbf{0}$,  $= \mathbf{1}$





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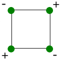
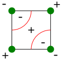
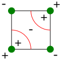
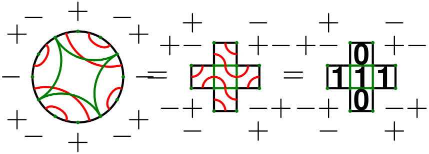
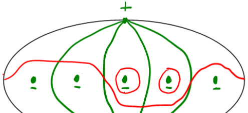
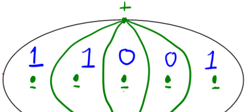
$SFH($  $\times S^1) = \mathbb{V} = \mathbb{Z}\mathbf{0} \oplus \mathbb{Z}\mathbf{1}$,  $= \mathbf{0}$,  $= \mathbf{1}$

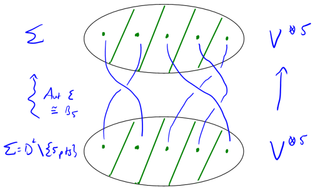






Mapping class group action

Taking (Σ, V) a punctured disc (following ideas of Tian)...



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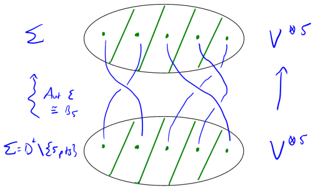

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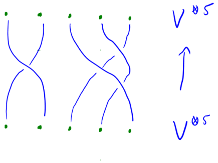

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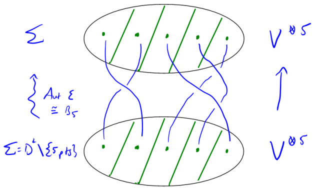

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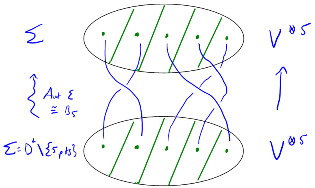

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Theorem (M.)

Let (Σ, V) be the disc with n punctures, so $MCG^+(\Sigma, V) \cong B_n$. The action of B_n on $SFH(\Sigma \times S^1, V \times S^1) \cong \mathbb{V}^{\otimes n}$ is isomorphic to the R-matrix action of $U_q\mathfrak{sl}(1|1)$ on $V_2^{\otimes n}$

So SFH obeys Yang-Baxter $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$.

Genralised Yang-Baxter

Some observations:

- Squares of surface decomposition can be regarded as fundamental representations of $U_q\mathfrak{sl}(1|1)$.
- A direct connection between $U_q\mathfrak{sl}(1|1)$ and Floer homology. (Also related work of Tian.)

Genralised Yang-Baxter

Some observations:

- Squares of surface decomposition can be regarded as fundamental representations of $U_q\mathfrak{sl}(1|1)$.
- A direct connection between $U_q\mathfrak{sl}(1|1)$ and Floer homology. (Also related work of Tian.)
- We actually obtain a *parametrised Yang-Baxter* equation

$$R_{12}(q_2)R_{23}(q_1)R_{12}(q_1) = R_{23}(q_1)R_{12}(q_1)R_{23}(q_2)$$

(as *SFH* is a $\mathbb{Z}[H_1(\Sigma)]$-module.)

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Further connections to quantum information theory, quantum gravity, statistical mechanics, representation theory, categorification, combinatorics...

Thanks for listening!

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