The University of Sydney

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH2008

Introduction to Modern Algebra

November, 2000

Time allowed: two hours

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This examination has 3 printed components:

1. *An extended answer question paper (this booklet, white, 80/33A). It has 4 pages numbered 1 to 4, with 7 questions numbered 1 to 7;*
2. *A multiple choice question paper (yellow, 80/33B). It has 4 pages numbered 1 to 4, with 15 questions numbered 1 to 15;*
3. *A multiple choice answer form (white, 80/33C, one page only).*

Components 2 and 3 must not be removed from the examination room.

Candidates should attempt both the extended answer section and the multiple choice section.

The extended answer section is worth 70% of the total marks for the paper, each of the 7 questions being worth 10%.

The multiple choice section is worth 30% of the the total marks for the paper, each of the 15 questions being worth 2%.

No notes or books are to be taken into the examination room.

1. (*i*) Write the MAGMA commands to do the following steps: define `R` to be the field of all real numbers, define `V` to be the vector space of all 5-component row vectors over the real field, define `u` and `v` to be the vectors $(1, 1, 3, -1, -2)$ and $(2, 2, 0, -3, 3)$ in `V`, define `W` to be the subspace of `V` spanned by `u` and `v`, and print out the dimension of `W`.

(*ii*) Write the code to define a MAGMA function `Length` on the space `V` defined in Part (*i*), so that for all vectors `x` in `V` the command `Length(x)` will return the length of `x`. Then write out MAGMA commands to compute the angle (in radians) between the vectors `u` and `v` as defined in Part (*i*).

(*iii*) Is the angle between the vectors `u` and `v` defined above greater than $\pi/2$ or less than $\pi/2$? Justify your answer.

(*iv*) Compute the projection of `u` onto the 1-dimensional space spanned by `v`, where `u` and `v` are as above, and draw a sketch that illustrates the relationship between `u`, `v` and this projection.

(*v*) Write out MAGMA commands that would compute the projection of `u` onto the space spanned by `v`.

2. (*i*) Find the least squares line of best fit for the four points

$$(2,3),\ (3,-1),\ (4,1),\ (5,-1).$$

(*ii*) Let $\underset{\sim}{x}$ and $\underset{\sim}{y}$ be any vectors in $\mathbb{R}^n$. Show that $\underset{\sim}{x} + 2\underset{\sim}{y}$ is orthogonal to $\underset{\sim}{x} - 2\underset{\sim}{y}$ if and only if $\|\underset{\sim}{x}\| = 2\|\underset{\sim}{y}\|$.

(*iii*) Suppose that v_1, v_2, v_3 form a basis for a subspace V of an inner product space. Write down formulas for calculating (successively) vectors u_1, u_2, u_3 that form an orthogonal basis for V, and formulas for then calculating an orthonormal basis.

3. (*i*) State the definition of the term *group*.

(*ii*) Write down MAGMA code to perform the following sequence of steps: define `S` to be the group of all permutations of $\{1, 2, 3, 4\}$, define `x` to be the element of `S` that interchanges 2 and 3 while fixing 1 and 4, define `G` to be the subgroup of `S` consisting of all elements that fix 2, and print out all the elements in the right coset of `G` containing `x`.

(*iii*) Let `x` and `G` be as in Part (*ii*) and let `y` be the 4-cycle that takes 1 to 2, 2 to 3, 3 to 4 and 4 to 1. Do `x` and `y` lie in the same right coset of `G`? Justify your answer.

4. (*i*) Let the vertices of a square be numbered 1, 2, 3, and 4, with 3 opposite 1. Write down all the permutations of $\{1, 2, 3, 4\}$ that correspond to symmetries of the square, and identify them all geometrically (using a diagram if you wish).

(*ii*) Suppose that S is a set and $\circ$ an operation on S (meaning that $a \circ b$ is an element of S whenever a and b are elements of S). Suppose also that $\circ$ is associative, and that e, f are elements of S such that e is a left identity element for $\circ$ and f is a right identity element for $\circ$. Prove that $e = f$.

5. (*i*) Let G be a group and x, y, $z \in G$ elements such that $xy = xz$. Prove that $y = z$.

(*ii*) Give an example of a group G and elements x, y, $z \in G$ such that $xy = zx$ but $y \neq z$.

(*iii*) Write down MAGMA code to perform the following sequence of steps: define `S` to be the group of all permutations of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, define `A` to be the group of all even permutations of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, define `g` to be the element $(1, 2)(3, 4, 5)(6, 7, 8, 9, 10)$ of `S`, and print out the i-th power of g for each i from 1 to 30.

(*iv*) Is the element `g` defined in Part (*iii*) in the subgroup `A`? What MAGMA code could you use to confirm your answer to this?

6. (*i*) Suppose that a MAGMA session commences with the commands

```
> G := Sym(4);
> p1 := {{1,4},{2,3}};
> parts := p1^G;
> f,L,K := Action(G,parts);
```

What are `f`, `L` and `K`?

(*ii*) Continuing from Part (*i*), if the next command is:

```
> Set(K);
```

what will MAGMA print in response?

(*iii*) Let H and K be subgroups of a group G. Prove that $H \cap K$ is also a subgroup of G.

7. Suppose that G is a group with 80 elements.

(i) What does Sylow's Theorem tell us about subgroups of G?

(ii) Use Lagrange's Theorem to show that if H and K distinct Sylow 5-subgroups of G then $H \cap K = \{e\}$.

(iii) Suppose that G exactly 16 Sylow 5-subgroups. How many elements of G do not have order 5? Justify your answer.

(iv) Prove that if G has exactly 16 Sylow 5-subgroups then it has a normal subgroup not equal to G or to the trivial subgroup $\{e\}$.

This is the last page of the extended answer section