THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION, ENGINEERING, AND SCIENCE

MATH2902
LINEAR ALGEBRA (Advanced)

Semester 1, 2000

Time allowed: Two hours
Lecturer: Steve Lack

This examination paper consists of two sections.

Section A consists of two questions and Section B consists of five questions.

All questions may be attempted.

All questions are of equal value.

Write your answers to Section A on the green sheet in the places indicated. Write your answers to Section B in the answer booklet provided. Write your name and student number on the green sheet. At the end of the exam place the green sheet inside the answer booklet.

Calculators may not be used.

No written material may be brought into the examination room.

B1. (i) The following is an incomplete list of axioms for a vector space V over a field F.

(a) $(\forall x, y, z \in V) \quad x + (y + z) = (x + y) + z$

(b) $(\forall x, y \in V) \quad x + y = y + x$

(c) $(\exists 0 \in V)(\forall x \in V) \quad x + 0 = 0 + x = x$

(d) $(\forall x \in V)(\exists x' \in V) \quad x + x' = x' + x = 0$

(e) $(\forall \lambda, \mu \in F)(\forall x \in V) \quad \lambda(\mu x) = (\lambda\mu)x$

Provide the missing axioms.

(ii) Prove, using only the vector space axioms, that $\lambda 0_V = 0_V$ for all $\lambda \in F$, where 0_V denotes the zero vector.

(iii) Hence or otherwise, show that for $\lambda \in F$ and $v \in V$, if $\lambda v = 0_V$ then $\lambda = 0$ or $v = 0_V$.

(iv) For each of the following fields F find a vector space V over F and a non-zero vector $v \in V$ such that $v = -v$, or explain why this is impossible: (a) $F = \mathbb{C}$, (b) $F = \mathbb{Z}_2$, (c) $F = \mathbb{Z}_3$.

[2+2+3+3=10 marks]

B2. Solve the following system of differential equations

$$\begin{aligned} x' &= x + 2z \\ y' &= \ \ y + z \\ z' &= -2y + 4z \end{aligned}$$

where $x = x(t)$, $y = y(t)$, $z = z(t)$ are differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.

[10 marks]

B3. Let $tr : \mathrm{Mat}_{2,2}(\mathbb{Z}_3) \to \mathbb{Z}_3$ be the function sending a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $a + d$, the trace of the matrix.

(i) Prove that tr is a linear transformation;

(ii) What is the image of tr?

(iii) State the rank-nullity theorem for linear transformations, and use it to calculate the dimension of the kernel of tr;

(iv) Find a basis for the kernel of tr;

(v) How many bases for $\mathrm{Mat}_{2,2}(\mathbb{Z}_3)$ contain your basis for the kernel of tr?

[2+1+2+2+3=10 marks]

B4. (i) What does it mean for matrices to be similar?

(ii) Prove that similar matrices have the same eigenvalues.

(iii) Let V be the subspace of $\mathbb{R}^{\mathbb{R}}$ spanned by $B = (e^x, xe^x, x^2e^x)$. Since B is linearly independent it is an (ordered) basis for V. Let $T : V \to V$ be differentiation. Find the matrix $[T]_B^B$.

(iv) Does V have a basis D for which $[T]_D^D$ is diagonal? Explain briefly.

(v) Find a basis D for which $[T]_D^D$ is in Jordan form.

[1+2+2+3+2=10 marks]

B5. Let V be an inner product space with inner product $\langle\ ,\ \rangle$.

(i) What does it mean for vectors u and v to be orthogonal?

(ii) Let $X = \{v_1, \dots, v_n\}$ be a pairwise orthogonal set of vectors in V. Show that X is a linearly independent subset of V.

(iii) If now V is finite-dimensional and $\langle v_i, v_i\rangle = 1$ for $i = 1, \dots, n$, show that there is an orthonormal basis for V containing X. You may state without proof theorems regarding bases for vector spaces (without inner product).

(iv) Give an example of an infinite-dimensional inner product space. You should describe briefly the vector space operations and the inner product but you need not verify any of the axioms or the fact that the vector space is infinite dimensional.

[1+3+4+2=10 marks]

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