THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 4

1. Use Theorem 3.13 to prove that the solution set of the system of equations

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is a subspace of $\mathbb{R}^3$.

Solution.

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for all x, y, $z \in \mathbb{R}$. As we have seen in lectures, as well as in Tutorial 3, multiplication by a $m \times n$ matrix over F is always a linear transformation from F^n to F^m, and so T defined above is linear. By Theorem 3.9 the kernel of T must be a subspace of $\mathbb{R}^3$, and hence must be a vector space. But by definition of the kernel,

$$\ker T = \{\, v \in \mathbb{R}^3 \mid T(v) = 0 \,\},$$

which is exactly the solution set of the given system of equations.

See also §3b#9 of the book.

2. (*i*) Let A be an $n \times n$ matrix over a field F and let λ be an arbitrary element of F. The λ-*eigenspace* of A is defined to be the set of all $v \in F^n$ such that $Av = \lambda v$. Prove that the λ-eigenspace is a subspace of F^n, and is nonzero if and only if λ is an eigenvalue of A.

(*ii*) Calculate the 1-eigenspace of $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Solution.

(*i*) We must prove that the λ-eigenspace of A is nonempty and closed under addition and scalar multiplication.

First of all, since $A\underset{\sim}{0} = \underset{\sim}{0} = \lambda\underset{\sim}{0}$ it follows that $\underset{\sim}{0}$, the zero n-tuple, is in the eigenspace. Hence the eigenspace is nonempty.

Let u and v be arbitrary elements of the eigenspace. Then by the definition we have $Au = \lambda u$ and $Av = \lambda v$, and by elementary properties of matrix multiplication it follows that

$$A(u + v) = Au + Av = \lambda u + \lambda v = \lambda(u + v),$$

whence $u + v$ is also in the eigenspace.

Let v be an arbitrary element of the eigenspace and let α be an arbitrary scalar. Then

$$A(\alpha v) = \alpha(Av) = \alpha(\lambda v) = (\alpha\lambda)v = (\lambda\alpha)v = \lambda(\alpha v).$$

Hence αv is in the eigenspace.

Note that we could have alternatively used the same method as in Exercise 1: the λ-eigenspace of A is the kernel of the linear transformation $T\colon F^n \to F^n$ defined by $T(v) = (A - \lambda I)v$.

We have proved that the eigenspace is a subspace of F^n. It is quite possible that this subspace consists of the zero element alone—the set $\{0\}$ is always a subspace. By definition λ is an eigenvalue of A if and only if there is a nonzero v satisfying $Av = \lambda v$; that is, if and only if the λ-eigenspace contains a nonzero element.

(*ii*) We must solve the equations

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If we let y and z take the arbitrary values α and β then we see that $x = -\alpha - \beta$ solves the system, and we deduce that the general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus the 1-eigenspace of the given matrix is the span of the two columns ${}^{t}(\,-1 \quad 1 \quad 0\,)$ and ${}^{t}(\,-1 \quad 0 \quad 1\,)$.

3. (*i*) Is $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ in the column space of $\begin{pmatrix} 1 & -3 & -4 \\ 5 & -14 & -13 \\ 2 & -2 & 20 \end{pmatrix}$?

(*ii*) Is $(1,1,1,1)$ in $\mathrm{Span}\big((5,-7,2,-13),(-3,5,-1,9)\big)$?

Solution.

(*i*) In view of 3.20.1 (page 74) of the text, the question can be rephrased as follows: do the equations

$$\begin{pmatrix} 1 & -3 & -4 \\ 5 & -14 & -13 \\ 2 & -2 & 20 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

have a solution? To find out, we apply row operations to the augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & -3 & -4 & 1 \\ 5 & -14 & -13 & 3 \\ 2 & -2 & 20 & -2 \end{array}\right) \xrightarrow[R_3:=R_3-2R_1]{R_2:=R_2-5R_1} \left(\begin{array}{ccc|c} 1 & -3 & -4 & 1 \\ 0 & 1 & 7 & -2 \\ 0 & 4 & 28 & -4 \end{array}\right)$$

$$\xrightarrow{R_3:=R_3-4R_2} \left(\begin{array}{ccc|c} 1 & -3 & -4 & 1 \\ 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

We have derived the equation $0x+0y+0z = 4$, which is clearly impossible to solve. Hence $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ is not in the column space of $\begin{pmatrix} 1 & -3 & -4 \\ 5 & -14 & -13 \\ 2 & -2 & 20 \end{pmatrix}$.

(*ii*) Again the question is whether there is a solution to a system of equations; in this case the equations are

$$x(5,-7,2,-13) + y(-3,5,-1,9) = (1,1,1,1),$$

or, in matrix notation,

$$\begin{pmatrix} 5 & -3 \\ -7 & 5 \\ 2 & -1 \\ -13 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Form the augmented matrix and use row operations.

$$\left(\begin{array}{cc|c} 5 & -3 & 1 \\ -7 & 5 & 1 \\ 2 & -1 & 1 \\ -13 & 9 & 1 \end{array}\right) \xrightarrow[\substack{R_3:=R_3-(2/5)R_1 \\ R_4:=R_4+(13/5)R_1}]{R_2:=R_2+(7/5)R_1} \left(\begin{array}{cc|c} 5 & -3 & 1 \\ 0 & 4/5 & 12/5 \\ 0 & 1/5 & 3/5 \\ 0 & 6/5 & 18/5 \end{array}\right)$$

$$\xrightarrow[R_4:=R_4-(3/2)R_2]{R_3:=R_3-(1/4)R_2} \left(\begin{array}{cc|c} 5 & -3 & 1 \\ 0 & 4/5 & 12/5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So this system of equations is consistent. Indeed, $x = 2$ and $y = 3$ is a solution. Thus $(1,1,1,1)$ is in the space spanned by $(5,-7,2,-13)$ and $(-3,5,-1,9)$.

4. Suppose that (v_1, v_2, v_3) is a basis for a vector space V, and define elements $w_1,\ w_2,\ w_3 \in V$ by $w_1 = v_1 - 2v_2 + 3v_3$, $w_2 = -v_1 + v_3$, $w_3 = v_2 - v_3$.

(*i*) Express $v_1,\ v_2,\ v_3$ in terms of $w_1,\ w_2,\ w_3$.

(*ii*) Prove that $w_1,\ w_2,\ w_3$ are linearly independent.

(*iii*) Prove that $w_1,\ w_2,\ w_3$ span V.

Solution.

(*i*) We have

$$\begin{pmatrix} 1 & -2 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

and performing the same row operations on both coefficient matrices will preserve the equality.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2:=R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 4 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_3:=R_3+2R_2 \\ R_1:=R_1+2R_2}]{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow[\substack{R_2:=R_2+R_3 \\ R_1:=R_1-R_3}]{R_3:=(1/2)R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 & 2 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1 \end{array}\right)$$

Thus we have shown that

$$\begin{aligned} v_1 &= (1/2)w_1 - (1/2)w_2 + w_3 \\ v_2 &= (1/2)w_1 + (1/2)w_2 + 2w_3 \\ v_3 &= (1/2)w_1 + (1/2)w_2 + w_3. \end{aligned}$$

(*ii*) Assume that $\lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 = 0$. Using the given expressions for the w_i in terms of the v_i gives

$$(\lambda_1 - \lambda_2)v_1 + (-2\lambda_1 + \lambda_3)v_2 + (3\lambda_1 + \lambda_2 - \lambda)v_3 = 0,$$

and since the v_i are linearly independent all the coefficients are zero. In matrix notation this gives

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} A = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

where
$$A = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$
By our row operations in (i) above we know that
$$B = \begin{pmatrix} 1/2 & -1/2 & 1 \\ 1/2 & 1/2 & 2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$$
is the inverse of A, and we deduce that
$$\begin{aligned}(\,\lambda_1 \quad \lambda_2 \quad \lambda_3\,) &= (\,\lambda_1 \quad \lambda_2 \quad \lambda_3\,)\,AB \\ &= (\,0 \quad 0 \quad 0\,)\,B \\ &= (\,0 \quad 0 \quad 0\,).\end{aligned}$$
Hence the w_i are linearly independent.

(iii) Let $v \in V$. Since the v_i span V there exist scalars λ_1, λ_2, λ_3 such that $v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$, and substituting our expressions for the v_i in terms of the w_i gives $v = \mu_1 w_1 + \mu_2 w_2 + \mu_3 w_3$ where
$$(\,\mu_1 \quad \mu_2 \quad \mu_3\,) = (\,\lambda_1 \quad \lambda_2 \quad \lambda_3\,)\,B.$$

5. Let V and W be vector spaces and let $T\colon V \to W$ be a linear transformation.

(i) Prove that if T is injective and $v_1, v_2, \ldots, v_n \in V$ are linearly independent then $T(v_1), T(v_2), \ldots, T(v_n)$ are linearly independent.

(ii) Prove that if T is surjective and $v_1, v_2, \ldots, v_n$ span V then $T(v_1), T(v_2), \ldots, T(v_n)$ span W.

Solution.

(i) Suppose that T is injective and $v_1, v_2, \ldots, v_n$ are linearly independent. Assume that
$$(*) \qquad \lambda_1 T(v_1) + \lambda_2 T(v_2) + \cdots + \lambda_n T(v_n) = 0.$$
We see that
$$\begin{aligned}T(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n) &= \lambda_1 T(v_1) + \lambda_2 T(v_2) + \cdots + \lambda_n T(v_n) \\ &= 0 = T(0)\end{aligned}$$
and since T is injective it follows that $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$. By linear independence of $v_1, v_2, \ldots, v_n$ we get $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. So the only solution of $(*)$ is the trivial solution, and therefore $T(v_1), T(v_2), \ldots, T(v_n)$ are linearly independent.

(ii) Suppose that T is surjective and $v_1, v_2, \ldots, v_n$ span V.

Let $w \in W$. Since T is surjective there exists $v \in V$ with $w = T(v)$. Since $v_1, v_2, \ldots, v_n$ span V we have $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$ for some scalars λ_i. Now
$$w = T(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n) = \lambda_1 T(v_1) + \lambda_2 T(v_2) + \cdots + \lambda_n T(v_n)$$
and we have shown that every element of W is expressible as a linear combination of $T(v_1), T(v_2), \ldots, T(v_n)$. Thus these elements span W.

6. Determine whether or not the following two subspaces of $\mathbb{R}^3$ are the same:
$$\text{Span}\left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}\right) \quad \text{and} \quad \text{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}\right).$$

Solution.

Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$, $w_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $w_2 = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$. By solving simultaneous equations we find that
$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = (-7/3)\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + (5/3)\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$
and since this gives
$$\lambda_1 w_1 + \lambda_2 w_2 = ((-7/3)\lambda_1 + 4\lambda_2)v_1 + ((5/3)\lambda_1 - \lambda_2)v_2$$
it follows that every linear combination of w_1 and w_2 is also a linear combination of v_1 and v_2. That is, if $T = \text{span}(w_1, w_2)$ and $S = \text{span}(v_1, v_2)$ then $T \subseteq S$. Similarly, if we can express v_1 and v_2 as linear combinations of w_1 and w_2 it will follow that $S \subseteq T$. Solving equations again gives
$$\begin{aligned}v_1 &= (3/13)w_1 + (5/13)w_2 \\ v_2 &= (8/13)w_1 + (9/13)w_2.\end{aligned}$$
Hence it is indeed true that $S \subseteq T$, and so $S = T$.