THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 12

1. Let A be an $n \times n$ matrix whose rank is less than n. Prove that 0 is an eigenvalue of A.

Solution.

Since the rank of A plus the nullity of A is n, the assumption that the rank is less than n gives that the nullity is nonzero. Hence the (right) null space of A contains a nonzero vector. If $\underset{\sim}{v}$ is any such, then $A\underset{\sim}{v} = \underset{\sim}{0} = 0\underset{\sim}{v}$, which shows that v is an eigenvector of A with 0 the corresponding eigenvalue.

2. Let V be a vector space and S and T subspaces of V such that $V = S \oplus T$. Prove or disprove the following assertion:

If U is any subspace of V then $U = (U \cap S) \oplus (U \cap T)$.

Solution.

Let $V = \mathbb{R}^2$, let S be the set of all scalar multiples of $\begin{pmatrix}1\\0\end{pmatrix}$ and let T be the set of all scalar multiples of $\begin{pmatrix}0\\1\end{pmatrix}$. Then $V = S \oplus T$. (We have chopped the standard basis of $\mathbb{R}^2$ into two pieces and defined S and T to be the spaces spanned by these pieces.) Now if we define U to be the set of all scalar multiples of $\begin{pmatrix}1\\1\end{pmatrix}$ we see that $U \cap S$ and $U \cap T$ both consist of the zero vector alone, and so it is certainly not true that $U = (U \cap S) \oplus (U \cap T)$.

3. (*i*) Let A, B and C be $n \times n$ matrices, and suppose that the column space of B equals the column space of C. Prove that the column space of AB equals that of AC. (Hint: Use Proposition 7.16 of the text.)

(*ii*) Let A be an $n \times n$ matrix and suppose that the rank of A^4 is the same as the rank of A^3. Prove that A^5 and all higher powers of A also have this same rank. (Hint: Apply Part (*i*) with $B = A^3$ and $C = A^4$.)

Solution.

(*i*) Proposition 7.16 says that $u \mapsto Au$ defines a surjective map from CS(B) to CS(AB). Hence

$$\text{(1)} \qquad \mathrm{CS}(AB) = \{\, Au \mid u \in \mathrm{CS}(B)\,\}.$$

Applying the same proposition with C in place of B gives

$$\text{(2)} \qquad \mathrm{CS}(AC) = \{\, Au \mid u \in \mathrm{CS}(C)\,\}.$$

Since $\mathrm{CS}(B) = \mathrm{CS}(C)$ the right hand side of (1) equals the right hand side of (2), and so the left hand sides are equal too, as required.

(*ii*) Recall that the rank of a matrix is the dimension of its column space. Thus we are given that the column spaces of A^3 and A^4 have the same dimension. But Proposition 3.21 tell us that the column space of A^4 is contained in the column space of A^3. (You can see it as follows. The equation $A^4 = A^3A$ shows that the j^{th} column of A^4 is A^3a_j, where a_j is the j^{th} column of A. Hence the columns of A^4 are linear combinations of the columns of A^3.) A subspace whose dimension equals that of the whole space must equal the whole space (see Proposition 4.11), and so we can conclude that $\mathrm{CS}(A^4) = \mathrm{CS}(A^3)$. Now Part (*i*) gives $\mathrm{CS}(A^5) = \mathrm{CS}(A^4)$, and applying it again gives $\mathrm{CS}(A^6) = \mathrm{CS}(A^5)$, and so on. Taking dimensions gives the result.

4. Let V and W be vector spaces over the field F and let $\boldsymbol{b} = (v_1, v_2, \ldots, v_n)$ and $\boldsymbol{c} = (w_1, w_2, \ldots, w_m)$ be bases of V and W respectively. Let $\mathrm{L}(V,W)$ be the set of all linear transformations from V to W, and let $\mathrm{Mat}(m \times n, F)$ be the set of all $m \times n$ matrices over F. We know that $\mathrm{Mat}(m \times n, F)$ is a vector space over F, and we have seen in Question 3 of Tutorial 5 that $\mathrm{L}(V,W)$ is too. Let $\Omega\colon \mathrm{L}(V,W) \to \mathrm{Mat}(m \times n, F)$ be the function defined by $\Omega(\theta) = \mathrm{M}_{\boldsymbol{cb}}(\theta)$ for all $\theta \in \mathrm{L}(V,W)$.

(*i*) Prove that Ω is a linear transformation. (Hint: The task is to prove that $\mathrm{M}_{\boldsymbol{cb}}(\phi+\theta) = \mathrm{M}_{\boldsymbol{cb}}(\phi) + \mathrm{M}_{\boldsymbol{cb}}(\theta)$ and $\mathrm{M}_{\boldsymbol{cb}}(\lambda\phi) = \lambda\,\mathrm{M}_{\boldsymbol{cb}}(\phi)$. Now the j^{th} column of $\mathrm{M}_{\boldsymbol{cb}}(\phi+\theta)$ is $\mathrm{cv}_{\boldsymbol{c}}((\phi+\theta)(v_j))$ while the j^{th} columns of $\mathrm{M}_{\boldsymbol{cb}}(\phi)$ and $\mathrm{M}_{\boldsymbol{cb}}(\theta)$ are $\mathrm{cv}_{\boldsymbol{c}}(\phi(v_j))$ and $\mathrm{cv}_{\boldsymbol{c}}(\theta(v_j))$. Use the definition of $\phi+\theta$ and fact that $x \mapsto \mathrm{cv}_{\boldsymbol{c}}(x)$ is linear to prove that the j^{th} column of $\mathrm{M}_{\boldsymbol{cb}}(\phi+\theta)$ is the sum of the j^{th} columns of $\mathrm{M}_{\boldsymbol{cb}}(\phi)$ and $\mathrm{M}_{\boldsymbol{cb}}(\theta)$.)

(*ii*) Prove that the kernel of Ω is $\{z\}$, where $z\colon V \to W$ is the zero function.

(*iii*) Prove that Ω is a vector space isomorphism. (Hint: By the first two parts we know that Ω is linear and injective; so surjectivity is all that remains. That is, given a $m \times n$ matrix M we must show that there is a linear transformation θ from V to W having M as its matrix. Now the coefficients of M determine what $\theta(v_i)$ has to be for each i, and Theorem 4.18 guarantees that such a linear transformation exists.)

(*iv*) Find a basis for $\mathrm{L}(V,W)$. (Hint: (Find a basis of $\mathrm{Mat}(m \times n, F)$ first. The corresponding linear transformations will give the desired basis of $\mathrm{L}(V,W)$.)

Solution.

Let ϕ, $\theta \in \mathrm{L}(V,W)$ and let $\lambda \in F$. For each j (from 1 to n) we have

$$\begin{aligned}\mathrm{cv}_{\boldsymbol{c}}((\phi+\theta)(v_j)) &= \mathrm{cv}_{\boldsymbol{c}}(\phi(v_j)+\theta(v_j)) \quad \text{(by definition of } \phi+\theta\text{)}\\ &= \mathrm{cv}_{\boldsymbol{c}}(\phi(v_j)) + \mathrm{cv}_{\boldsymbol{c}}(\theta(v_j))\end{aligned}$$

since the mapping $\mathrm{cv}_{\boldsymbol{c}}\colon W \to F^m$ given by $x \mapsto \mathrm{cv}_{\boldsymbol{c}}(x)$ an isomorphism. Hence the j^{th} column of $\mathrm{M}_{\boldsymbol{cb}}(\phi+\theta)$ is the sum of the j^{th} columns of $\mathrm{M}_{\boldsymbol{cb}}(\phi)$ and $\mathrm{M}_{\boldsymbol{cb}}(\theta)$; hence $\mathrm{M}_{\boldsymbol{cb}}(\phi+\theta) = \mathrm{M}_{\boldsymbol{cb}}(\phi)+\mathrm{M}_{\boldsymbol{cb}}(\theta)$. That is, $\Omega(\phi+\theta) = \Omega(\phi)+\Omega(\theta)$. Similarly, for each j the j^{th} column of $\mathrm{M}_{\boldsymbol{cb}}(\lambda\phi)$ is $\mathrm{cv}_{\boldsymbol{c}}((\lambda\phi)(v_j))$, which equals $\mathrm{cv}_{\boldsymbol{c}}(\lambda(\phi(v_j))) = \lambda\,\mathrm{cv}_{\boldsymbol{c}}(\phi(v_j))$ (by definition of $\lambda\phi$ and linearity of the mapping $\mathrm{cv}_{\boldsymbol{c}}$). Hence $\mathrm{M}_{\boldsymbol{cb}}(\lambda\phi) = \lambda\,\mathrm{M}_{\boldsymbol{cb}}(\phi)$; that is, $\Omega(\lambda\phi) = \lambda\Omega(\phi)$. This proves (i).

The kernel of Ω is the set of all ϕ in $\mathrm{L}(V,W)$ such that $\mathrm{M}_{\boldsymbol{cb}}(\phi)$ is the zero matrix. The fact that Ω is linear guarantees that z, the zero of $\mathrm{L}(V,W)$, is in the kernel. If ϕ is an arbitrary element of the kernel then $\mathrm{cv}_{\boldsymbol{c}}(\phi(v_j)) = 0$ for each j, since $\mathrm{cv}_{\boldsymbol{c}}(\phi(v_j))$ is the j^{th} column of $\mathrm{M}_{\boldsymbol{cb}}(v_j)$. Since $\mathrm{cv}_{\boldsymbol{c}}$ is an isomorphism we deduce that $\phi(v_j) = 0$ for all j, and it follows by linearity of ϕ that $\phi(v) = 0$ for all $v \in V$. That is, $\phi = z$. So z is the only element of the kernel.

Let $A \in \mathrm{Mat}(m \times n, F)$ be arbitrary and for each j let $\alpha_j \in F^m$ be the j^{th} column of A. Since $\mathrm{cv}_{\boldsymbol{c}}$ is an isomorphism there exist $w_j \in W$ such that $\mathrm{cv}_{\boldsymbol{c}}(w_j) = \alpha_j$, and since linear transformations can be defined arbitrarily on a basis (Theorem 4.18) there is a $\phi \in \mathrm{L}(V,W)$ such that $\phi(v_j) = w_j$ for each j. Clearly now $\mathrm{M}_{\boldsymbol{cb}}(\phi) = A$; that is, $\Omega(\phi) = A$. So Ω is surjective.

If E_{kl} is the matrix in $\mathrm{Mat}(m \times n, F)$ which has 1 as the $(k,l)^{\text{th}}$ entry and zeros elsewhere then the matrices $(\, E_{kl} \mid 1 \le k \le m, 1 \le l \le n\,)$ form a basis for $\mathrm{L}(V,W)$. In fact if $A \in \mathrm{Mat}(m \times n, F)$ has $(i,j)^{\text{th}}$ entry α_{ij} then $A = \sum \alpha_{ij} E_{ij}$, and this is the unique way of expressing A as a linear combination of the E_{kl}. It is a general fact that an isomorphism of vector spaces will map a basis of one space to a basis of the other. (See Theorem 4.19 and Exercise 3 of Tutorial 4.) So to find a basis of $\mathrm{L}(V,W)$ it suffices to find linear transformations $\phi_{kl}\colon V \to W$ such that $\Omega(\phi_{kl}) = E_{kl}$ (for $1 \le k \le m$, $1 \le l \le n$). By Theorem 4.18 we know that (for each k and l) there is a linear transformation ϕ_{kl} satisfying

$$\phi_{kl}(v_j) = \begin{cases} 0 & \text{if } j \ne l \\ w_k & \text{if } j = l \end{cases}$$

and, by the definition of the matrix of a linear transformation, we see that the matrix of ϕ_{kl} relative to $\boldsymbol{b}$ and $\boldsymbol{c}$ has its j^{th} column equal to zero unless $j = l$, while the l^{th} column is $\mathrm{cv}_{\boldsymbol{c}}(v_k)$, which has 1 as its k^{th} entry and all other entries zero. Thus $\mathrm{M}_{\boldsymbol{cb}}(\phi_{kl}) = E_{kl}$, and, by the remarks above, $(\, \phi_{kl} \mid 1 \le k \le m,\ 1 \le l \le n\,)$ is a basis for $\mathrm{L}(V,W)$.