The University of Sydney

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

Semester1, 2001 Lecturer: R. Howlett

Tutorial 3

1. Which of the following functions are linear transformations?

(*i*) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(*ii*) $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(*iii*) $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ y \\ x - y \end{pmatrix}$

(*iv*) $f\colon \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = \begin{pmatrix} x \\ x + 1 \end{pmatrix}$

2. Let $\mathcal{A}$ be the set of all 2-component column vectors whose entries are differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Thus, for example, if h and k are the functions defined by $h(t) = \cos t$ and $k(t) = t^2 + 1$ for all $x \in \mathbb{R}$ then $\begin{pmatrix} h \\ k \end{pmatrix}$ is an element of $\mathcal{A}$.

(*i*) How should addition and scalar multiplication be defined so that $\mathcal{A}$ becomes a vector space over $\mathbb{R}$?

(*ii*) If f and g are real-valued functions on $\mathbb{R}$ then their *pointwise product* is the function $f \cdot g$ defined by $(f \cdot g)(t) = f(t)g(t)$ for all $t \in \mathbb{R}$. Prove that

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto h \cdot f + g'$$

(where h is as above and g' is the derivative of g) defines a linear transformation from $\mathcal{A}$ to the space of all real-valued functions on $\mathbb{R}$.

3. Let V be a vector space and let S and T be subspaces of V.

(*i*) Prove that $S \cap T$ is a subspace of V.

(*ii*) Let $S + T = \{\, x + y \mid x \in S \text{ and } y \in T \,\}$. Prove that $S + T$ is a subspace of V.

4. Let V be a vector space over the field F and let $v_1,\ v_2, \dots, v_n$ be arbitrary elements of V. Prove that the *span* of $\{v_1, v_2, \dots, v_n\}$

$$\operatorname{Span}(v_1, v_2, \dots, v_n) = \{\, \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \mid \lambda_1,\ \lambda_2, \dots, \lambda_n \in F \,\}$$

is a subspace of V.

5. Let A and B be $n \times n$ matrices over the field F. We say that B is *similar* to A if there exists a nonsingular matrix T such that $B = T^{-1}AT$. Prove

(*i*) every $n \times n$ matrix is similar to itself,

(*ii*) if B is similar to A then A is similar to B,

(*iii*) if C is similar to B and B is similar to A then C is similar to A.