The University of Sydney

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 4

1. Use Theorem 3.13 to prove that the solution set of the system of equations

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is a subspace of $\mathbb{R}^3$.

2. (*i*) Let A be an $n \times n$ matrix over a field F and let λ be an arbitrary element of F. The λ-*eigenspace* of A is defined to be the set of all $v \in F^n$ such that $Av = \lambda v$. Prove that the λ-eigenspace is a subspace of F^n, and is nonzero if and only if λ is an eigenvalue of A.

(*ii*) Calculate the 1-eigenspace of $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

3. (*i*) Is $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ in the column space of $\begin{pmatrix} 1 & -3 & -4 \\ 5 & -14 & -13 \\ 2 & -2 & 20 \end{pmatrix}$?

(*ii*) Is $(1, 1, 1, 1)$ in $\mathrm{Span}\big((5, -7, 2, -13), (-3, 5, -1, 9)\big)$?

4. Suppose that (v_1, v_2, v_3) is a basis for a vector space V, and define elements w_1, w_2, $w_3 \in V$ by $w_1 = v_1 - 2v_2 + 3v_3$, $w_2 = -v_1 + v_3$, $w_3 = v_2 - v_3$.

(*i*) Express v_1, v_2, v_3 in terms of w_1, w_2, w_3.

(*ii*) Prove that w_1, w_2, w_3 are linearly independent.

(*iii*) Prove that w_1, w_2, w_3 span V.

5. Let V and W be vector spaces and let $T\colon V \to W$ be a linear transformation.

(*i*) Prove that if T is injective and $v_1, v_2, \ldots, v_n \in V$ are linearly independent then $T(v_1)$, $T(v_2)$, ..., $T(v_n)$ are linearly independent.

(*ii*) Prove that if T is surjective and v_1, v_2, ..., v_n span V then $T(v_1)$, $T(v_2)$, ..., $T(v_n)$ span W.

6. Determine whether or not the following two subspaces of $\mathbb{R}^3$ are the same:

$$\mathrm{Span}\left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right) \quad \text{and} \quad \mathrm{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} \right).$$