Summary of week 9 (Lectures 25, 26 and 27)

Lecture 25 and the first part of Lecture 26 were concerned with permutations. See Definitions 8.1, 8.2 and 8.3 of [VST]. The notation we use is that introduced on p. 171 of [VST]. See page 173 for examples of multiplication of permutations.

The *identity permutation* on the set $\{1, 2, \ldots, n\}$ is the identity function $\mathtt{i}\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ (defined by $\mathtt{i}(j) = j$ for all $j \in \{1, 2, \ldots, n\}$). By the properties of left and right inverses discussed in Tutorial 1 we know that every permutation has a two-sided inverse. That is, for each $\sigma \in S_n$ there is a $\sigma^{-1} \in S_n$ such that $\sigma\sigma^{-1} = \sigma^{-1}\sigma = \mathtt{i}$. To write down the inverse of a given permutation, simply swap the rows. For example, if

$$\tau = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

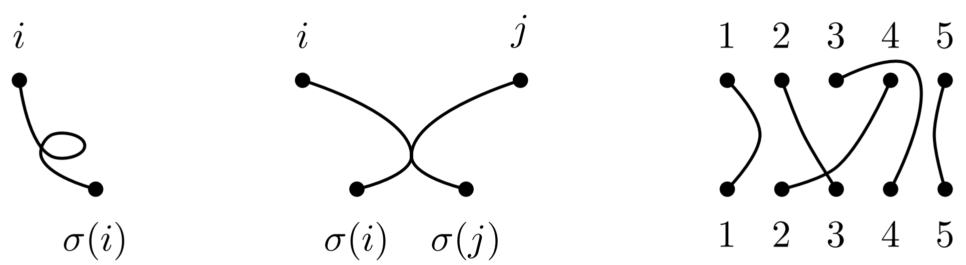
then the inverse of τ is

$$\tau^{-1} = \begin{bmatrix} 4 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix},$$

which would usually be written as

$$\tau^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{bmatrix}.$$

Our discussion of the parity of a permutation follows that used in MATH1902. A given permutation $\sigma \in S_n$ can be represented by a diagram constructed according to the following rules. Draw two horizontal rows of n dots, one underneath the other, both labelled 1, 2, $\ldots$, n from left to right. For each i draw a line joining the dot in the upper row that is labelled i to the dot in the lower row that is labelled $\sigma(i)$. The lines do not have to be straight, but they must remain within the horizontal strip whose edges are the horizontal lines containing the two rows of dots. Moreover, no line is permitted to cross itself, and two lines are not permitted to touch at a point without crossing there. Thus



are all illegitimate.

Let D be a diagram for the permutation σ, drawn so as to satisfy the above rules. As a temporary notation, let C_i denote the line from dot i in the upper row to dot $\sigma(i)$ in the lower row. Note that C_i divides the horizontal strip between the rows of dots into two pieces. Dot j in the upper row is in the left hand piece if $j < i$ and in the right hand piece if $j > i$; similarly, lower row dots with label less than $\sigma(i)$ are in the left hand piece, while those with label greater than $\sigma(i)$ are in the right hand piece. So if j is such that $i < j$ and $\sigma(i) > \sigma(j)$ then C_j starts in the right hand piece and finishes in the left hand piece; consequently, it crosses C_i (the dividing line separating the pieces) an odd number of times. On the other hand, if $i < j$ and $\sigma(i) < \sigma(j)$ then the number of times C_j crosses C_i must be even (possibly zero). Define $\mathrm{cross}(i, j)$ to be the number of times that C_i and C_j cross, and put

$$\mathrm{Crossings}(D) = \sum_{\{(i,j)\,|\,i<j\}} \mathrm{cross}(i, j), \tag{1}$$

the total number of crossings in the diagram.

We are interested in whether $\mathrm{Crossings}(D)$ is even or odd. Even terms in the sum in Eq. (1) do not affect this, and so may be ignored. Furthermore, the sum of an even number of odd numbers is even, while the sum of an odd number of odd numbers is odd. So we conclude that $\mathrm{Crossings}(D)$ is odd if and only if there are an odd number of pairs (i, j) with $i < j$ and $\sigma(i) > \sigma(j)$. Thus Definition 8.2 says that σ is odd if and only if $\mathrm{Crossings}(D)$ is odd, where D is any diagram associated with σ.

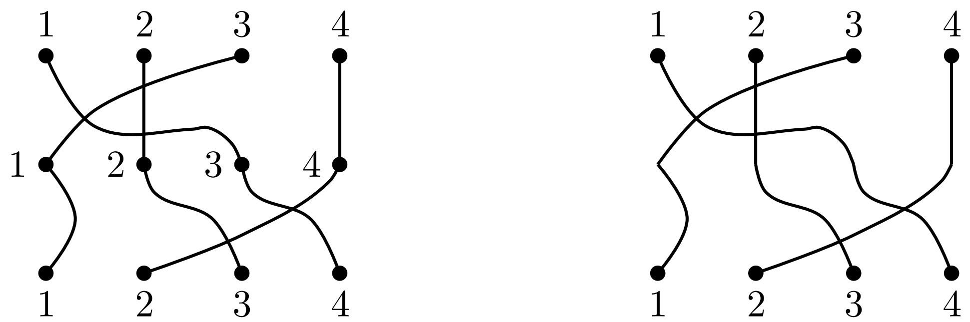
If D is a diagram associated with the permutation σ and D' a diagram associated with a permutation τ then a diagram D'' for the product $\sigma\tau$ can be obtained by identifying the upper row of σ with the lower row of τ and then removing the middle row of dots. We illustrate this for the permutations

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix} \qquad \tau = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{bmatrix},$$

noting that this gives

$$\sigma\tau = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{bmatrix}$$

as is easily checked.



Since $\mathrm{Crossings}(D'') = \mathrm{Crossings}(D') + \mathrm{Crossings}(D)$, we conclude that $\sigma\tau$ is even if σ and τ are both even or both odd, while it is odd if σ is even and τ odd or if σ is odd and τ even.

The above reasoning has proved Corollary 8.9 of [VST]: if σ, $\tau \in S_n$ then $\varepsilon(\sigma\tau) = \varepsilon(\sigma)\varepsilon(\tau)$. The material on pp. 175–177 of [VST], which gives an alternative proof of this (and a few other things) was not done in lectures.

It is clear that $\varepsilon(\mathtt{i}) = 1$, since there are no pairs (i, j) with $i < j$ and $\mathtt{i}(i) > \mathtt{i}(j)$ (since $\mathtt{i}(i) = i$ and $\mathtt{i}(j) = j$). Combined with Corollary 8.9 this shows that $\varepsilon(\sigma^{-1}) = \varepsilon(\sigma)$ for all $\sigma \in S_n$.

If i, $j \in \{1, 2, \ldots, n\}$ with $i \neq j$ then $\tau_{ij} \in S_n$ is the permutation defined by

$$\tau_{ij}(k) = \begin{cases} j & \text{if } k = i, \\ i & \text{if } k = j, \\ k & \text{if } k \neq i \text{ and } k \neq j, \end{cases}$$

for all $k \in \{1, 2, \ldots, n\}$. Permutations of this form are called *transpositions*. Now if l, $m \in \{1, 2, \ldots, n\}$ with $l \neq m$ then there are precisely $2|l - m| - 1$ pairs (i, j) with $i < j$ and $\tau_{lm}(i) > \tau_{lm}(j)$. Indeed, assuming that $l < m$—which involves no loss of generality, since $\tau_{lm} = \tau_{ml}$—the pairs (i, j) with this property are those with $i = l$ and $j \in \{l + 1, l + 2, \ldots, m - 1\}$, or with $i \in \{l + 1, l + 2, \ldots, m - 1\}$ and $j = m$, or with $i = l$ and $j = m$. It follows that transpositions are odd permutations.

The Pi notation for products is analogous to the Sigma notation for sums:

$$\prod_{i=1}^{n} a_i \stackrel{\text{def}}{=} a_1 a_2 \cdots a_n.$$

In this notation, $n! \stackrel{\text{def}}{=} \prod_{i=1}^{n} i$. Note that $n!$ is the total number of permutations of $\{1, 2, \ldots, n\}$ (since if $\sigma \in S_n$ then there are n possible values for $\sigma(1)$, for each of these there are $n - 1$ possibilities for $\sigma(2)$, then $n - 2$ possibilities for $\sigma(3)$, and so on).

The remainder of Lectures 26 and 27 dealt with determinants, following closely the discussion on pp. 179–187 of [VST]. The only difference between the treatment in the book and the treatment given in lectures is that the book mentions the *row space*, $\mathrm{RS}(A)$, of a matrix A, and the *rank* of A, two concepts that have not yet been discussed in lectures (although they will be discussed soon). The row space of $A \in \mathrm{Mat}(n \times m, F)$ is the subspace of ${}^{t}F^{m}$ spanned by the rows of A, and the rank of A is the dimension of its row space. It is fairly easy to prove (and we shall do so in lectures) that elementary row operations do not change the row space, and hence that the rank of an $n \times n$ matrix is n if and only if the matrix is expressible as a product of elementary matrices.