Induced metric

If (X, d) is a metric space and $Y \subseteq X$ we can define

$$d_Y : Y \times Y \to \mathbb{R}$$

to be the restriction of the function $d: X \times X \to \mathbb{R}$. That is,

$$d_Y(a, b) = d(a, b) \qquad \text{for all } a,\ b \in Y.$$

In words this says that distance between two points of Y is just the same as the distance between the same two points considered as points of X. The fact that d is a metric on X trivially implies that d_Y is a metric on Y. It is called the metric on Y induced by the metric on X. We say that the metric space (Y, d_Y) is a *subspace* of the metric space (X, d).

Topology on metric spaces

Let (X, d) be a metric space and $A \subseteq X$. Recall that $\mathrm{Int}(A)$ is defined to be the set of all interior points of A. It is obvious therefore that $\mathrm{Int}(A) \subseteq A$.

Lemma. *Let A be an arbitrary subset of the metric space X. Then* $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$.

Proof. As noted above, $\mathrm{Int}(B) \subseteq B$ for all $B \subseteq X$, and so putting $B = \mathrm{Int}(A)$ we deduce that $\mathrm{Int}(\mathrm{Int}(A)) \subseteq \mathrm{Int}(A)$. So we just have to prove the reverse inclusion.

Let $a \in \mathrm{Int}(A)$ be arbitrary. Choose $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq A$. We shall show that in fact $B(a, \varepsilon) \subseteq \mathrm{Int}(A)$. For, suppose that $b \in B(a, \varepsilon)$. Since $B(a, \varepsilon)$ is open, there exists $\delta > 0$ such that $B(b, \delta) \subseteq B(a, \varepsilon)$. Thus $B(b, \delta) \subseteq A$, which shows that $b \in \mathrm{Int}(A)$. As this holds for all $b \in B(a, \varepsilon)$, we have shown that $B(a, \varepsilon) \subseteq \mathrm{Int}(A)$, as claimed. However, this statement says that a is an interior point of $\mathrm{Int}(A)$, and since a was originally chosen as an arbitrary point of $\mathrm{Int}(A)$, we have shown that all points of $\mathrm{Int}(A)$ are interior points of $\mathrm{Int}(A)$, as required. □

A subset of a metric space is open if and only if every point of the set is an interior point. That is, a set is open if and only if it equals its own interior. The lemma above shows that $\mathrm{Int}(A)$ has this property for any A. Thus $\mathrm{Int}(A)$ is always an open set. Note, however, that $\mathrm{Int}(A)$ could well be empty. For example, if $X = \mathbb{R}^2$ and d is the Euclidean metric—that is, the usual metric for $\mathbb{R}^2$—and $A = \{\, (x, y) \mid x^2 + y^2 = 1 \,\}$ then $\mathrm{Int}\, A = \emptyset$ (since there is no open ball consisting of points which all lie on the circumference of the unit circle. Similarly, if $B = \{\, (x, 0) \mid x \in \mathbb{R} \,\}$ then $\mathrm{Int}(B) = \emptyset$, since there is no open ball consisting entirely of points on the X-axis. An example of a set which does have a nonempty interior is $C = \{\, (x, y) \mid -1 \le x \le 1 \text{ and } -1 \le y \le 1 \}$. In fact $\mathrm{Int}(C) = \{\, (x, y) \mid -1 < x < 1 \text{ and } -1 < y < 1 \,\}$.

It is immediate from the above discussion that $\mathrm{Int}(A)$ is the largest open set which is a subset of A. It can also be described as the union of all the open balls that are subsets of A, or indeed as the union of all the open sets which are subsets of A. It is an important property of open sets that the union of any collection of open sets is again open.

Lemma. *If A, B are subsets of X such that $A \subseteq B$ then* $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$.

Proof. Let $a \in \mathrm{Int}(A)$. Then we may choose $\varepsilon > 0$ with $B(a, \varepsilon) \subseteq A$. Since $A \subseteq B$ this gives $B(a, \varepsilon) \subseteq B$, and so $a \in \mathrm{Int}(B)$. Since a was an arbitrary point of $\mathrm{Int}(A)$ we have shown that all points of $\mathrm{Int}(A)$ are in $\mathrm{Int}(B)$; that is, $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$. □

More background

It is worth noting that a sequence is really a kind of function. Thus a sequence $(a_k)_{k=1}^{\infty} = (a_1, a_2, \dots)$ is simply a function whose domain is $\mathbb{Z}^+$ (defined by the rule $k \mapsto a_k$ for all $k \in \mathbb{Z}^+$). An element of $\mathbb{R}^n$ is the same thing as a finite sequence $(x_1, x_2, \dots, x_n)$ of real numbers, which is just a function from the set $\{\, i \in \mathbb{Z} \mid 1 \leq i \leq n \,\}$ to $\mathbb{R}$.

Often mathematicians talk of an *indexed family* of objects. Notation like $(S_i)_{i \in I}$ is commonly used for indexed families; the idea is that there is an object S_i for each i in the *indexing set* I. For example, the infinite sequence $(a_k)_{k=1}^{\infty}$ is a family indexed by the set of positive integers. In fact, a family indexed by a set I is exactly the same thing as a function with domain I. But when we write a_i rather than $a(i)$ for the value of the function at $i \in I$ then we usually write $a = (a_i)_{i \in I}$ and call a a family indexed by I rather than a function defined on I.

Back to topology

Proposition. *Let $(A_i)_{i \in I}$ be an arbitrary family of open subsets of the metric space X. Then $A = \bigcup_{i \in I} A_i$ is an open subset of X.*

Proof. For each $i \in I$ we have $A_i \subseteq A$; so, by the lemma above, we have $\mathrm{Int}(A_i) \subseteq \mathrm{Int}(A)$. But A_i is open; so $A_i = \mathrm{Int}(A_i)$, and thus $A_i \subseteq \mathrm{Int}(A)$. Since this holds for all $i \in I$ it follows that $\bigcup_{i \in I} A_i \subseteq \mathrm{Int}(A)$. That is, $A \subseteq \mathrm{Int}(A)$. The reverse inclusion is trivial; so $A = \mathrm{Int}(A)$. Hence A is open, as required. □

It is not true that the intersection of an arbitrary family of open sets is open. Indeed, in Question 3 of Tutorial 1 we saw an example of an infinite sequence of open intervals in $\mathbb{R}$ whose intersection is the singleton set $\{0\}$. Open intervals in $\mathbb{R}$ are open sets (relative to the usual metric), but $\{0\}$ is not open.

It is true that the intersection of two open sets is open. This is easily proved, as follows. Let U_1 and U_2 be open sets and let $a \in U_1 \cap U_2$ be arbitrary. Then $a \in U_1 = \mathrm{Int}(U_1)$, and so there exists $\varepsilon_1 > 0$ such that $B(a, \varepsilon_1) \subseteq U_1$. Similarly, there exists $\varepsilon_2 > 0$ such that $B(a, \varepsilon_2) \subseteq U_2$. If we put $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ then $\varepsilon \leq \varepsilon_1$ and $\varepsilon \leq \varepsilon_2$), and so

$$B(a, \varepsilon) \subseteq B(a, \varepsilon_1) \subseteq U_1$$

and

$$B(a, \varepsilon) \subseteq B(a, \varepsilon_2) \subseteq U_2.$$

Thus $B(a, \varepsilon) \subseteq U_1 \cap U_2$, which shows that $a \in \mathrm{Int}(U_1 \cap U_2)$. This holds for all $a \in U_1 \cap U_2$; so $U_1 \cap U_2$ is open.

It follows from the above, by a straightforward induction, that the intersection of finitely many open sets is always open. For suppose that U_1, U_2, …, U_n are open sets. Define sets V_i recursively, as follows: let $V_1 = U_1$, and for $2 \leq i \leq n$ let $V_i = V_{i-1} \cap U_i$. Thus $V_2 = U_1 \cap U_2$, and $V_3 = (U_1 \cap U_2) \cap U_3$, and so on. We have that V_1 is open, and if V_{i-1} is open then so is V_i, being the intersection of the open sets V_{i-1} and U_i. So all the V_i are open, and since $V_n = U_1 \cap U_2 \cap \cdots \cap U_n$ this completes the proof of the following proposition.

Proposition. *The intersection of any finite collection of open sets is open.*

The interior of the empty set is obviously empty; so $\emptyset = \mathrm{Int}(\emptyset)$, which shows that $\emptyset$ is open. The whole metric space X is also an open subset of X, since by the definition of an open ball $B(a, \varepsilon)$ in X it is true that $B(a, \varepsilon) \subseteq X$ for all $a \in X$ and all $\varepsilon > 0$. Thus, for example, for all $a \in X$ we have $B(a, 1) \subseteq X$, showing that $a \in \text{Int}(X)$.

It transpires that the properties of open sets mentioned in the preceding paragraph and in the two propositions above are the key properties of open sets, in the sense that many of the proofs in this subject need only these properties. Moreover, even when dealing with a set S for which there is no concept of distance it sometimes turns out that there is a natural collection of subsets of S satisfying these properties. A set S together with such a collection of subsets of S is called a *topological space*; we shall have more to say about these in the next several lectures.