Lemma. *Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be points in $\mathbb{R}^n$. Then*

$$d_\infty(x, y) \le d_p((x, y) \le \sqrt[p]{n}\, d_\infty(x, y)$$

for all $p \ge 1$.

Proof. Choose $i \in \{1, 2, \ldots, n\}$ so that

$$|x_i - y_i| = \max(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|).$$

Thus, $|x_i - y_i| = d_\infty(x, y)$. Now, clearly

$$|x_i - y_i|^p \le |x_1 - y_1|^p + |x_2 - y_2|^p + \cdots + |x_n - y_n|^p \le n|x_i - y_i|^p,$$

and so, taking p-th roots,

$$|x_i - y_i| \le \sqrt[p]{|x_1 - y_1|^p + |x_2 - y_2|^p + \cdots + |x_n - y_n|^p} \le \sqrt[p]{n}\, |x_i - y_i|,$$

That is,

$$d_\infty(x, y) \le d_p(x, y) \le \sqrt[p]{n}\, d_\infty(x, y)$$

as claimed. □

Corollary. *For all $x \in \mathbb{R}^n$ and all $\varepsilon > 0$,*

$$B_{d_\infty}(x, n^{-1/p}\varepsilon) \subseteq B_{d_p}(x, \varepsilon) \subseteq B_{d_\infty}(x, \varepsilon).$$

Proof. If $y \in B_{d_p}(x, \varepsilon)$ then $d_p(x, y) < \varepsilon$, and so, by the lemma, $d_\infty(x, y) < \varepsilon$, giving $y \in B_{d_\infty}(x, \varepsilon)$. Since this is true for all $y \in B_{d_p}(x, \varepsilon)$, the second of the above inclusions holds. Similarly, if $y \in B_{d_\infty}(x, n^{-1/p}\varepsilon)$ then $n^{1/p} d_\infty(x, y) < \varepsilon$ and, by the lemma, $d_p(x, y) < \varepsilon$, whence $y \in B_{d_p}(x, \varepsilon)$. So the first inclusion holds too. □

Corollary. *Let $A \subseteq \mathbb{R}^n$ and $x \in A$, and let $p \ge 1$ be arbitrary. If there exists $\varepsilon > 0$ with $B_{d_p}(x, \varepsilon) \subseteq A$ then there exists $\varepsilon' > 0$ with $B_{d_\infty}(x, \varepsilon') \subseteq A$, and vice versa.*

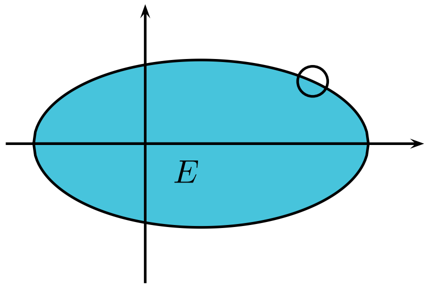
Proof. By the previous corollary we see that if $B_{d_\infty}(x, \varepsilon) \subseteq A$ then $B_{d_p}(x, \varepsilon) \subseteq A$, since $B_{d_p}(x, \varepsilon) \subseteq B_{d_\infty}(x, \varepsilon)$. So $B_{d_p}(x, \varepsilon') \subseteq A$ holds with $\varepsilon' = \varepsilon$. Conversely, if $B_{d_p}(x, \varepsilon) \subseteq A$ then, since $B_{d_\infty}(x, n^{-1/p}\varepsilon) \subseteq B_{d_p}(x, \varepsilon)$, the desired conclusion $B_{d_\infty}(x, \varepsilon') \subseteq A$ follows if we put $\varepsilon' = n^{-1/p}\varepsilon$. □

Given $A \subseteq \mathbb{R}^n$, let $\mathrm{Int}_\infty(A)$ be the interior of A when we use d_∞ as the metric on $\mathbb{R}^n$, and let $\mathrm{Int}_p(A)$ be the interior of A when d_p is used as the metric. By the definition of interior, $x \in \mathrm{Int}_p(A)$ if and only if there exists $\varepsilon > 0$ with $B_{d_p}(x, \varepsilon) \subseteq A$, and similarly $x \in \mathrm{Int}_\infty(A)$ if and only if there exists $\varepsilon' > 0$ with $B_{d_\infty}(x, \varepsilon') \subseteq A$. But the corollary above shows that these two conditions are equivalent, and therefore $\mathrm{Int}_p(A) = \mathrm{Int}_\infty(A)$. A subset of a metric space is open if and only if it coincides with its interior; thus A is an open set of the metric space $(\mathbb{R}^n, d_p)$ if and only if it is an open set of the metric space $(\mathbb{R}^n, d_\infty)$ (since $A = \mathrm{Int}_p(A)$ if and only if $A = \mathrm{Int}_\infty(A)$).

The conclusion of this reasoning can be summarized as follows: for all $p > 1$ the metrics d_p and d_∞ are topologically equivalent, in the sense that they give rise to the same collection of open sets. We shall refer to this topology the usual topology on $\mathbb{R}^n$.

(Note also that everything that has been said above works just as well for $\mathbb{C}^n$ as it does for $\mathbb{R}^n$.)

Intuitively, a subset A of $\mathbb{R}^2$ whose boundary is a continuous curve is an open set (in the usual topology) if the boundary points themselves are not part of the set. Thus, for example, the set $E = \{\,(x_1, x_2) \mid x_1^2 - 2x_1 + 4x_2^2 < 8\,\}$, consisting of the points of the plane that are inside but not on the ellipse $x_1^2 - 2x_1 + 4x_2^2 = 8$ is an open set.



It was shown in Lecture 6 that the boundary of a set, in the technical sense we have defined, consists of all points a with the property that every neighbourhood of a has nonempty intersection with both the set and its complement. This clearly accords with the everyday meaning of the word "boundary", at least for sets like E above: the points $(x_1, x_2) \in \mathbb{R}^2$ such that every circle with centre (x_1, x_2) contains points of E and points of $\mathbb{R}^2 \setminus E$ are clearly those points which lie on the ellipse $x_1^2 - 2x_1 + 4x_2^2 = 8$. Note also that the closure of E is the union of E and its boundary, which coincides with $\{\,(x, y) \mid x^2 - 2x + 4y^2 \le 8\,\}$.

If $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is any continuous function, and $K \in \mathbb{R}$, then the set A given by $A = \{\,(x_1, x_2) \in \mathbb{R}^2 \mid f(x_1, x_2) < K\,\}$ is always open. To prove this we must show that for all points $(x_1, x_2) \in A$ there is an $\delta > 0$ such that $B((x_1, x_2), \delta) \subseteq A$. In other words, whenever $f(x_1, x_2) < K$ there exists $\delta > 0$ such that if (y_1, y_2) is any point whose distance from (x_1, x_2) is less than δ then $f(y_1, y_2) < K$. Since all the metrics d_p give the same collection of open sets, it makes no difference which we use as our measure of distance. For convenience, we shall use d_∞.

Suppose that $f(x_1, x_2) < K$, and put $\varepsilon = K - f(x_1, x_2)$. Note that $\varepsilon > 0$. Since f is continuous there is a $\delta > 0$ such that $|f(y_1, y_2) - f(x_1, x_2)| < \varepsilon$ whenever $|y_1 - x_1| < \delta$ and $|y_2 - x_2| < \delta$. That is,

$$f(x_1, x_2) - \varepsilon < f(y_1, y_2) < f(x_1, x_2) + \varepsilon = K$$

whenever $d_\infty((y_1, y_2), (x_1, x_2)) < \delta$. In particular, $(y_1, y_2) \in B((x_1, x_2), \varepsilon)$ implies that $f(y_1, y_2) < K$, as required.

Observe that the set A above can be described as $\{\,x \in \mathbb{R}^2 \mid f(x) \in I\,\}$, where $I = \{\,t \in \mathbb{R} \mid t < K\,\} = (-\infty, K)$. That is, $A = f^{-1}(I)$. Thus the proof we have just been through could have been circumvented by proving that I is an open subset of $\mathbb{R}$, and using the fact (which we proved in Lecture 2) that the preimage of an open set under a continous function is open.

Taking f to be the function defined by $f(x_1, x_2) = x_1^2 - 2x_1 + 4x_2^2$ and taking K to be 8 yields a proof that the set E above is indeed open. Similarly, if we instead take $f(x_1, x_2) = -x_1^2 + 2x_2 - 4x_2^2$ then it follows that the set $\{\,(x_1.x_2) \in \mathbb{R}^2 \mid x_1^2 - 2x_1 + 4x_2^2 > 8\,\}$ is also open. In other words, since closed sets are complements of open sets, the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 - 2x_1 + 4x_2^2 \le 8\,\}$ is closed. To prove that this set is $\overline{E}$, the closure of E, requires proving that every neighbourhood of a point on the ellipse contains a point of E. We omit this, although it is not difficult.

Limits of sequences

Let (X, d) be a metric space, and $(x_n)_{n=1}^{\infty}$ a sequence of points of X. We say that (x_n) *converges* to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$. If (x_n) converges to x then we write $x_n \to x$ as $n \to \infty$. The notation $\lim_{n\to\infty} x_n = x$ is also used.

From the definition of convergence for sequences of real numbers, $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if for all $\varepsilon > 0$ there is an $N \in \mathbb{Z}$ such that $d(x_n, x) < \varepsilon$ for all $n > N$. Combining this with the above definition of convergence in a metric space X, we see that $x_n \to x$ as $n \to \infty$ if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{Z}$ such that $d(x_n, x) < \varepsilon$ for all $n > N$. Equivalently again, $x_n \to x$ as $n \to \infty$ if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{Z}$ such that $x_n \in B_d(x, \varepsilon)$ for all $n > N$.

For example, let $X = \mathbb{R}^3$ and $d = d_p$, as defined in an earlier lecture. Suppose that $(x^{(k)})_{k=1}^{\infty}$ is a sequence in X, and let $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})$ for each k. Suppose that $x^{(k)} \to x = (x_1, x_2, x_3)$ as $k \to \infty$. Then for all k, and for each $l \in \{1, 2, 3\}$,

$$0 \le |x_l^{(k)} - x_l| = \sqrt[p]{|x_l^{(k)} - x_l|^p} \le \sqrt[p]{\sum\nolimits_{i=1}^{3} |x_i^{(k)} - x_i|^p} = d_p(x^{(k)}, x) \to 0 \text{ as } k \to \infty.$$

Hence $x^{(k)} \to x$ as $k \to \infty$ implies that $x_l^{(k)} \to x_l$ as $k \to \infty$ for each l. Conversely, if $x_l^{(k)} \to x_l$ as $k \to \infty$ for each $l \in \{1, 2, 3\}$, then $d_p(x^{(k)}, x) = \sqrt[p]{\sum_{i=1}^{3} |x_i^{(k)} - x_i|^p} \to 0$ as $k \to \infty$; that is, $x^{(k)} \to x$.

It is clear that the same works for $\mathbb{R}^n$ for any value of n:

$$(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}) \to (x_1, x_2, \ldots, x_n) \text{ as } k \to \infty$$

if and only if

$$x_l^{(k)} \to x_l \text{ as } k \to \infty \text{ for all } l.$$

A sequence in $\mathbb{R}^n$ converges, for the d_p metric, if and only if each sequence of coordinates converges in $\mathbb{R}$. The same statement applies if we use d_∞ instead of d_p, and the proof is much the same as the proof above.

Recall that ℓ^p is the space of all sequences $a = (a_1, a_2, a_3, \ldots)$ of real numbers such that $\sum_{k=1}^{\infty} |a_k|^p$ converges, with metric d_p given by $d_p(a, b) = \sqrt[p]{\sum_{k=1}^{\infty} |a_k - b_k|^p}$ (where $a = (a_k)$ and $b = (b_k)$ are arbitrary elements of ℓ^p). Also, ℓ^∞ is the space of all bounded sequences (a_k), with metric d_∞ defined by $d_\infty(a, b) = \sup_k |a_k - b_k|$. (To say that the sequence (a_k) is bounded is to say that there exists a number $B \in \mathbb{R}$ such that $|a_k| < B$ for all $k \in \mathbb{Z}^+$.)

Suppose that $a^{(k)} = (a_1^{(k)}, a_2^{(k)}, a_3^{(k)}, \ldots,)$ is in ℓ^p for all $k \in \mathbb{Z}^+$, and suppose that $\lim_{k\to\infty} a^{(k)} = a = (a_1, a_2, a_3, \ldots)$. Then

$$0 \le |a_i^{(k)} - a_i| = \sqrt[p]{|a_i^{(k)} - a_i|^p} \le \sqrt[p]{\sum\nolimits_{l=1}^{\infty} |a_l^{(k)} - a_l|^p} \quad \longrightarrow 0 \quad \text{as } k \to \infty.$$

So $a_i^{(k)} \to a_i$ as $k \to \infty$ for all i. That is, we have shown that if $a^{(k)} \to a$ as $k \to \infty$ then (ith term of $a^{(k)}$) $\to$ (ith term of a) as $k \to \infty$, for each i. The converse, however, does not hold. For example, define

$$
\begin{aligned}
a^{(1)} &= (1,0,0,0,\dots) \\
a^{(2)} &= (0,1,0,0,\dots) \\
a^{(3)} &= (0,0,1,0,\dots) \\
\cdots \quad & \cdots \quad \cdots \quad \cdots.
\end{aligned}
$$

That is,

$$
a_i^{(k)} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}
$$

If i is fixed then $\lim_{k\to\infty} a_i^{(k)} = 0$, since $a_i^{(k)} = 0$ for all $k > i$. But it is not true that $a^{(k)} \to a = (0,0,0,\dots)$ as $k \to \infty$, since

$$
\begin{aligned}
d_p(a^{(k)}, a) &= \sqrt[p]{\sum\nolimits_{l=1}^{\infty} |a_i^{(k)} - a_i|^p} \\
&= 1 \qquad (\text{since } a_i^{(k)} - a_i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}) \\
&\not\to 0 \qquad \text{as } k \to \infty.
\end{aligned}
$$

(The same works for ℓ^∞. In fact, exactly the same example is applicable.)