The completion of a metric space

Our next objective is to show that every metric space can be embedded in a complete metric space. More precisely, we wish to prove the following theorem.

Theorem. *For every metric space (X, d) there is a metric space $(\widehat{X}, \widehat{d})$ such that*
(1) $(\widehat{X}, \widehat{d})$ is complete,
(2) (X, d) is a subspace of $(\widehat{X}, \widehat{d})$, and
(3) X is dense in $\widehat{X}$.

As a first step we prove a straightforward lemma that will be used several times.

Lemma. *Let (X, d) be a metric space and a, b, a', $b' \in X$. Then*

$$|d(a', b') - d(a, b)| \leq d(a, a') + d(b, b').$$

Proof. By the triangle inequality,

$$d(a', b') \leq d(a', a) + d(a, b') \leq d(a', a) + d(a, b) + d(b, b'),$$

and by the same reasoning with a swapped with a' and b swapped with b',

$$d(a, b) \leq d(a, a') + d(a', b') + d(b', b).$$

Since $d(a, a') = d(a', a)$ and $d(b, b') = d(b', b)$ these two inequalities combine to give

$$-d(a, a') - d(b, b') \leq d(a', b') - d(a, b) \leq d(a, a') + d(b, b'),$$

which gives us the desired conclusion. □

Intuitively, if a Cauchy sequence (x_n) in X does not converge, then as $n \to \infty$ the points x_n must be clustering around a "hole" in the space X, and we should be able to "fill in the hole" by adding to X a new point that will serve as the limit of (x_n). If we do this for all the non-convergent Cauchy sequences then the space we end up with will be the completion of X.

We shall have to extend the definition of the distance function to include distances between these new points. Clearly, if (x_n) and (y_n) are Cauchy sequences then the distance between the limit of (x_n) and the limit of (y_n) ought to equal $\lim_{n\to\infty} d(x_n, y_n)$. So the following lemma, which says that this limit always exists, is important to our cause.

Lemma. *Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be Cauchy sequences in a metric space (X, d). Then $\lim_{n\to\infty} d(x_n, y_n)$ exists.*

Proof. Note that $(d(x_n, y_n))_{n=1}^{\infty}$ is a sequence of real numbers. To prove that it converges we can use the Cauchy convergence criterion: every Cauchy sequence in $\mathbb{R}$ converges. (That is, $\mathbb{R}$ is a complete metric space. We proved this in Lecture 11.)

Let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence there exists a number N such that $d(x_n, x_m) < \varepsilon/2$ for all n, $m > N$. Since (y_n) is a Cauchy sequence there exists an M such that $d(y_n, y_m) < \varepsilon/2$ for all n, $m > M$. Put $K = \max(N, M)$. Then for all n, $m > K$ we have (using the lemma proved above)

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $(d(x_n, y_n))$ is a Cauchy sequence, as required. □

Let (X, d) be a metric space and define $\mathrm{CS}(X)$ to be the set of all Cauchy sequences in X. To construct the completion, $(\widehat{X}, \widehat{d})$, of (X, d) we need to construct a set $\widehat{X} \supseteq X$ such that

$$(a_n) \in \mathrm{CS}(X) \textit{ implies that } \lim_{n\to\infty} a_n \textit{ exists in } \widehat{X}.$$

That is, there should be a function $L\colon \mathrm{CS}(X) \to \widehat{X}$, taking each Cauchy sequence to its limit. We do not want the set $\widehat{X}$ to be any bigger than necessary; so, preferably, the function L will be surjective (meaning that all the points in $\widehat{X}$ are limits of Cauchy sequences in X, just as all real numbers are limits of sequences of rationals). Note, however, that L probably will not be injective, as it is quite possible for different Cauchy sequences to have the same limit. Thus if (x_n) and (y_n) are non-convergent Cauchy sequences in X such that $\lim_{n\to\infty} d(x_n, y_n) = 0$ then the new point which we create to be the limit of (x_n) will also serve as the limit of (y_n). Thus the new points to be added to X should correspond not to individual Cauchy sequences, but rather to equivalence classes of Cauchy sequences, where by definition (x_n) and (y_n) are equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition. If (x_n), $(y_n) \in \mathrm{CS}(X)$, write $(x_n) \sim (y_n)$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$. If this holds we shall say that (x_n) and (y_n) are *equivalent*.

Lemma. *The relation $\sim$ defined above is an equivalence relation on* $\mathrm{CS}(X)$*.*

Proof. Let $\alpha = (x_n) \in \mathrm{CS}(X)$. Since $\lim_{n\to\infty} d(x_n, x_n) = \lim_{n\to\infty} 0 = 0$ it follows that $\alpha \sim \alpha$. Since this holds for all $\alpha \in \mathrm{CS}(X)$, the relation $\sim$ is reflexive.

Let $\alpha = (x_n)$ and $\beta = (y_n)$ be Cauchy sequences in X such that $\alpha \sim \beta$. By the definition, $\lim_{n\to\infty} d(x_n, y_n) = 0$. But since $d(y_n, x_n) = d(x_n, y_n)$ for all n it follows that $\lim_{n\to\infty} d(y_n, x_n) = \lim_{n\to\infty} d(x_n, y_n) = 0$. So $\beta \sim \alpha$. Hence $\sim$ is symmetric.

If $\alpha = (x_n), \beta = (y_n)$ and $\gamma = (z_n)$ are Cauchy sequnces such that $\alpha \sim \beta$ and $\beta \sim \gamma$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$. Now

$$0 \le d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n) \to 0 \quad \text{as } n \to \infty,$$

and thus $\alpha \sim \gamma$. So $\sim$ is transitive. □

We now make use of the following general fact about equivalence relations:

If $\equiv$ is an equivalence relation on a set S then there is a set Q and a function $\pi\colon S \to Q$ such that

(1) π is surjective, and

(2) $\pi(s) = \pi(t)$ if and only if $s \equiv t$ (for all s, $t \in S$).

Indeed, we may take Q to be the set of all equivalence classes and $\pi\colon S \to Q$ the function which takes each $s \in S$ to the $\equiv$-equivalence class in which it lies. This set Q is called the *quotient* of S by the equivalence relation $\equiv$, and π is called the natural map, or projection, of S onto Q.

Applying this to our situation, the fact that $\sim$ is an equivalence relation on $\mathrm{CS}(X)$ enables us to draw the following conclusion:

> *There exists a set $\widehat{X}$ and a map $L\colon \mathrm{CS}(X) \to \widehat{X}$ such that L is surjective and, for all Cauchy sequences (x_n) and (y_n), we have $L((x_n)) = L((y_n))$ if and only if $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

We now aim to define a metric $\widehat{d}$ on the set $\widehat{X}$, and show that $(\widehat{X}, \widehat{d})$ has a subspace which is a copy of (X, d). This means that we must find a one to one, distance-preserving correspondence between X and a subset of $\widehat{X}$. That is, we must find an injective function $\eta\colon X \to \widehat{X}$ with the property that $\widehat{d}(\eta(x), \eta(y)) = d(x, y)$ for all x, $y \in X$.†

Let α, $\beta \in \widehat{X}$. Since L is surjective there exist (a_n), $(b_n) \in \mathrm{CS}(X)$ such that $\alpha = L((a_n))$ and $\beta = L((b_n))$. We know that $\lim_{n\to\infty} d(a_n, b_n)$ exists, and we would like to define $\widehat{d}(\alpha, \beta)$ to be equal to this limit. But since the sequences (a_n) and (b_n) may not be uniquely determined by α and β, we need to prove that alternative choices for the sequences give rise to the same limit. Our next lemma does this.

Lemma. *Suppose that (a_n), (a'_n), (b_n), $(b'_n) \in \mathrm{CS}(X)$, and suppose that $(a_n) \sim (a'_n)$ and $(b_n) \sim (b'_n)$. Then $\lim_{n\to\infty} d(a_n, b_n) = \lim_{n\to\infty} d(a'_n, b'_n)$.*

Proof. By the Lemma from the start of this lecture,

$$|d(a_n, b_n) - d(a'_n, b'_n)| < d(a_n, a'_n) + d(b_n, b'_n) \to 0 \quad \text{as } n \to \infty,$$

whence the result follows. □

This lemma shows that there is a well-defined function $\widehat{d}\colon \widehat{X} \times \widehat{X} \to \mathbb{R}$ such that $\widehat{d}(\alpha, \beta) = \lim_{n\to\infty} d(a_n, b_n)$ whenever (a_n), $(b_n) \in \mathrm{CS}(X)$ with $\alpha = L((a_n))$ and $\beta = L((b_n))$. We defer for a moment the proof that $\widehat{d}$ is actually a metric on $\widehat{X}$, and turn to the question of identifying X with a subset of $\widehat{X}$.

For each $x \in X$ let $c(x)$ be the corresponding constant sequence, al of whose terms are x. That is, $c(x) = (x_n)$, where $x_n = x$ for all $n \in \mathbb{Z}^+$. Clearly $c(x)$ is a Cauchy sequence, since for any $\varepsilon > 0$ we have $d(x_n, x_m) = 0 < \varepsilon$ for all n, $m > 0$. (Indeed, since $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} 0 = 0$, the sequence (x_n) is convergent, with limit x. We know that all convergent sequences are Cauchy sequences.) We can now define a function $\eta\colon X \to \widehat{X}$ by $\eta(x) = L(c(x))$ for all $x \in X$. It is easily seen that η is injective, and therefore defines a one to one correspondence between X and $\eta(X) \subseteq \widehat{X}$. For suppose that x, $y \in X$ satisfy $\eta(x) = \eta(y)$. Then $L(c(x)) = L(c(y))$, and by the definition of L this implies that $c(x) \sim c(y)$. So $\lim_{n\to\infty} d(x_n, y_n) = 0$, where x_n is the n-th term of $c(x)$ and y_n the n-th term of $c(y)$. But $x_n = x$ and $y_n = y$ for all n; so $d(x, y) = \lim_{n\to\infty} d(x, y) = \lim_{n\to\infty} d(x_n, y_n) = 0$, and therefore $x = y$. Since this holds whenever $\eta(x) = \eta(y)$ we have shown that η is injective, as claimed.

As explained above, we need to check that $\widehat{d}(\eta(x), \eta(y)) = d(x, y)$ for all x, $y \in X$. Since $\eta(x) = L(c(x)) = L((x_n))$ where $x_n = x$ for all n, and similarly $\eta(y) = L((y_n))$ where $y_n = y$ for all n, the definition of $\widehat{d}$ gives

$$\widehat{d}(\eta(x), \eta(y)) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x, y) = d(x, y),$$

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† In this web version of the lecture notes the proof differs somewhat from the proof that was actually given in the lecture. There $\widehat{X}$ was defined to be the disjoint union of X with a set $\mathcal{L}$ in one-to-one correspondence with the equivalence classes of sequences in $\mathrm{CS}(X)$ that do not have limits in X; this approach had the advantage that X really is a subset of $\widehat{X}$, but some of the proofs had to be split into two cases, depending on whether or not a Cauchy sequence had a limit. Here, though our proofs work more smoothly, X is not strictly a subspace of $\widehat{X}$, and so we have to argue that it can be identified with a subset of $\widehat{X}$: in effect, it is as good as a subset of X.

as required. We conclude that the subspace $\eta(X)$ of $(\widehat{X}, \widehat{d})$ is indeed a copy of the space (X, d) that we started with.

It is time we proved that $\widehat{d}$ really is a metric. So let α, β, $\gamma \in \widehat{X}$ be arbitrary, and choose Cauchy sequences (a_n), (b_n) and (c_n) with $L((a_n)) = \alpha$, $L((b_n)) = \beta$ and $L((c_n)) = \gamma$. Since $\widehat{d}(\alpha, \beta) = \lim_{n\to\infty} d(a_n, b_n)$, and since $d(a_n, b_n) \geq 0$ for all n, it is clear that $\widehat{d}(\alpha, \beta) \geq 0$. Moreover, $\widehat{d}(\alpha, \alpha) = \lim_{n\to\infty} d(a_n, a_n) = \lim_{n\to\infty} 0 = 0$. Just as simply,

$$\widehat{d}(\alpha, \beta) = \lim_{n\to\infty} d(a_n, b_n) = \lim_{n\to\infty} d(b_n, a_n) = \widehat{d}(\beta, \alpha).$$

And since $d(b_n, c_n) \leq d(a_n, b_n) + d(a_n, c_n)$ for all n,

$$\widehat{d}(\beta, \gamma) = \lim_{n\to\infty} d(b_n, c_n) \leq \lim_{n\to\infty} d(a_n, b_n) + \lim_{n\to\infty} d(a_n, c_n) = \widehat{d}(\alpha, \beta) + \widehat{d}(\alpha, \gamma),$$

so that the triangle inequality is satisfied. Now all that remains is to prove that $\widehat{d}(\alpha, \beta) = 0$ implies $\alpha = \beta$. But $\widehat{d}(\alpha, \beta) = 0$ says that $\lim_{n\to\infty} d(a_n, b_n) = 0$, which means that $(a_n) \sim (b_n)$, and thus, by the definition of the function L,

$$\alpha = L((a_n)) = L((b_n)) = \beta,$$

as required. We have now proved that $(\widehat{X}, \widehat{d})$ is a metric space and that (X, d) is (as good as) a subspace of $(\widehat{X}, \widehat{d})$.

Assertion (3) of the theorem is that X is a dense subspace of $\widehat{X}$. By the definition of dense, this means that the closure of X in $\widehat{X}$ is the whole of $\widehat{X}$. Since the closure of a set consists of all points that are limits of convergent sequences whose terms lie in the set, our task is to prove that every point of $\widehat{X}$ is the limit of some sequence of points of X. Here, of course, we must identify each point $x \in X$ with the point $\eta(x) \in \widehat{X}$.

Lemma. *Let $s = (a_n)_{n=1}^{\infty} \in \mathrm{CS}(X)$. Then the sequence $(\eta(a_n))_{n=1}^{\infty}$ converges in $\widehat{X}$, and its limit is $L(s)$.*

Proof. If x is any point of X then, by the way $\widehat{d}$ is defined,

$$\widehat{d}(\eta(x), L(s)) = \widehat{d}(L(c(x), L(s)) = \lim_{n\to\infty} d(x, a_n).$$

Thus, in particular, $\widehat{d}(\eta(a_m), L(s)) = \lim_{n\to\infty} d(a_m, a_n)$ for each positive integer m. Now let $\varepsilon > 0$ be arbitrary. Since (a_n) is a Cauchy sequence, we may choose a number N such that $d(a_m, a_n) < \varepsilon/2$ for all $m, n > N$. It follows that if $m > N$ then $\lim_{n\to\infty} d(a_m, a_n) \leq \varepsilon/2 < \varepsilon$. So there exists an N such that $0 \leq \widehat{d}(\eta(a_m), L(s)) < \varepsilon$ for all $m > N$. Thus $\lim_{m\to\infty} \widehat{d}(\eta(a_m), L(s)) = 0$, and, by the definition of limits in metric spaces, this means that $\eta(a_m) \to L(s)$ as $m \to \infty$. □

Since every point of $\widehat{X}$ has the form $L(s)$ for some $s \in \mathrm{CS}(X)$, it follows from the lemma that every point of $\widehat{X}$ is the limit of some sequence of points $\eta(X)$. Thus $\eta(X)$ is dense in $\widehat{X}$, as required.