INDUCED W-GRAPHS

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Abstract. Let $\mathscr{H}$ be the Hecke algebra associated with a Coxeter group W. Many interesting $\mathscr{H}$-modules can be described using the concept of a W-graph, as introduced in the influential paper [6] of Kazhdan and Lusztig. In particular, Kazhdan and Lusztig showed that the regular representation of $\mathscr{H}$ has an associated W-graph. In [5] it is shown that if W_J is a parabolic subgroup of W and V is a module for the corresponding Hecke algebra $\mathscr{H}_J$, then a W_J-graph structure for V gives rise to a W-graph structure for the induced module $\mathscr{H}\otimes_{\mathscr{H}_J}V$. In the case that W_J is the identity subgroup and V has dimension 1, the construction coincides with that given by Kazhdan and Lusztig for the regular representation, while for arbitrary J and V of dimension 1 it coincides with constructions given by Couillens [1] and Deodhar [3]. The present paper includes a minor reformulation of the results of [5] and some additional results; notably, we describe how cells in the W_J-graph naturally give rise to subsets of the induced W-graph that are unions of cells.

1. Preliminaries

Let W be a Coxeter group with S the set of simple reflections, and let $\mathscr{H}$ be the corresponding Hecke algebra. We use a variation of the definition given in [6], taking $\mathscr{H}$ to be an algebra over $\mathcal{A} = \mathbb{Z}[q^{-1}, q]$, the ring of Laurent polynomials with integer coefficients in the indeterminate q, having an $\mathcal{A}$-basis $\{\, T_w \mid w \in W \,\}$ satisfying

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ T_{sw} + (q - q^{-1})T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for all $w \in W$ and $s \in S$. We also define $\mathcal{A}^+ = \mathbb{Z}[q]$, the ring of polynomials in q with integer coefficients, and let $a \mapsto \overline{a}$ be the involutory automorphism of $\mathcal{A}$ such that $\overline{q} = q^{-1}$. This involution on $\mathcal{A}$ extends to an involution on $\mathscr{H}$ satisfying $\overline{T_s} = T_s^{-1} = T_s + (q^{-1} - q)$ for all $s \in S$. This gives $\overline{T_w} = T_{w^{-1}}^{-1}$ for all $w \in W$.

For each $J \subseteq S$ define $W_J = \langle J \rangle$, the corresponding parabolic subgroup of W, and let $D_J = \{\, w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in J \,\}$, the set of minimal coset representatives of W/W_J. Let $\mathscr{H}_J$ be the Hecke algebra associated with W_J. As is well known, $\mathscr{H}_J$ can be identified with a subalgebra of $\mathscr{H}$.

2. Definition of W-graph

Modifying the definitions in [6] to suit our modified definition of the Hecke algebra, a W-graph is a set Γ (the vertices of the graph) with a set Θ of two-element subsets of Γ (the edges) together with the following additional data: for each vertex γ we are given a subset I_γ of S, and for each ordered pair of vertices δ, γ we are

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given an integer $\mu(\delta,\gamma)$ which is nonzero if and only if $\{\delta,\gamma\}\in\Theta$. These data are subject to the requirement that $\mathcal{A}\Gamma$, the free $\mathcal{A}$-module on Γ, has an $\mathscr{H}$-module structure satisfying

$$T_s\gamma=\begin{cases}-q^{-1}\gamma & \text{if } s\in I_\gamma\\ q\gamma+\sum_{\{\delta\in\Gamma\mid s\in I_\delta\}}\mu(\delta,\gamma)\delta & \text{if } s\notin I_\gamma,\end{cases}\tag{1}$$

for all $s\in S$ and $\gamma\in\Gamma$. If τ_s is the $\mathcal{A}$-endomorphism of $\mathcal{A}\Gamma$ such that $\tau_s(\gamma)$ is the right-hand side of Eq. (1) then this requirement is equivalent to the condition that for all s, $t\in S$ such that st has finite order,

$$\underbrace{\tau_s\tau_t\tau_s\ldots}_{m\text{ factors}}=\underbrace{\tau_t\tau_s\tau_t\ldots}_{m\text{ factors}}$$

where m is the order of st.

To avoid over-proliferation of symbols, we shall use the name of the vertex set of a W-graph to also refer to the W-graph itself.

Given a W-graph Γ we define

$$\Gamma_s^-=\{\,\gamma\in\Gamma\mid s\in I_\gamma\,\},$$
$$\Gamma_s^+=\{\,\gamma\in\Gamma\mid s\notin I_\gamma\,\}.$$

Observe that the involution $a\mapsto\overline{a}$ on $\mathcal{A}$ determines a semilinear involution $v\mapsto\overline{v}$ on $\mathcal{A}\Gamma$ with the property that $\overline{\gamma}=\gamma$ for all $\gamma\in\Gamma$. If $s\in S$ and $\gamma\in\Gamma$ then

$$\overline{T_s}\,\overline{\gamma}=\overline{T_s}\gamma=T_s\gamma+(q^{-1}-q)\gamma;$$

thus if $\gamma\in\Gamma_s^-$ it follows that

$$\overline{T_s}\,\overline{\gamma}=-q^{-1}\gamma+(q^{-1}-q)\gamma=-q\gamma=\overline{T_s\gamma},$$

while if $\gamma\in\Gamma_s^+$ we find that

$$\begin{aligned}\overline{T_s}\,\overline{\gamma}&=\big(q\gamma+\sum_{\delta\in\Gamma_s^-}\mu(\delta,\gamma)\delta\big)+(q^{-1}-q)\gamma\\&=q^{-1}\gamma+\sum_{\delta\in\Gamma_s^-}\mu(\delta,\gamma)\delta\\&=\overline{T_s\gamma}.\end{aligned}$$

Since $\mathscr{H}$ is generated by $\{\,T_s\mid s\in S\,\}$, the following proposition is an immediate consequence of these calculations.

Proposition 2.1. *If Γ is a W-graph then the associated $\mathscr{H}$-module $\mathcal{A}\Gamma$ admits an involution $v\mapsto\overline{v}$ that fixes all elements of Γ and is compatible with the involution $h\mapsto\overline{h}$ of $\mathscr{H}$, in the sense that $\overline{hv}=\overline{h}\,\overline{v}$ for all $h\in\mathscr{H}$ and $v\in E$.*

For use in the final sections of this paper, we make the following definition.

Definition 2.2. An *ordered W-graph* set Γ with a W-graph structure and a partial order $\leqslant$ satisfying the following conditions:

(i) for all θ, $\gamma\in\Gamma$ such that $\mu(\theta,\gamma)\neq 0$, either $\theta<\gamma$ or $\gamma<\theta$;
(ii) for all $s\in S$ and $\gamma\in\Gamma_s^+$ the set $\{\,\theta\in\Gamma_s^-\mid\gamma<\theta \text{ and } \mu(\theta,\gamma)\neq 0\,\}$ is either empty or consists of a single element $s\gamma$;
(iii) for all $s\in S$ and $\gamma\in\Gamma_s^+$, if $s\gamma$ exists then $\mu(s\gamma,\gamma)=1$.

3. Induced modules

Suppose now that Γ is a W_J-graph (so that $\mathcal{A}\Gamma$ is an $\mathscr{H}_J$-module) and let M be the $\mathscr{H}$-module induced from the $\mathscr{H}_J$-module $\mathcal{A}\Gamma$. Thus, identifying $\mathcal{A}\Gamma$ with an $\mathcal{A}$-submodule of M in the obvious way, M has an $\mathcal{A}$-basis $\{\, T_d\gamma \mid d \in D_J,\ \gamma \in \Gamma \,\}$, and we can define an involution on M by setting $\overline{T_d\gamma} = \overline{T_d}\gamma$ for all $d \in D_J$ and $\gamma \in \Gamma$. Since T_1 is the identity element of $\mathscr{H}$ this extends the involution on $\mathcal{A}\Gamma$ described in Proposition 2.1, and clearly $\overline{T_d v} = \overline{T_d}\,\overline{v}$ for all $d \in D_J$ and $v \in \mathcal{A}\Gamma$. Thus for all $d \in D_J$ and $u \in W_J$ we have

$$\overline{T_{du}\gamma} = \overline{T_d T_u\gamma} = \overline{T_d(T_u\gamma)} = \overline{T_d}(\overline{T_u\gamma}) = \overline{T_d}(\overline{T_u}\gamma) = \overline{T_{du}}\,\gamma \qquad (\text{for all } \gamma \in \Gamma),$$

and hence $\overline{T_{du}v} = \overline{T_{du}}\,\overline{v}$ for all $v \in \mathcal{A}\Gamma$. Thus $\overline{hv} = \overline{h}\,\overline{v}$ for all $h \in \mathscr{H}$ and $v \in \mathcal{A}\Gamma$, and so we obtain the following result.

Proposition 3.1. *The involution on M defined above is compatible with the involution on $\mathscr{H}$.*

Proof. Let $h \in \mathscr{H}$ and $m \in M$ be arbitrary. Then $m = kv$ for some $k \in \mathscr{H}$ and $v \in \mathcal{A}\Gamma$, and so

$$\overline{h}\,\overline{m} = \overline{h}(\overline{kv}) = \overline{h}(\overline{k}\,\overline{v}) = (\overline{hk})\overline{v} = \overline{hkv} = \overline{hm},$$

as required. □

Our aim is to associate M with a W-graph by finding an appropriate basis of M. In particular, elements of this W-graph basis will be fixed by the involution.

The following result is well known.

Lemma 3.2 (Deodhar [2, Lemma 3.2])**.** *Let $J \subseteq S$ and $s \in S$, and define*

$$D^-_{J,s} = \{\, d \in D_J \mid \ell(sd) < \ell(d) \,\},$$
$$D^+_{J,s} = \{\, d \in D_J \mid \ell(sd) > \ell(d) \text{ and } sd \in D_J \,\},$$
$$D^0_{J,s} = \{\, d \in D_J \mid \ell(sd) > \ell(d) \text{ and } sd \notin D_J \,\},$$

so that D_J is the disjoint union $D^-_{J,s} \cup D^+_{J,s} \cup D^0_{J,s}$. Then $sD^+_{J,s} = D^-_{J,s}$, and if $d \in D^0_{J,s}$ then $sd = dt$ for some $t \in J$.

4. The elements $R_{x,\delta,y,\gamma}$

For all x, $y \in D_J$ and γ, $\delta \in \Gamma$ we define elements $R_{x,\delta,y,\gamma} \in \mathcal{A}$ by the formula

$$\overline{T_y}\gamma = \sum_{x \in D_J,\, \delta \in \Gamma} R_{x,\delta,y,\gamma} T_x \delta. \tag{2}$$

We begin by deriving formulas which provide an inductive procedure for calculating these elements.

If $y = 1$ then $\overline{T_y}\gamma = \gamma$, and hence

$$R_{x,\delta,1,\gamma} = \begin{cases} 1 & \text{if } x = 1 \text{ and } \delta = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Suppose now that $1 \neq y \in D_J$, so that we may choose $s \in S$ with $\ell(sy) = \ell(y) - 1$. Then by Lemma 3.2 we have $y = sv$ with $v \in D^+_{J,s}$ and $\ell(y) = \ell(v) + 1$, and

$$\overline{T_y}\gamma = \overline{T_s}(\overline{T_v}\gamma) = \sum_{x \in D_J,\, \delta \in \Gamma} R_{x,\delta,v,\gamma} T_s^{-1} T_x \delta.$$

Each x in this expression lies in exactly one of the sets $D^+_{J,s}$, $D^-_{J,s}$ or $D^0_{J,s}$. When $x \in D^0_{J,s}$ we write $t = x^{-1}sx$ (an element of J); in this case $T_s^{-1}T_x = T_xT_t^{-1}$. When $x \in D^-_{J,s}$ we have $T_s^{-1}T_x = T_{sx}$, while $x \in D^+_{J,s}$ gives $T_s^{-1}T_x = T_{sx} + (q^{-1} - q)T_x$. Thus we obtain

$$
\begin{aligned}
\overline{T_y}\gamma &= \sum_{\delta\in\Gamma}\sum_{x\in D^+_{J,s}} R_{x,\delta,v,\gamma}(T_{sx} + (q^{-1}-q)T_x)\delta \\
&\quad + \sum_{\delta\in\Gamma}\sum_{x\in D^-_{J,s}} R_{x,\delta,v,\gamma}T_{sx}\delta \ + \sum_{\delta\in\Gamma}\sum_{x\in D^0_{J,s}} R_{x,\delta,v,\gamma}T_xT_t^{-1}\delta \\
&= \sum_{\delta\in\Gamma}\sum_{x\in D^-_{J,s}} R_{sx,\delta,v,\gamma}T_x\delta \ + \sum_{\delta\in\Gamma}\sum_{x\in D^+_{J,s}} (q^{-1}-q)R_{x,\delta,v,\gamma}T_x\delta \\
&\quad + \sum_{\delta\in\Gamma}\sum_{x\in D^+_{J,s}} R_{sx,\delta,v,\gamma}T_x\delta \ + \sum_{x\in D^0_{J,s}}\sum_{\delta\in\Gamma^-_t} R_{x,\delta,v,\gamma}T_xT_t^{-1}\delta \\
&\quad + \sum_{x\in D^0_{J,s}}\sum_{\delta\in\Gamma^+_t} R_{x,\delta,v,\gamma}T_xT_t^{-1}\delta \\
&= \sum_{\delta\in\Gamma}\sum_{x\in D^-_{J,s}} R_{sx,\delta,v,\gamma}T_x\delta \ + \sum_{\delta\in\Gamma}\sum_{x\in D^+_{J,s}} (q^{-1}-q)R_{x,\delta,v,\gamma}T_x\delta \\
&\quad + \sum_{\delta\in\Gamma}\sum_{x\in D^+_{J,s}} R_{sx,\delta,v,\gamma}T_x\delta \ - \sum_{x\in D^0_{J,s}}\sum_{\delta\in\Gamma^-_t} qR_{x,\delta,v,\gamma}T_x\delta \\
&\quad + \sum_{x\in D^0_{J,s}}\sum_{\delta\in\Gamma^+_t} R_{x,\delta,v,\gamma}T_x\Big(q^{-1}\delta + \sum_{\theta\in\Gamma^-_t}\mu(\theta,\delta)\theta\Big) \\
&= \sum_{\delta\in\Gamma}\sum_{x\in D^-_{J,s}} R_{sx,\delta,v,\gamma}T_x\delta \ + \sum_{\delta\in\Gamma}\sum_{x\in D^+_{J,s}} (q^{-1}-q)R_{x,\delta,v,\gamma}T_x\delta \\
&\quad + \sum_{\delta\in\Gamma}\sum_{x\in D^+_{J,s}} R_{sx,\delta,v,\gamma}T_x\delta \ - \sum_{x\in D^0_{J,s}}\sum_{\delta\in\Gamma^-_t} qR_{x,\delta,v,\gamma}T_x\delta \\
&\quad + \sum_{x\in D^0_{J,s}}\sum_{\delta\in\Gamma^+_t} q^{-1}R_{x,\delta,v,\gamma}T_x\delta \ + \sum_{x\in D^0_{J,s}}\sum_{\theta\in\Gamma^+_t}\sum_{\delta\in\Gamma^-_t}\mu(\delta,\theta)R_{x,\theta,v,\gamma}T_x\delta.
\end{aligned}
$$

Comparing this with Eq. (2) gives us the following result.

Proposition 4.1. *Let γ, $\delta \in \Gamma$ and x, $y \in D_J$. If $s \in S$ is such that $\ell(sy) < \ell(y)$ then*

$$
R_{x,\delta,y,\gamma} = \begin{cases} R_{sx,\delta,sy,\gamma} & \text{if } x \in D^-_{J,s} \\ R_{sx,\delta,sy,\gamma} + (q^{-1}-q)R_{x,\delta,sy,\gamma} & \text{if } x \in D^+_{J,s} \\ q^{-1}R_{x,\delta,sy,\gamma} & \text{if } x \in D^0_{J,s} \text{ and } \delta \in \Gamma^+_t \\ -qR_{x,\delta,sy,\gamma} + \sum_{\theta\in\Gamma^+_t}\mu(\delta,\theta)R_{x,\theta,sy,\gamma} & \text{if } x \in D^0_{J,s} \text{ and } \delta \in \Gamma^-_t, \end{cases}
$$

where $t = x^{-1}sx$.

We can use induction on $\ell(y)$ to establish that $R_{x,\delta,y,\gamma} = 0$ unless $x \leqslant y$ in the Bruhat partial order on W; this follows from the fact that if $sy \leqslant y$ and $x \leqslant sy$ then both $x \leqslant y$ and $sx \leqslant y$ (see Deodhar [2, Theorem 1.1]). It is also easily seen that

$$R_{x,\delta,x,\gamma} = \begin{cases} 1 & \text{if } \delta = \gamma \\ 0 & \text{if } \delta \neq \gamma, \end{cases}$$

and if $\ell(y) - \ell(x) = k$ then the coefficient of q^j in $R_{x,\delta,y,\gamma}$ is zero for $|j| > k$, and also zero for $|j| = k$ if $\delta \neq \gamma$.

5. The construction of the W-graph basis

As is the preceding section, we assume that Γ is a W_J-graph and M the induced $\mathscr{H}$-module.

Theorem 5.1. *The module M has a unique basis $\{\, C_{w,\gamma} \mid w \in D_J,\ \gamma \in \Gamma \,\}$ such that $\overline{C_{w,\gamma}} = C_{w,\gamma}$ for all $w \in D_J$ and $\gamma \in \Gamma$, and*

$$C_{w,\gamma} = \sum_{y \in D_J, \delta \in \Gamma} P_{y,\delta,w,\gamma} T_y \delta$$

for some elements $P_{y,\delta,w,\gamma} \in \mathcal{A}^+$ with the following properties:

(i) $P_{y,\delta,w,\gamma} = 0$ *if* $y \nleqslant w$;

(ii) $P_{w,\delta,w,\gamma} = \begin{cases} 1 & \text{if } \delta = \gamma, \\ 0 & \text{if } \delta \neq \gamma; \end{cases}$

(iii) $P_{y,\delta,w,\gamma}$ *has zero constant term if* $(y, \delta) \neq (w, \gamma)$.

We shall show that the basis elements $C_{w,\gamma}$ can be identified with the vertices of a W-graph for the module M; we shall denote this W-graph by Λ. Before giving the proof of Theorem 5.1, we describe the additional data associated with Λ.

Given $y, w \in D_J$ and $\delta, \gamma \in \Gamma$ with $(y, \delta) \neq (w, \gamma)$, we define an integer $\mu(y, \delta, w, \gamma)$ as follows. If $y < w$ then $\mu(y, \delta, w, \gamma)$ is the coefficient of q in $-P_{y,\delta,w,\gamma}$, and if $w < y$ then it is the coefficient of q in $-P_{w,\gamma,y,\delta}$. If neither $y < w$ nor $w < y$ then

$$\mu(y, \delta, w, \gamma) = \begin{cases} \mu(\delta, \gamma) & \text{if } y = w, \\ 0 & \text{if } y \neq w. \end{cases}$$

We write $(y, \delta) \prec (w, \gamma)$ if $y < w$ and $\mu(y, \delta, w, \gamma) \neq 0$.

The subset of S associated with the vertex $C_{w,\gamma}$ of Λ is

$$I(w, \gamma) = \{\, s \in S \mid \ell(sw) < \ell(w) \text{ or } sw = wt \text{ for some } t \in I_\gamma \,\}$$

and the integer associated with the pair of vertices $(C_{y,\delta}, C_{w,\gamma})$ is $\mu(y, \delta, w, \gamma)$ (as defined above). Thus $\{C_{y,\delta}, C_{w,\gamma}\}$ is an edge of Λ if and only if $\mu(y, \delta, w, \gamma) \neq 0$, and this occurs if and only if either $(y, \delta) \prec (w, \gamma)$ or $(w, \gamma) \prec (y, \delta)$, or $y = w$ and $\{\delta, \gamma\}$ is an edge of Γ.

Modifying slightly the notation introduced in Section 2, we define

$$\begin{aligned} \Lambda_s^- &= \{\, (w, \gamma) \in D_J \times \Gamma \mid s \in I(w, \gamma) \,\} \\ &= \{\, (w, \gamma) \mid w \in D_{J,s}^- \text{ or } w \in D_{J,s}^0 \text{ with } t \in I_\gamma \,\}, \end{aligned}$$

and similarly $\Lambda_s^+ = \{\, (w, \gamma) \in D_J \times \Gamma \mid s \notin I(w, \gamma) \,\}$.

Our proof of Theorem 5.1 will also incorporate a proof of the following result, which will be an important component of the subsequent proof that Λ is a W-graph.

Theorem 5.2. *Let $v \in D_J$ and $\gamma \in \Gamma$. Then for all $s \in S$ such that $\ell(sv) > \ell(v)$ and $sv \in D_J$ we have*

$$T_s C_{v,\gamma} = qC_{v,\gamma} + C_{sv,\gamma} + \sum \mu(z, \delta, v, \gamma) C_{z,\delta},$$

where the sum is over all $(z, \delta) \in \Lambda_s^-$ such that $(z, \delta) \prec (v, \gamma)$.

Proof. We address the uniqueness part of Theorem 5.1 first. Fix $w \in D_J$ and $\gamma \in \Gamma$, and observe that the equation $C_{w,\gamma} = \overline{C_{w,\gamma}}$ can be written in the form

$$\sum_{\substack{x \in D_J \\ \delta \in \Gamma}} P_{x,\delta,w,\gamma} T_x \delta = \sum_{\substack{y \in D_J \\ \theta \in \Gamma}} \overline{P_{y,\theta,w,\gamma}} \sum_{\substack{x \in D_J \\ \delta \in \Gamma}} R_{x,\delta,y,\theta} T_x \delta,$$

or, equivalently, as

$$P_{x,\delta,w,\gamma} = \sum_{y \in D_J} \sum_{\theta \in \Gamma} \overline{P_{y,\theta,w,\gamma}} R_{x,\delta,y,\theta}$$

for all $x \in D_J$ and $\delta \in \Gamma$. Recall that $R_{x,\delta,x,\delta} = 1$, and if $(y, \theta) \neq x, \delta)$ then $R_{x,\delta,y,\theta} = 0$ unless $x < y$. Since also $P_{y,\theta,w,\gamma}$ is required to be zero unless $y \leqslant w$, we obtain

$$P_{x,\delta,w,\gamma} - \overline{P_{x,\delta,w,\gamma}} = \sum_{\{y,\theta \,|x<y\leqslant w\}} \overline{P_{y,\theta,w,\gamma}} R_{x,\delta,y,\theta}. \tag{3}$$

Conditions (ii) and (iii) in Theorem 5.1 specify the elements $P_{x,\delta,w,\gamma}$ when $x = w$, and in view of Condition (iii) they are then recursively determined for $x < w$ by Eq. (3): the point is that the right hand side is known by the inductive hypothesis, and since $P_{x,\delta,w,\gamma}$ is required to be in $\mathcal{A}^+$ and have zero constant term it must equal the sum of the terms on the right hand side of Eq. (3) with positive exponent of q. So there is at most one family of elements $P_{x,\delta,w,\gamma}$ satisfying the required conditions.

Turning now to the existence part of the proof, we give a recursive procedure for constructing elements $P_{x,\delta,w,\gamma}$ satisfying the requirements of Theorem 5.1. We start with the definition

$$P_{y,\delta,1,\gamma} = \begin{cases} 0 & \text{if } (y,\delta) \neq (1,\gamma), \\ 1 & \text{if } (y,\delta) = (1,\gamma). \end{cases}$$

for all $y \in D_J$ and γ, $\delta \in \Gamma$. This gives $C_{1,\gamma} = \gamma$, so that $\overline{C_{w,\gamma}} = C_{w,\gamma}$ holds for $w = 1$, as do Conditions (i), (ii) and (iii).

Now assume that $w \neq 1$ and that for all $v \in D_J$ with $\ell(v) < \ell(w)$ the elements $P_{x,\delta,v,\gamma}$ have been defined (for all $x \in D_J$ and γ, $\delta \in \Gamma$) so that the requirements of Theorem 5.1 are satisfied. Thus the elements $C_{v,\gamma}$ are known when $\ell(v) < \ell(w)$. We may choose $s \in S$ such that $w = sv$ with $\ell(w) = \ell(v) + 1$; note that $v \in D_J$ by Lemma 3.2. In accordance with the formula in Theorem 5.2 we define

$$C_{w,\gamma} = (T_s - q)C_{v,\gamma} - \sum_{\substack{(z,\theta)\prec(v,\gamma) \\ (z,\theta)\in\Lambda_s^-}} \mu(z, \theta, v, \gamma) C_{z,\theta}. \tag{4}$$

Since $\overline{T_s - q} = T_s - q$, induction immediately gives $\overline{C_{w,\gamma}} = C_{w,\gamma}$. We define $P'_{y,\delta,w,\gamma}$ and $P''_{y,\delta,w,\gamma}$ by

$$(T_s - q)C_{v,\gamma} = \sum_{\substack{y\in D_J\\ \delta\in\Gamma}} P'_{y,\delta,w,\gamma} T_y \delta \tag{5}$$

$$\sum_{\substack{(z,\theta)\prec(v,\gamma)\\ (z,\theta)\in\Lambda_s^-}} \mu(z,\theta,v,\gamma)C_{z,\theta} = \sum_{\substack{y\in D_J\\ \delta\in\Gamma}} P''_{y,\delta,w,\gamma} T_y \delta \tag{6}$$

and define $P_{y,\delta,w,\gamma} = P'_{y,\delta,w,\gamma} - P''_{y,\delta,w,\gamma}$.

If $y \in D_J$ then

$$(T_s - q)T_y = \begin{cases} T_{sy} - qT_y & \text{if } y \in D^+_{J,s} \\ T_{sy} - q^{-1}T_y & \text{if } y \in D^-_{J,s} \\ T_y(T_t - q) & \text{if } y \in D^0_{J,s} \end{cases}$$

where we have written $t = y^{-1}sy \in J$ in the case $y \in D^0_{J,s}$. Thus we see that

$$\begin{aligned}(T_s - q)C_{v,\gamma} &= \sum_{\substack{y\in D^+_{J,s}\\ \delta\in\Gamma}} P_{y,\delta,v,\gamma}(T_{sy} - qT_y)\delta \ + \sum_{\substack{y\in D^-_{J,s}\\ \delta\in\Gamma}} P_{y,\delta,v,\gamma}(T_{sy} - q^{-1}T_y)\delta \\ &\qquad + \sum_{\substack{y\in D^0_{J,s}\\ \delta\in\Gamma}} P_{y,\delta,v,\gamma}T_y(T_t - q)\delta \\ &= \sum_{\substack{y\in D^-_{J,s}\\ \delta\in\Gamma}} (P_{sy,\delta,v,\gamma} - q^{-1}P_{y,\delta,v,\gamma})T_y\delta \ + \sum_{\substack{y\in D^+_{J,s}\\ \delta\in\Gamma}} (P_{sy,\delta,v,\gamma} - qP_{y,\delta,v,\gamma})T_y\delta \\ &\qquad + \sum_{\substack{y\in D^0_{J,s}\\ \theta\in\Gamma}} P_{y,\theta,v,\gamma}T_y(T_t - q)\theta.\end{aligned}$$

Now for all $t \in J$ and $\theta \in \Gamma$,

$$(T_t - q)\theta = \begin{cases} (-q - q^{-1})\theta & \text{if } \theta \in \Gamma_t^- \\ \sum\limits_{\delta\in\Gamma_t^-} \mu(\delta,\theta)\delta & \text{if } \theta \in \Gamma_t^+, \end{cases}$$

and therefore

$$\begin{aligned}\sum_{\theta\in\Gamma} P_{y,\theta,v,\gamma}T_y(T_t - q)\theta &= \sum_{\theta\in\Gamma_t^-} (-q - q^{-1})P_{y,\theta,v,\gamma}T_y\theta \ + \sum_{\substack{\theta\in\Gamma_t^+\\ \delta\in\Gamma_t^-}} \mu(\delta,\theta)P_{y,\theta,v,\gamma}T_y\delta \\ &= \sum_{\delta\in\Gamma_t^-} \Big((-q - q^{-1})P_{y,\delta,v,\gamma} + \sum_{\theta\in\Gamma_t^+} \mu(\delta,\theta)P_{y,\theta,v,\gamma}\Big)T_y\delta.\end{aligned}$$

Now comparing Eq. (5) with the expression for $(T_s - q)C_{v,\gamma}$ obtained above we obtain the following formulas:

$$(7)\quad P'_{y,\delta,w,\gamma} = \begin{cases} P_{sy,\delta,v,\gamma} - qP_{y,\delta,v,\gamma} & \text{if } y \in D^+_{J,s},\\ P_{sy,\delta,v,\gamma} - q^{-1}P_{y,\delta,v,\gamma} & \text{if } y \in D^-_{J,s},\\ (-q-q^{-1})P_{y,\delta,v,\gamma} + \sum_{\theta\in\Gamma^+_t} \mu(\delta,\theta)P_{y,\theta,v,\gamma} & \text{if } y \in D^0_{J,s} \text{ and } \delta \in \Gamma^-_t,\\ 0 & \text{if } y \in D^0_{J,s} \text{ and } \delta \in \Gamma^+_t. \end{cases}$$

Since $C_{z,\theta} = \sum_{y,\delta} P_{y,\delta,z,\theta}T_y\delta$, we have

$$\sum_{\substack{(z,\theta)\prec(v,\gamma)\\(z,\theta)\in\Lambda^-_s}} \mu(z,\theta,v,\gamma)C_{z,\theta} = \sum_{\substack{y\in D_J\\ \delta\in\Gamma}} \sum_{\substack{(z,\theta)\prec(v,\gamma)\\(z,\theta)\in\Lambda^-_s}} \mu(z,\theta,v,\gamma)P_{y,\delta,z,\theta}T_y\delta$$

and by comparison with Eq. (6)

$$(8)\qquad P''_{y,\delta,w,\gamma} = \sum_{\substack{(z,\theta)\prec(v,\gamma)\\(z,\theta)\in\Lambda^-_s}} \mu(z,\theta,v,\gamma)P_{y,\delta,z,\theta}.$$

We must check that with $P'_{y,\delta,w,\gamma}$ and $P''_{y,\delta,w,\gamma}$ given by Eq's (7) and (8), the elements $P_{y,\delta,w,\gamma} = P'_{y,\delta,w,\gamma} - P''_{y,\delta,w,\gamma}$ lie in $\mathcal{A}^+$ and satisfy Conditions (i), (ii) and (iii) of Theorem 5.1.

In the second and third cases of Eq. (7), observe that $y \notin \Gamma^+_t$ whereas $v \in \Gamma^+_t$. Hence $y \neq v$, and the inductive hypothesis guarantees that $P_{y,\delta,v,\gamma}$ is an element of $\mathcal{A}^+$ with zero constant term; so $q^{-1}P_{y,\delta,v,\gamma} \in \mathcal{A}^+$. It follows that $P'_{y,\delta,w,\gamma} \in \mathcal{A}^+$ in all cases, and since also $P''_{y,\delta,w,\gamma} \in \mathcal{A}^+$ we deduce that $P_{y,\delta,w,\gamma} \in \mathcal{A}^+$.

With regard to Condition (i), the inductive hypothesis tells us that the right hand side of Eq. (7) is nonzero only if $y \leqslant v$ or $sy \leqslant v$. Since $w = sv$ with $\ell(w) = \ell(v) + 1$, both of these conditions imply that $y \leqslant w$. Hence $P'_{y,\delta,w,\gamma} = 0$ unless $y \leqslant w$. Similarly, the right hand side of Eq. (8) is nonzero only if $y \leqslant z$ for some $z < w$; so $P''_{y,\delta,w,\gamma} = 0$ unless $y < w$. Hence Condition (i) is satisfied.

The above remarks show, in particular, that $P''_{w,\delta,w,\gamma} = 0$ in all cases. Since $w \nleqslant v$ we see that $P_{w,\delta,v,\gamma} = 0$, and since $w \in D^-_{J,s}$ (by the choice of s) the second case in Eq. (7) gives

$$P_{w,\delta,w,\gamma} = P'_{w,\delta,w,\gamma} = P_{v,\delta,v,\gamma} = \begin{cases} 1 & \text{if } \delta = \gamma\\ 0 & \text{if } \delta \neq \gamma. \end{cases}$$

Hence Condition (ii) is satisfied.

It remains to check that Condition (iii) is satisfied. We may assume that $y < w$, since otherwise the required conclusion follows from Conditions (i) and (ii).

So suppose that $y < w$, and consider first the case that $y \in D^+_{J,s}$. Then $(z,\theta) = (y,\delta)$ is not permitted in the sum in Eq. (8), since $(z,\theta) \in \Lambda^-_s$ implies that $z \notin D^+_{J,s}$. Hence all the summands have zero constant term (by the inductive hypothesis), and so $P''_{y,\delta,w,\gamma}$ has zero constant term. Furthermore, $y \neq w$ gives $sy \neq v$; so $P_{sy,\delta,v,\gamma}$ has zero constant term, and hence so does $P'_{y,\delta,w,\gamma}$. So Condition (iii) holds in this case.

Next, suppose that $y \in D^-_{J,s}$ and $(y,\delta) \nprec (v,\gamma)$. In this case it is again true that $(z,\theta) = (y,\delta)$ cannot occur in Eq. (8), and so $P''_{y,\delta,w,\gamma}$ has zero constant term.

Furthermore, $(y,\delta) \not\prec (v,\gamma)$ also implies that the coefficient of q in $P_{y,\delta,v,\gamma}$ is zero, whence $q^{-1}P_{y,\delta,v,\gamma}$ has zero constant term. Again $P_{sy,\delta,v,\gamma}$ has zero constant term since $sy \neq v$; so $P'_{y,\delta,w,\gamma}$ has zero constant term, and the desired conclusion follows.

If $y \in D^-_{J,s}$ and $(y,\delta) \prec (v,\gamma)$ then $(z,\theta) = (y,\delta)$ does arise in Eq. (8). Since $P_{y,\delta,y,\delta} = 1$, the corresponding summand is exactly $\mu(y,\delta,v,\gamma)$. Since all the other summands have zero constant term it follows that the constant term of $P''_{y,\delta,w,\gamma}$ is $\mu(y,\delta,v,\gamma)$. This is also the constant term of $P'_{y,\delta,w,\gamma}$, since $\mu(y,\delta,v,\gamma)$ is the coefficient of q in $-P_{y,\delta,v,\gamma}$ while $P_{sy,\delta,v,\gamma}$ has zero constant term. So $P_{y,\delta,w,\gamma}$ has zero constant term.

Finally, suppose that $y \in D^0_{J,s}$. If $\delta \in \Gamma^+_t$—that is, $t \notin I_\delta$—then $(y,\delta) \notin \Lambda^-_s$, and so $(z,\theta) = (y,\delta)$ is not allowed in Eq. (8). Hence $P''_{y,\delta,w,\gamma}$ has zero constant term. Since in this case we also have that $P'_{y,\delta,w,\gamma} = 0$, the desired conclusion follows. So it remains to consider $\delta \in \Gamma^-_t$. In this case $(z,\theta) = (y,\delta)$ occurs in Eq. (8) if and only if $(y,\delta) \prec (v,\gamma)$. So, as above, we see that $P''_{y,\delta,w,\gamma}$ has constant term $\mu(y,\delta,v,\gamma)$ if $(y,\delta) \prec (v,\gamma)$, and zero in the other case. Turning to $P'_{y,\delta,w,\gamma}$, we see that the summands $\mu(\delta,\theta)P_{y,\theta,v,\gamma}$ all have zero constant term, while the constant term of $(-q-q^{-1})P_{y,\delta,v,\gamma}$ is the coefficient of q in $P_{y,\delta,v,\gamma}$, which is $\mu(y,\delta,v,\gamma)$ if $(y,\delta) \prec (v,\gamma)$ and zero otherwise. So $P_{y,\delta,w,\gamma} = P'_{y,\delta,w,\gamma} - P''_{y,\delta,w,\gamma}$ has zero constant term in either case, as required. □

Observe that the formula for $C_{w,\gamma}$ in Theorem 5.1 may be written as

$$C_{w,\gamma} = T_w\gamma + \sum_{\{y,\delta \mid y<w\}} P_{y,\delta,w,\gamma} T_y \delta,$$

and inverting this gives

$$T_w\gamma = C_{w,\gamma} + \sum_{\{y,\delta \mid y<w\}} Q_{y,\delta,w,\gamma} C_{y,\delta} \tag{9}$$

where the elements $Q_{y,\delta,w,\gamma}$ (defined whenever $y < w$) are given recursively by

$$Q_{y,\delta,w,\gamma} = -P_{y,\delta,w,\gamma} - \sum_{\{z,\theta \mid y<z<w\}} Q_{y,\delta,z,\theta} P_{z,\theta,w,\gamma}.$$

In particular, $Q_{y,\delta,w,\gamma}$ is in $\mathcal{A}^+$, has zero constant term, and has coefficient of q equal to $\mu(y,\delta,w,\gamma)$.

We now state the main result of this paper.

Theorem 5.3. *The elements $C_{w,\gamma}$ give M a W-graph structure, as described above.*

Proof. For all (z,θ), $(w,\gamma) \in D_J \times \Gamma$ we define

$$\xi(z,\theta,w,\gamma) = \begin{cases} \mu(z,\theta,w,\gamma) & \text{if } (z,\theta) \prec (w,\gamma) \text{ or } z = w, \\ 1 & \text{if } (z,\theta) = (rw,\gamma) \text{ and } \ell(z) > \ell(w) \text{ for some } r \in S, \\ 0 & \text{otherwise.} \end{cases}$$

We start by using induction on $\ell(w)$ to prove that for all $s \in S$

$$T_s C_{w,\gamma} = \begin{cases} -q^{-1} C_{w,\gamma} & \text{if } (w,\gamma) \in \Lambda^-_s, \\ qC_{w,\gamma} + \sum_{(z,\theta)\in\Lambda^-_s} \xi(z,\theta,w,\gamma) C_{z,\theta} & \text{if } (w,\gamma) \notin \Lambda^-_s. \end{cases} \tag{10}$$

If $w \in D^+_{J,s}$ then $(w,\gamma) \notin \Lambda^-_s$, and Eq. (10) follows immediately from Theorem 5.2 (applied with v replaced by w), since the only $(z,\theta) \in \Lambda^-_s$ with $\xi(z,\theta,w,\gamma) \neq 0$ and $\ell(z) \geqslant \ell(w)$ is $(z,\theta) = (sw,\gamma)$.

If $w \in D^-_{J,s}$, which implies that $(w,\gamma) \in \Lambda^-_s$, then writing $v = sw$ and applying Theorem 5.2 gives

$$C_{w,\gamma} = (T_s - q)C_{v,\gamma} - \sum \mu(z,\delta,v,\gamma)C_{z,\delta},$$

where $(z,\delta) \prec (v,\gamma)$ and $(z,\delta) \in \Lambda^-_s$ for all terms in the sum. The inductive hypothesis thus gives $T_s C_{z,\delta} = -C_{z,\delta}$, and since also $T_s(T_s - q) = -q^{-1}(T_s - q)$ it follows that $T_s C_{w,\gamma} = -q^{-1}C_{w,\gamma}$, as required.

Now suppose that $w \in D^0_{J,s}$, and as usual let us write $sw = wt$, where $t \in J$. Suppose first that $t \in I_\gamma$, so that $(w,\gamma) \in \Lambda^-_s$. By Eq. (9) above,

$$C_{w,\gamma} = T_w\gamma - \sum_{\{y,\delta \mid y<w\}} Q_{y,\delta,w,\gamma} C_{y,\delta},$$

and since $T_s T_w \gamma + q^{-1} T_w \gamma = T_w(T_t\gamma + q^{-1}\gamma) = 0$ we find that

$$T_s C_{w,\gamma} + q^{-1} C_{w,\gamma} = - \sum_{\{y,\delta \mid y<w\}} Q_{y,\delta,w,\gamma}(T_s C_{y,\delta} + q^{-1} C_{y,\delta}). \tag{11}$$

By the inductive hypothesis,

$$T_s C_{y,\delta} + q^{-1} C_{y,\delta} = \begin{cases} 0 & \text{if } (y,\delta) \in \Lambda^-_s \\ (q+q^{-1})C_{y,\delta} + \sum\limits_{(z,\theta)\in\Lambda^-_s} \xi(z,\theta,y,\delta)C_{z,\theta} & \text{if } (y,\delta) \notin \Lambda^-_s, \end{cases}$$

and so Eq. (11) gives

$$T_s C_{w,\gamma} + q^{-1} C_{w,\gamma} = - \sum_{\substack{(y,\delta)\notin\Lambda^-_s \\ y<w}} Q_{y,\delta,w,\gamma}(q+q^{-1})C_{y,\delta} + X \tag{12}$$

for some X in the $\mathcal{A}$-module spanned by the elements $C_{z,\theta}$ for $(z,\theta) \in \Lambda^-_s$. Now since $T_s = T_s^{-1} + (q - q^{-1})$ it follows that

$$\begin{aligned} (T_s + q^{-1})C_{w,\gamma} &= \overline{(T_s + q^{-1})C_{w,\gamma}} \\ &= - \sum_{\substack{(y,\delta)\notin\Lambda^-_s \\ y<w}} \overline{Q_{y,\delta,w,\gamma}}(q^{-1}+q)C_{y,\delta} + \overline{X}, \end{aligned}$$

and comparing with Eq. (12) shows that for all (y,δ) with $y < w$ and $(y,\delta) \notin \Lambda^-_s$,

$$\overline{Q_{y,\delta,w,\gamma}} = Q_{y,\delta,w,\gamma}. \tag{13}$$

Since $Q_{y,\delta,w,\gamma}$ is in $\mathcal{A}^+$ and has zero constant term, Eq. (13) forces $Q_{y,\delta,w,\gamma} = 0$ whenever $y < w$ and $(y,\delta) \notin \Lambda^-_s$. Thus the right hand side of Eq. (11) is zero, since $T_s C_{y,\delta} + C_{y,\delta} = 0$ whenever $(y,\delta) \in \Lambda^-_s$. So

$$T_s C_{w,\gamma} = -q^{-1} C_{w,\gamma},$$

as required.

Now suppose that $t \notin I_\gamma$, so that $(w,\gamma) \notin \Lambda^-_s$. Replacing γ by θ in Eq. (9) we obtain

$$C_{w,\theta} = T_w\theta - \sum_{\{y,\delta \mid y<w\}} Q_{y,\delta,w,\theta} C_{y,\delta},$$

for all $\theta \in \Gamma$. It follows that

$$(T_s - q)C_{w,\gamma} - \sum_{\theta \in \Gamma_t^-} \mu(\theta,\gamma)C_{w,\theta} \tag{14}$$

is the sum of

$$(T_s - q)T_w\gamma - \sum_{\theta \in \Gamma_t^-} \mu(\theta,\gamma)T_w\theta \tag{15}$$

and

$$-\sum_{\{y,\delta \mid y<w\}} \Big(Q_{y,\delta,w,\gamma}(T_s - q)C_{y,\delta} - \sum_{\theta \in \Gamma_t^-} \mu(\theta,\gamma)Q_{y,\delta,w,\theta}C_{y,\delta} \Big).$$

Using the inductive hypothesis to evaluate $(T_s - q)C_{y,\delta}$, this last expression can be written as the sum of the following three terms:

$$-\sum_{\substack{(y,\delta)\in\Lambda_s^+ \\ y<w}} \sum_{(z,\theta)\in\Lambda_s^-} Q_{y,\delta,w,\gamma}\xi(z,\theta,y,\delta)C_{z,\theta}, \tag{16}$$

$$\sum_{\substack{(y,\delta)\in\Lambda_s^- \\ y<w}} Q_{y,\delta,w,\gamma}(q^{-1} + q)C_{y,\delta}, \tag{17}$$

$$\sum_{\substack{\{y,\delta) \mid y<w\} \\ \theta \in \Gamma_t^-}} \mu(\theta,\gamma)Q_{y,\delta,w,\theta}C_{y,\delta}. \tag{18}$$

Now the expression (15) is zero, since

$$(T_s - q)T_w\gamma - \sum_{\theta \in \Gamma_t^-} \mu(\theta,\gamma)T_w\theta = T_w\Big(T_t\gamma - q\gamma - \sum_{\theta \in \Gamma_t^-} \mu(\theta,\gamma)\theta\Big),$$

and $t \notin I_\gamma$. Observe that the coefficient of each $C_{y,\delta}$ in the sum of the expressions (16), (17) and (18) is in $\mathcal{A}^+$, and the only contributions to the constant terms of these coefficients come from (17) when $(y,\delta) \prec (w,\gamma)$. However, the expression (14) is invariant under the involution $m \mapsto \overline{m}$; hence the total coefficient of each $C_{y,\delta}$ in the sum of (16), (17) and (18) must be a constant (since no other elements of $\mathcal{A}^+$ are invariant under the involution). So we conclude that

$$(T_s - q)C_{w,\gamma} - \sum_{\theta \in \Gamma_t^-} \mu(\theta,\gamma)C_{w,\theta} = \sum_{\substack{(y,\delta)\in\Lambda_s^- \\ (y,\delta)\prec(w,\gamma)}} \mu(y,\delta,w,\gamma)C_{y,\delta}.$$

Since $\mu(\theta,\gamma) = \mu(w,\theta,w,\gamma)$, and the condition $\theta \in \Gamma_t^-$ is equivalent to $(w,\theta) \in \Lambda_s^-$, this may be rewritten as

$$T_sC_{w,\gamma} = qC_{w,\gamma} + \sum \mu(y,\delta,w,\gamma)C_{y,\delta}.$$

where the sum is over all $(y,\delta) \in \Lambda_s^-$ such that $(y,\delta) \prec (w,\gamma)$ or $y = w$. To deduce that Eq. (10) holds, it remains to check that that there is no $z \in D_J$ such that $(z,\gamma) \in \Lambda_s^-$ and $\ell(z) = \ell(w) + 1$, with $z = rw$ for some $r \in S$.

Clearly these conditions cannot hold with $r = s$, as $sw \notin D_J$; so we may suppose that $r \neq s$. Now $(z,\gamma) \in \Lambda_s^-$ implies that either $\ell(sz) < \ell(z)$ or $sz = zu$ for some $u \in I_\gamma$. In the former case we would have both $\ell(sz) < \ell(z)$ and $\ell(rz) < \ell(z)$, implying that $\ell(srz) = \ell(z) - 2$, a contradiction since $srz = rzt$ and $rz \in D_J$. The other case gives a similar contradiction, since $\ell(s(zu)) = \ell(z) < \ell(zu)$ and

$\ell(r(zu)) = \ell(wu) < \ell(zu)$, whereas the length of $srzu = rztu$ is greater than or equal to $\ell(rz)$, and hence is not $\ell(zu) - 2$.

We have now completed the proof of Eq. (10), and to complete the proof of Theorem 5.3 it remains to show that for all $s \in S$ we have $\xi(z,\theta,w,\gamma) = \mu(z,\theta,w,\gamma)$ whenever $(z,\theta) \in \Lambda_s^-$ and $(w,\gamma) \notin \Lambda_s^-$. This is true by definition whenever $\ell(z) \leqslant \ell(w)$, both sides being zero unless $(z,\theta) \prec (w,\gamma)$ or $z = w$. If $\ell(z) > \ell(w)$ then both sides are zero unless $(w,\gamma) \prec (z,\theta)$.

So we must show that $(w,\gamma) \prec (z,\theta)$ with $(z,\theta) \in \Lambda_s^-$ and $(w,\gamma) \notin \Lambda_s^-$ implies that $(z,\theta) = (rw,\gamma)$, where $r \in S$ and $\ell(z) = \ell(w) + 1$, and $\mu(z,\theta,w,\gamma) = 1$. In fact we shall show that this holds with $r = s$ (which is the only possibility, as could be shown directly by an argument similar to the one used above).

Since $(z,\theta) \in \Lambda_s^-$ we have that $T_s C_{z,\theta} = -C_{z,\theta}$, whence

$$\sum_{y\in D_J,\delta\in\Gamma} P_{y,\delta,z,\theta} T_s T_y \delta = -\sum_{y\in D_J,\delta\in\Gamma} P_{y,\delta,z,\theta} T_y \delta. \tag{19}$$

If $w \in D_{J,s}^0$, so that $(w,\gamma) \notin \Lambda_s^-$ gives $\gamma \notin \Gamma_t^-$ (where $t = w^{-1}sw$), then comparing the coefficients of $T_w\gamma$ gives $P_{w,\gamma,z,\theta} = 0$ (since $T_sT_w\gamma = T_wT_t\gamma = qT_w\gamma + X$, where X is a combination of terms of the form $T_w\delta$ with $\delta \in \Gamma_t^-$). This contradicts $(w,\gamma) \prec (z,\theta)$. The only alternative is $w \in D_{J,s}^+$, and in this case comparison of the coefficients of $T_{sw}\gamma$ on the two sides of Eq. (19) gives

$$(q - q^{-1})P_{sw,\gamma,z,\theta} + P_{w,\gamma,z,\theta} = -q^{-1}P_{sw,\gamma,z,\theta},$$

which reduces to

$$qP_{sw,\gamma,z,\theta} = -P_{w,\gamma,z,\theta}.$$

Since $(w,\gamma) \prec (z,\theta)$ the coefficient of q in $P_{w,\gamma,z,\theta}$ is nonzero; so the constant term of $P_{sw,\gamma,z,\theta}$ is nonzero. So $(sw,\gamma) = (z,\theta)$ and $-P_{w,\gamma,z,\theta} = q$, whence $\mu(w,\gamma,z,\theta) = 1$, as required. □

It is convenient to distinguish three kinds of edges of the W-graph Λ. Firstly, there is an edge from the vertex $C_{w,\gamma}$ to the vertex $C_{w,\delta}$ whenever there is an edge from γ to δ in Γ. We call these *horizontal* edges. Next, if $s \in S$ and w is in either $D_{J,s}^+$ or $D_{J,s}^-$ then there is an edge joining $C_{w,\gamma}$ and $C_{sw,\gamma}$. We call these *vertical* edges. All other edges are called *transverse*.

Proposition 5.4. *Suppose that vertices $C_{w,\gamma}$ and $C_{z,\theta}$ of Λ are joined by a transverse edge, and suppose that $\ell(w) \leqslant \ell(z)$. Then $I(z,\theta) \subseteq I(w,\gamma)$.*

Proof. Let $s \in I(z,\theta)$, and suppose, for a contradiction, that $s \notin I(w,\gamma)$. Since the edge is not horizontal we have either $(w,\gamma) \prec (z,\theta)$ or $(z,\theta) \prec (w,\gamma)$, and the assumption $\ell(w) \leqslant \ell(z)$ means that the former alternative holds. So we have $(w,\gamma) \prec (z,\theta)$, with $(z,\theta) \in \Lambda_s^-$ and $(w,\gamma) \in \Lambda_s^+$. We showed in the course of the previous proof that these conditions imply that $(z,\theta) = (sw,\gamma)$. This means that the edge $\{C_{w,\gamma}, C_{z,\theta}\}$ is vertical rather than transverse, and so we have the desired contradiction. □

Proposition 5.5. *Suppose that the W_J-graph Γ admits a partial order $\leqslant$ satisfying the conditions of Definition 2.2. Then the induced W-graph Λ admits a partial order $\leqslant$ satisfying Definition 2.2 and having the following properties*:

(i) *if δ, $\gamma \in \Gamma$ and y, $w \in D_J$ are such that $y \leqslant w$ and $\delta \leqslant \gamma$, then $C_{y,\delta} \leqslant C_{w,\gamma}$;*

(ii) *if* $\delta,\,\gamma\in\Gamma$ *and* $y,\,w\in D^+_{J,s}$ *for some* $s\in S$*, then* $C_{y,\delta}\leqslant C_{w,\gamma}$ *implies that* $C_{sy,\delta}\leqslant C_{sw,\gamma}$*;*

(iii) *if* $y\in D^0_{J,s}$ *and* $w\in D^+_{J,s}$ *for some* $s\in S$*, then* $C_{y,\delta}\leqslant C_{w,\gamma}$ *implies that* $C_{y,t\delta}\leqslant C_{sw,\gamma}$*, for all* $\gamma\in\Gamma$ *and* $\delta\in\Gamma^+_t$ *such that* $t\delta$ *exists, where* $t=y^{-1}sy$*;*

(iv) *if* (y,δ), $(w,\gamma)\in D_J\times\Gamma$ *satisfy* $P_{y,\delta,w,\gamma}\neq 0$ *then* $C_{y,\delta}\leqslant C_{w,\gamma}$*.*

Proof. We define $\leqslant$ on Λ to be the minimal transitive relation satisfying the requirements (i), (ii) and (iii). It is clear that $C_{y,\delta}\leqslant C_{w,\gamma}$ implies that $y\leqslant w$, with equality only if $\delta\leqslant\gamma$. Hence the fact that the relation $\leqslant$ on Γ is antisymmetric implies the same for the relation $\leqslant$ on Λ.

We prove first that Condition (iv) is satisfied, using induction on $\ell(w)$. In the case $\ell(w)=0$ the assumption that $P_{y,\delta,w,\gamma}\neq 0$ forces $(y,\delta)=(w,\gamma)$, and so $C_{y,\delta}\leqslant C_{w,\gamma}$. So suppose that $\ell(w)>0$, and choose $s\in S$ with $\ell(sw)<\ell(w)$. Recall that $P_{y,\delta,w,\gamma}=P'_{y,\delta,w,\gamma}-P''_{y,\delta,w,\gamma}$; hence either $P''_{y,\delta,w,\gamma}\neq 0$ or $P'_{y,\delta,w,\gamma}\neq 0$.

If $P''_{y,\delta,w,\gamma}\neq 0$ then by Eq. (8) there exists a pair (z,θ) with $(z,\theta)\prec(sw,\gamma)$ and $P_{y,\delta,z,\theta}\neq 0$. The inductive hypothesis then yields both $C_{y,\delta}\leqslant C_{z,\theta}$ and $C_{z,\theta}\leqslant C_{sw,\gamma}$, and since also $C_{sw,\gamma}\leqslant C_{w,\gamma}$ it follows that $C_{y,\delta}\leqslant C_{w,\gamma}$, as required. So we may assume that $P'_{y,\delta,w,\gamma}\neq 0$.

Suppose first that $y\in D^+_{J,s}$. By Eq. (7) either $P_{y,\delta,sw,\gamma}\neq 0$ or $P_{sy,\delta,sw,\gamma}\neq 0$, and so the inductive hypothesis yields that either $C_{y,\delta}\leqslant C_{sw,\gamma}$ or $C_{sy,\delta}\leqslant C_{sw,\gamma}$. Since $C_{y,\delta}\leqslant C_{sy,\delta}$ we obtain $C_{y,\delta}\leqslant C_{sw,\gamma}$ in either case, and hence $C_{y,\delta}\leqslant C_{w,\gamma}$.

Now suppose that $y\in D^-_{J,s}$. Again Eq. (7) and the inductive hypothesis combine to yield that either $C_{y,\delta}\leqslant C_{sw,\gamma}$ or $C_{sy,\delta}\leqslant C_{sw,\gamma}$. The former alternative yields $C_{y,\delta}\leqslant C_{w,\gamma}$ as in the previous cases, while the latter alternative yields the same result since (ii) above holds.

Finally, suppose that $y\in D^0_{J,s}$, and let $t=y^{-1}sy\in J$. By Eq. (7) we see that either $P_{y,\delta,sw,\gamma}\neq 0$, which yields $C_{y,\delta}\leqslant C_{w,\gamma}$ as in the previous cases, or else $\delta\in\Gamma^-_t$ and $\mu(\delta,\theta)P_{y,\theta,sw,\gamma}\neq 0$ for some $\theta\in\Gamma^+_t$. Thus $\{\theta,\delta\}$ is an edge of Γ with $t\in I_\delta$ and $t\notin I_\theta$, and by Conditions (i), (ii) of Definition 2.2 it follows that either $\delta=t\theta$ or $\delta\leqslant\theta$. Moreover, since $P_{y,\theta,sw,\gamma}\neq 0$ the inductive hypothesis yields that $C_{y,\theta}\leqslant C_{sw,\gamma}$. If $\delta\leqslant\theta$ then $C_{y,\delta}\leqslant C_{y,\theta}$, and so $C_{y,\delta}\leqslant C_{sw,\gamma}\leqslant C_{w,\gamma}$. If $\delta=t\theta$ then $C_{y,\delta}\leqslant C_{w,\gamma}$ follows from $C_{y,\theta}\leqslant C_{sw,\gamma}$, in view of (iii) above.

It remains to show that Λ is an ordered W-graph in the sense of Definition 2.2.

Let $C_{y,\delta}$, $C_{w,\gamma}\in\Lambda$ with $\mu(y,\delta,w,\gamma)\neq 0$. If $y=w$ then $\mu(y,\delta,w,\gamma)=\mu(\delta,\gamma)$, and since Γ is an ordered W_J-graph it follows that γ and δ are comparable, whence so are (w,γ) and $(w,\delta)=(y,\delta)$. If $y\neq w$ then $\mu(y,\delta,w,\gamma)$ is a coefficient of one or other of the polynomials $P_{y,\delta,w,\gamma}$ and $P_{w,\gamma,y,\delta}$, and so (iv) above implies that (w,γ) and (y,δ) are comparable. So Condition (i) of Definition 2.2 holds.

Let $s\in S$ and $(w,\gamma)\in\Lambda^+_s$, and suppose that $(y,\delta)\in\Lambda^-_s$ with $C_{w,\gamma}<C_{y,\delta}$ and $\mu(y,\delta,w,\gamma)\neq 0$. We must show that (y,δ) is the unique such element of Λ^-_s.

Suppose first that the edge $\{C_{y,\delta},C_{w,\gamma}\}$ is transverse. Since $s\in I(y,\delta)$ and $s\notin I(w,\gamma)$, it follows from Proposition 5.4 that $\ell(w)\nleqslant\ell(y)$, and so $(y,\delta)\prec(w,\gamma)$. But this implies that $P_{y,\delta,w,\gamma}\neq 0$, and in view of (iv) this contradicts the assumption that $C_{w,\gamma}<C_{y,\delta}$. So $\{C_{y,\delta},C_{w,\gamma}\}$ is either vertical or horizontal.

If the edge $\{C_{y,\delta},C_{w,\gamma}\}$ is vertical then $\delta=\gamma$ and $y=rw$ for some $r\in S$. Since $C_{w,\gamma}<C_{y,\gamma}$ we have $w\leqslant y$; so $\ell(w)\leqslant\ell(rw)$. Now since $s\in I(rw,\gamma)$ and

$s \notin I(w,\gamma)$ it follows readily that $r = s$. So $(y,\delta) = (sw,\gamma)$; moreover, this case can only arise if $w \in D^+_{J,s}$.

Now suppose that $\{C_{y,\delta}, C_{w,\gamma}\}$ is horizontal, so that $y = w$ and $\{\delta,\gamma\}$ is an edge of Γ. Since Γ is an ordered W_J-graph, Condition (i) of Definition 2.2 yields that either $\gamma < \delta$ or $\delta < \gamma$; however, the latter alternative would give $C_{w,\delta} < C_{w,\gamma}$, contradicting our assumption that $C_{w,\gamma} < C_{y,\delta} = C_{w,\delta}$. Now since $s \in I(w,\delta)$ and $s \notin I(w,\gamma)$ we see that $w \in D^0_{J,s}$, and $t = w^{-1}sw$ is in I_δ and not in I_γ. Since Γ satisfies Condition (ii) of Definition 2.2 it follows that $\delta = t\gamma$.

We have shown that

$$(y,\delta) = \begin{cases} (sw,\gamma) & \text{if } w \in D^+_{J,s} \\ (w,t\gamma) & \text{if } w \in D^0_{J,s} \end{cases}$$

where $t = w^{-1}sw$. So (y,δ) is uniquely determined. In accordance with Definition 2.2, we write $C_{y,\delta} = sC_{w,\gamma}$.

It remains to check that Λ satisfies Condition (iii) of Definition 2.2; that is, we must show that if $(w,\gamma) \in \Lambda^+_s$ and $C_{y,\delta} = sC_{w,\gamma}$ then $\mu(y,\delta,s,\gamma) = 1$. If $w \in D^0_{J,s}$ with $w^{-1}sw = t$ then $sC_{w,\gamma}$ is defined if and only if $t\gamma$ is defined, in which case $sC_{w,\gamma} = C_{w,t\gamma}$. Moreover, in this case we have that $\mu(w,t\gamma,w,\gamma) = \mu(t\gamma,\gamma) = 1$, since Γ satisfies Condition (iii) of Definition 2.2. On the other hand, if $w \in D^+_{J,s}$ then $s(w,\gamma) = (sw,\gamma)$, and the desired conclusion that $\mu(sw,\gamma,w,\gamma) = 1$ follows from Theorem 5.2. □

6. Inducing cells

Let $(w,\gamma) \in D_J \times \Gamma$, and let $s \in S$. If $(w,\gamma) \in \Lambda^-_s$ then $T_sC_{w,\gamma} = -q^{-1}C_{w,\gamma}$, and so

$$-q^{-1} \sum_{\substack{y \in D_J \\ \delta \in \Gamma}} P_{y,\delta,w,\gamma} T_y\delta = \sum_{\substack{y \in D_J \\ \delta \in \Gamma}} P_{y,\delta,w,\gamma} T_sT_y\delta. \tag{20}$$

We also have

$$T_sT_y\delta = \begin{cases} T_{sy}\delta & \text{if } y \in D^+_{J,s} \\ T_{sy}\delta + (q-q^{-1})T_y\delta & \text{if } y \in D^-_{J,s} \\ -q^{-1}T_y\delta & \text{if } y \in D^0_{J,s} \text{ and } \delta \in \Gamma^-_t \\ qT_y\delta + \sum_{\theta \in \Gamma^-_t} \mu(\theta,\delta)T_y\theta & \text{if } y \in D^0_{J,s} \text{ and } \delta \in \Gamma^+_t \end{cases}$$

where $t = y^{-1}sy$. Substituting this into Eq. (20) and equating coefficients yields a proof of the following result.

Proposition 6.1. *Let $s \in S$ and $(w,\gamma) \in \Lambda^-_s$. If $y \in D^0_{J,s}$ and $\delta \in \Gamma^+_t$, where $t = y^{-1}sy$, then $P_{y,\delta,w,\gamma} = 0$. If $y \in D^+_{J,s}$ then $P_{y,\delta,w,\gamma} = -qP_{sy,\delta,w,\gamma}$ for all $\delta \in \Gamma$.*

Note that this simplifies our original inductive formulas for the polynomials $P_{y,\delta,w,\gamma}$. In particular, in the situation of Eq. (8) we have that $P''(y,\delta,w,\gamma) = 0$ when $y \in D^0_{J,s}$ and $\delta \in \Gamma^+_t$.

Let $\leq_\Gamma$ be the preorder on Γ defined (as in [6]) by the rule that $\delta \leq_\Gamma \gamma$ if and only if there exists a finite sequence $\delta = \gamma_0, \gamma_1, \ldots, \gamma_k = \gamma$ of elements of Γ with $\mu(\gamma_{i-1},\gamma_i) \neq 0$ and $I(\gamma_{i-1}) \nsubseteq I(\gamma_i)$ for all $i \in \{1,2,\ldots,k\}$.

Proposition 6.2. *Let y, $w \in D_J$ and δ, $\gamma \in \Gamma$ with $\delta \not\leq_\Gamma \gamma$. Then $P_{y,\delta,w,\gamma} = 0$.*

Proof. Use induction on $\ell(w)$. Since $\delta \neq \gamma$ the case $\ell(w) = 0$ follows from (i) and (ii) of Theorem 5.1. So assume that $\ell(w) > 0$, and let $w = sv$ where $s \in S$ and $\ell(v) = \ell(w) - 1$.

The inductive hypothesis immediately implies that the terms on the right hand side of Eq. (7) are zero, with the possible exception of the terms $\mu(\delta,\theta)P_{y,\theta,v,\gamma}$ in the sum that appears in the third case (when $y \in D^0_{J,s}$ and $\delta \in \Gamma^-_t$). In all of these terms we have that $I_\delta \nsubseteq I_\theta$, since $t \in I_\delta$ and $t \notin I_\theta$. So either $\delta \leq_\Gamma \theta$ or else $\mu(\delta,\theta) = 0$. By the inductive hypothesis, either $\theta \leq_\Gamma \gamma$ or else $P_{y,\theta,v,\gamma} = 0$. But since $\delta \nleq_\Gamma \gamma$ we cannot have both $\delta \leq_\Gamma \theta$ and $\theta \leq_\Gamma \gamma$; so either $\mu(\delta,\theta) = 0$ or $P_{y,\theta,v,\gamma} = 0$. So all the terms $\mu(\delta,\theta)P_{y,\theta,v,\gamma}$ are zero, and so $P'_{y,\delta,w,\gamma} = 0$.

All the elements z appearing on the right hand side of Eq. (8) satisfy $z \leqslant v$, and so the inductive hypothesis tells us that if $\delta \nleq_\Gamma \theta$ then $P_{y,\delta,z,\theta} = 0$. Furthermore, if $\theta \nleq_\Gamma \gamma$ then $P_{z,\theta,v,\gamma} = 0$, and so $\mu(z,\theta,v,\gamma) = 0$. Since $\delta \nleq_\Gamma \gamma$ we must have either $\theta \nleq_\Gamma \gamma$ or $\delta \nleq_\Gamma \theta$, and so all the terms $\mu(z,\theta,v,\gamma)P_{y,\delta,z,\theta}$ are zero. So $P''_{y,\delta,w,\gamma} = 0$, and hence $P_{y,\delta,w,\gamma} = 0$, as required. □

Suppose now that $C_{z,\theta}$ and $C_{w,\gamma}$ vertices of Λ that are adjacent and satisfy $I(z,\theta) \nsubseteq I(w,\gamma)$. If $w = z$ then $s \in I(w,\theta)$ and $s \notin I(w,\gamma)$ forces $sw = wt$ for some $t \in I_\theta$ with $t \notin I_\gamma$. So in this case θ and γ are adjacent vertices of Γ with $I_\theta \nsubseteq I_\gamma$. In particular, $\theta \leq_\Gamma \gamma$. The same conclusion holds trivially if the edge $\{C_{z,\theta}, C_{w,\gamma}\}$ is vertical, since in this case $\theta = \gamma$. If the edge is transverse then by Proposition 5.4 we deduce that $\ell(z) < \ell(w)$, and so we must have $(z,\theta) \prec (w,\gamma)$. Thus $P_{z,\theta,w,\gamma} \neq 0$, and so $\theta \leq_\Gamma \gamma$ by Proposition 6.2.

Let $\leq_\Lambda$ be the preorder relation on the W-graph Λ generated by the requirement that $C_{z,\theta} \leq_\Lambda C_{w,\gamma}$ whenever $C_{z,\theta}$ and $C_{w,\gamma}$ are adjacent and $I(z,\theta) \nsubseteq I(w,\gamma)$. The above calculations have proved the following theorem.

Theorem 6.3. *If $C_{z,\theta}$ and $C_{w,\gamma}$ are vertices of Λ with $C_{z,\theta} \leq_\Lambda C_{w,\gamma}$ then $\theta \leq_\Gamma \gamma$.*

Vertices $\theta, \gamma \in \Gamma$ are said to be *equivalent* if $\theta \leq_\Gamma \gamma$ and $\gamma \leq_\Gamma \theta$, and the corresponding equivalence classes are called the *cells* of Γ. The cells of Λ are similarly defined, using the preorder $\leq_\Lambda$. Theorem 6.3 shows that if Δ is a cell in Γ then the set $\{\, C_{w,\gamma} \mid w \in D_J \text{ and } \gamma \in \Delta \,\}$ is a union of cells in Λ. In the case that Γ is the Kazhdan-Lusztig W_J-graph for the regular representation, this result (and Theorem 6.3) have been proved by Meinolf Geck [4].

7. Connection with Kazhdan-Lusztig polynomials

The following result, which follows from Theorem 5.1 above, is a reformulation of Theorem 1.1 of [6]:

Theorem 7.1. *The algebra $\mathscr{H}$ has a unique basis $\{\, C_w \mid w \in W \,\}$ such that $\overline{C_w} = C_w$ for all w and $C_w = \sum_{y \in W} p_{y,w} T_y$ for some elements $p_{y,w} \in \mathcal{A}^+$ with the following properties::*

(i) $p_{y,w} = 0$ *if* $y \nleqslant w$;
(ii) $p_{w,w} = 1$;
(iii) $p_{y,w}$ *has zero constant term if* $y \neq w$.

The polynomials $p_{y,w}$ are related to the polynomials $P_{y,w}$ of [6] (the genuine Kazhdan-Lusztig polynomials) by $p_{y,w}(q) = (-q)^{\ell(w)-\ell(y)}\overline{P_{y,w}(q^2)}$. That is, to get $p_{y,w}$ from $P_{y,w}$ replace q by q^2, apply the bar involution, and then multiply by $(-q)^{\ell(w)-\ell(y)}$. The quantity $\mu(y,w)$, which is the coefficient of $q^{\frac{1}{2}(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$, is the coefficient of q in $(-1)^{\ell(w)-\ell(y)}p_{y,w}$. However, since Kazhdan and Lusztig show that $\mu_{y,w}$ is nonzero only when $\ell(w)-\ell(y)$ is odd, $\mu(y,w)$ is the coefficient of q in $-p_{y,w}$.

The elements C_w form a W-graph basis for $\mathscr{H}$, and Eq. (2.3a) of [6] (or Theorem 5.2 above) shows the W-graph is ordered, in the sense of Definition 2.2, relative to the Bruhat order on W.

Applying Theorem 7.1 with W replaced by W_J yields a W_J-graph basis for the regular representation of $\mathscr{H}_J$. The representation of $\mathscr{H}$ obtained by inducing the regular representation of $\mathscr{H}_J$ is, of course, the regular representation of $\mathscr{H}$. Applying our procedure for inducing W-graphs yields a W-graph basis for $\mathscr{H}$ consisting of elements $C_{w,\gamma}$ (for $w \in D_J$ and $\gamma \in W_J$) such that $\overline{C_{w,\gamma}} = C_{w,\gamma}$ and

$$C_{w,\gamma} = \sum_{y\in D_J}\sum_{\delta\in W_J} P_{y,\delta,w,\gamma} T_y C_\delta, \tag{21}$$

where the polynomials $P_{y,\delta,w,\gamma}$ satisfy the conditions given in Theorem 5.1. By Proposition 5.5 there is a partial order on the set $\Lambda = \{\, C_{w,\gamma} \mid w \in D_J,\ \gamma \in W_J \,\}$ such that for all $y,\ w \in D_J$ and $\delta,\ \gamma \in W_J$,

(i) if $y \leqslant w$ and $\delta \leqslant \gamma$ then $C_{y,\delta} \leqslant C_{w,\gamma}$,
(ii) if $C_{y,\delta} \leqslant C_{w,\gamma}$ and if $y,\ w \in D^+_{J,s}$ for some $s \in S$, then $C_{sy,\delta} \leqslant C_{sw,\gamma}$,
(iii) if $C_{y,\delta} \leqslant C_{w,\gamma}$ with $w \in D^+_{J,s}$ and $y \in D^0_{J,s}$ for some $s \in S$, and if also $t\delta > \delta$ where $t = y^{-1}sy$, then $C_{y,t\delta} \leqslant C_{sw,\gamma}$.

Furthermore, the partial order on Λ is defined to be the minimal partial order satisfying these three properties.

Note that Λ is in bijective correspondence with W via $C_{w,\gamma} \leftrightarrow w\gamma$.

Proposition 7.2. *The above partial order on Λ corresponds exactly the Bruhat order on W, in the sense that $C_{y,\delta} \leqslant C_{w,\gamma}$ if and only if $y\delta \leqslant w\gamma$ in W.*

Proof. Let us check first that the Bruhat order on W does satisfy the properties (i), (ii) and (iii) above. With regard to (i), it is certainly true that $y \leqslant w$ and $\delta \leqslant \gamma$ implies that $y\delta \leqslant w\gamma$. Turning to (ii), suppose that $y,\ w \in D^+_{J,s}$ and $\delta,\ \gamma \in W_J$ with $y\delta \leqslant w\gamma$. Since $w < sw \in D_J$ we see that

$$\ell(sw\gamma) = \ell(sw) + \ell(\gamma) = 1 + \ell(w) + \ell(\gamma) = 1 + \ell(w\gamma),$$

and $\ell(sy\delta) = 1 + \ell(y\delta)$ similarly. So $sy\delta \leqslant sw\gamma$, by Deodhar [2, Theorem 1.1]. For (iii), suppose that $w \in D^+_{J,s}$ and $y \in D^0_{J,s}$, and let $\delta,\ \gamma \in W_J$ with $y\delta \leqslant w\gamma$. Suppose also that $t\delta > \delta$, where $t = y^{-1}sy \in J$. Then

$$\ell(sy\delta) = \ell(yt\delta) = \ell(y) + \ell(t\delta) = 1 + \ell(y) + \ell(\delta) = 1 + \ell(y\delta),$$

and since also $\ell(sw\gamma) = 1 + \ell(w\gamma)$ as above, Deodhar [2, Theorem 1.1] again gives the desired conclusion that $yt\delta = sy\delta \leqslant sw\gamma$.

Since the partial order on Λ is generated by the properties (i), (ii) and (iii), and since also the Bruhat order on W satisfies the same properties, it follows that $C_{y,\delta} \leqslant C_{w,\gamma}$ implies that $y\delta \leqslant w\gamma$ for all $y,\ w \in D_J$ and $\delta,\ \gamma \in W_J$.

We must show, conversely, that $y\delta \leqslant w\gamma$ implies that $C_{y,\delta} \leqslant C_{w,\gamma}$. In view of statement IV in [2, Theorem 1.1] it is sufficient to do this when $\ell(w\gamma) = \ell(y\delta) + 1$. Making this assumption, we argue by induction on $\ell(w)$. Observe that if $\ell(w) = 0$ then $w\gamma = \gamma \in W_J$, and since $y\delta \leqslant w\gamma$ it follows that $y\delta \in W_J$. Hence $y = 1$, and

$C_{y,\delta} \leqslant C_{w,\gamma}$ by Property (i). So suppose that $\ell(w) > 0$, and choose $s \in S$ with $sw < w$.

Consider first the possibility that $sy\delta > y\delta$. Then we must in fact have $sy\delta = w\gamma$, since, using the terminology of [2, Theorem 1.1], Property $Z(s, sy\delta, w\gamma)$ implies that $sy\delta \leqslant w\gamma$. So either $sy = w$ and $\delta = \gamma$, in which case $C_{y,\delta} \leqslant C_{w,\gamma}$ by Property (i), or else $y = w$ and $\gamma = t\delta$, where $t = y^{-1}sy \in J$, and again Property (i) gives $C_{y,\delta} \leqslant C_{w,\gamma}$.

The only alternative is that $sy\delta < y\delta$, and in this case we have that $sy\delta \leqslant sw\gamma$ (by $Z(s, y\delta, w\gamma)$, in Deodhar's terminology). If $y \in D^{-}_{J,s}$ then the inductive hypothesis yields that $C_{sy,\delta} \leqslant C_{sw,\gamma}$, and Property (ii) gives $C_{y,\delta} \leqslant C_{w,\gamma}$. Since $y \in D^{+}_{J,s}$ is not possible given $sy\delta < y\delta$, it remains to deal with the case $y \in D^{0}_{J,s}$. Writing $t = y^{-1}sy$ we have $sy\delta = yt\delta \leqslant sw\gamma$, and the inductive hypothesis gives $C_{y,t\delta} \leqslant C_{sw,\gamma}$. Note that here $t\delta < \delta$ and $sw \in D^{+}_{J,s}$; so applying Property (iii) we obtain the desired conclusion that $C_{y,\delta} \leqslant C_{w,\gamma}$. □

Equation (21) and Theorem 7.1 give $C_\delta = \sum_{\theta\in W_J} p_{\theta,\delta}T_\theta$, and we deduce that

$$C_{w,\gamma} = \sum_{y\in D_J} \sum_{\delta,\theta\in W_J} P_{y,\delta,w,\gamma}\, p_{\theta,\delta} T_{y\theta},$$

since $T_yT_\theta = T_{y\theta}$ for all $y \in D_J$ and $\theta \in W_J$. The coefficient of $T_{y\theta}$ in this expression is $\sum_{\delta\in W_J} P_{y,\delta,w,\gamma}p_{\theta,\delta}$, and for this to be nonzero there must exist a $\delta \in W_J$ such that $P_{y,\delta,w,\gamma}$ and $p_{\theta,\delta}$ are both nonzero. Now $p_{\theta,\delta} \neq 0$ implies that $\theta \leqslant \delta$ by Theorem 7.1, and $P_{y,\delta,w,\gamma} \neq 0$ gives $y\delta \leqslant w\gamma$, by Propositions 5.5 and 7.2. These combine to give $y\theta \leqslant y\delta \leqslant w\gamma$. So if the coefficient of $T_{y\theta}$ in $C_{w,\gamma}$ is nonzero then $y\theta \leqslant w\gamma$. Furthermore, the coefficient is a polynomial in q whose constant term is nonzero only if there exists a $\delta \in W_J$ such that $P_{y,\delta,w,\gamma}$ and $p_{\theta,\delta}$ both have nonzero constant terms. This only occurs when $(y, \delta) = (w, \gamma)$ and $\theta = \delta$; that is, the constant term is nonzero only if $y\theta = w\gamma$. Hence by the uniqueness assertion in Theorem 7.1 we deduce that $C_{w,\gamma} = C_{w\gamma}$, and

$$p_{y\theta,w\gamma} = \sum_{\delta\in W_J} P_{y,\delta,w,\gamma}\, p_{\theta,\delta} \tag{22}$$

for all y, $w \in D_J$ and θ, $\gamma \in W_J$.

Since the elements $C_{w,\gamma}$ produced by our construction coincide with the elements $C_{w\gamma}$ of the Kazhdan-Lusztig construction, the W-graph data of our construction must also agree with Kazhdan-Lusztig. So if $y\theta \leqslant w\gamma$ then $\mu(y\theta, w\gamma)$, the coefficient of q in $-p_{y\theta,w\gamma}$, must equal the element $\mu(y, \theta, w, \gamma)$ of our construction. That is, if $y < w$ then $\mu(y\theta, w\gamma)$ equals the coefficient of q in $-P_{y,\theta,w,\gamma}$, while if $y = w$ then it equals $\mu(\theta, \gamma)$, which is the coefficient of q in $-p_{\theta,\gamma}$. Eq. (22) above confirms this.

8. Concluding remarks

The computer algebra package Magma has been used to calculate the polynomials $P_{y,\delta,w,\gamma}$ when W is of type E_6 and W_J of type D_5, for W_J-graphs corresponding to each of the irreducible characters of W_J. Explicit matrices representing the generators of $\mathscr{H}$ in the induced representations were found, and the defining relations checked.

It seems plausible that Eq. (22) may be useful for computation of Kazhdan-Lusztig polynomials, but we are yet to investigate this.

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