Projective functors and their applications I

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1 Introduction and motivation

• Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of characteristic 0. Set $U = U(\mathfrak{g})$ and let $Z = Z(U)$ be its centre (the ring of Laplace operators).

• Following Bernstein's classic paper, we define and investigate *projective functors* arising from finite-dimensional $\mathfrak{g}$-modules V. These are endofunctors of the category $\mathscr{M}_{Zf}$ of Z-finite $\mathfrak{g}$-modules, occurring as direct summands of the functor

$$F_V : \mathscr{M}_{Zf} \to \mathscr{M}_{Zf}, \quad M \mapsto V \otimes M.$$

When restricted to a category $\mathscr{M}(\theta)$ of $\mathfrak{g}$-modules with fixed central character θ, projective functors and their morphisms are well behaved, and admit easy classifications.

• **Goal today:** See/prove the main theorems on projective functors, then apply them in two directions: finding equivalences $\mathscr{M}(\theta) \cong \mathscr{M}(\theta')$ for certain pairs (θ, θ'), and producing an easy proof of Duflo's theorem.

2 Preliminaries

2.1 Category theory

• All categories and functors are assumed to be k-linear, unless otherwise stated.

• If $\mathscr{B}$ is a complete subcategory of the abelian category $\mathscr{A}$, and $\mathscr{B}$ is closed under subquotients, then $\mathscr{B}$ is abelian too.

• Suppose $\mathscr{A}$ is an abelian category containing a class of objects $\mathscr{P}$ closed under direct sums. An object A is $\mathscr{P}$-generated in case there exists an exact sequence

$$P \to A \to 0$$

in $\mathscr{A}$, and $\mathscr{P}$-presented in case there is an exact sequence

$$P' \to P \to A \to 0$$

in $\mathscr{A}$. The full subcategory of $\mathscr{P}$-presentable objects in $\mathscr{A}$ is denoted $\mathscr{A}_{\mathscr{P}}$.

• The opposite algebra of an associative unital k-algebra A is denoted A°. Thus (A, B)-bimodules X may be identified with left $A \otimes B^\circ$-modules. Write A^2 for the algebra $A \otimes A^\circ$.

• Let us denote by $h(X)$ the functor of tensoring induced by X:

$$h(X) : B\text{-mod} \to A\text{-mod}, \quad M \mapsto X \otimes_B M.$$

Recall that, by definition, a *right continuous* functor is right exact and commutes with inductive limits.

• **Theorem 2.1 (Watt):** Let $\mathscr{C}$ be the full subcategory of right continuous functors within the category of functors B-mod $\to$ A-mod. Then the functor

$$h : (A, B)\text{-bimod} \to \mathscr{C}, \quad X \mapsto h(X)$$

is an equivalence of categories.

2.2 Lie theory

• **Standard notation:**

(i) $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, dual to the space $\mathfrak{h}^*$ of weights of $\mathfrak{g}$.

(ii) R^+ is a choice of positive roots inside the root system R, with half-sum ρ and corresponding nilpotent sublagebra $\mathfrak{n}^+$

(iii) To each $\gamma \in R$ corresponds the dual root $h_\gamma \in \mathfrak{h}$ and the reflection σ_γ,

$$\sigma_\gamma : \mathfrak{h}^* \to \mathfrak{h}^*, \quad \sigma_\gamma(\chi) = \chi - \chi(h_\gamma)\gamma;$$

these generate the Weyl group $W = \langle \sigma_\gamma \rangle$.

(iv) $\Lambda = \{\chi \in \mathfrak{h}^* : \chi(h_\gamma) \in \mathbb{Z} \text{ for all } \gamma \in R\}$ is the lattice of integer weights, containing the sublattice Γ generated by R.

(v) Given $\chi \in \mathfrak{h}^*$, let R_χ denote the set of $\gamma \in R$ for which $\chi(h_\gamma) \in \mathbb{Z}$, and let

$$W_\chi = \mathrm{Stab}_W(\chi), \quad W_{\chi+\Gamma} = \mathrm{Stab}_W(\chi + \Gamma)$$

be stabilisers with respect to the action of W on $\mathfrak{h}^*$ and $\mathfrak{h}^*/\Gamma$. Recall that we call χ regular in case W_χ is trivial.

(vi) $|\chi|$ denotes the length of $\chi \in \mathfrak{h}^*$ with respect to some W-invariant inner product on Λ.

• **A partial order on $\mathfrak{h}^*$:** Given $\gamma \in R^+$, write

$$\psi <_\gamma \chi \quad \text{for } \psi, \chi \in \mathfrak{h}^*$$

whenever $\psi = \sigma_\gamma(\chi)$ and $\chi(h_\gamma) \in \mathbb{Z}^+$. We then let $\psi < \chi$ whenever there exist

$$\psi = \psi_0, \ldots, \psi_n = \chi \in \mathfrak{h}^*, \quad \gamma_1, \ldots, \gamma_n$$

such that $\psi_i <_{\gamma_{i+1}} \psi_{i+1}$ for all i. (So $<$ is the transitive closure of all the $<_\gamma$.) Call χ dominant if it is $<$-maximal.

• **Central characters of $\mathfrak{g}$:** $\Theta = \mathrm{Hom}(Z, k)$. The kernel $J_\theta \subseteq Z$ of $\theta \in \Theta$ is clearly a maximal ideal.

• Denote by $\eta^* : Z \to S(\mathfrak{h})$ the Harish–Chandra homomorphism. Identifying $S(\mathfrak{h})$ with the set of polynomial functions on $\mathfrak{h}^*$, we obtain a dual map

$$\eta : \mathfrak{h}^* \to \Theta, \quad \eta(\chi)(z) = \eta^*(z)(\chi).$$

• **Theorem 2.2 (Harish–Chandra):** η is an epimorphism with fibres

$$\eta^{-1}(\eta(\chi)) = W(\chi).$$

• Any (U, U)-bimodule Y admits an adjoint action of $\mathfrak{g}$ given by

$$X \cdot u = Xu - uX, \quad X \in \mathfrak{g}, u \in U;$$

denote the resulting $\mathfrak{g}$-module by Y^{ad}.

• **Theorem 2.3 (Kostant):** For any finite-dimensional $\mathfrak{g}$-module U, $\mathrm{Hom}_{\mathfrak{g}}(L, U^{\mathrm{ad}})$ is naturally a free Z-module of rank equal to the multiplicity of the zero weight in L.

• **Some key categories of U-modules:** Full inside of $\mathscr{M} = U$-mod:

$$\mathscr{M}_f = \{\text{finitely generated } U\text{-modules}\}, \quad \mathscr{M}_{Zf} = \{Z\text{-finite } U\text{-modules}\}.$$

For $\theta \in \Theta$ and $n \geq 1$, set $U_\theta^n = U_\theta / J_\theta^n$ and

$$\mathscr{M}^n(\theta) = \{M \in \mathscr{M} : J_\theta^n M = 0\} = U_\theta^n\text{-mod}.$$

$$\mathscr{M}^\infty(\theta) = \{M \in \mathscr{M} : \text{for all } m \in M \text{ there exists } n \geq 1 \text{ such that } J_\theta^n m = 0\},$$

suppressing the superscript for the case $n = 1$.

• **Elementary fact:** each Z-finite module M admits a unique decomposition

$$M = \bigoplus_{\theta \in \Theta} M_\theta, \quad M_\theta \in \mathscr{M}^\infty(\theta).$$

• Hence $\mathscr{M}_{Zf} \cong \prod_\theta \mathscr{M}^\infty(\theta)$ and we obtain projection functors

$$\mathrm{Pr}(\theta) : \mathscr{M}_{Zf} \to \mathscr{M}^\infty(\theta).$$

• Also have subcategory $\mathcal{O} \subseteq \mathscr{M}_{Zf}$, containing the Verma module

$$M_\chi = U/U(I_{\chi-\rho} + \mathfrak{n});$$

$I_{\chi-\rho}$ is the ideal in $U(\mathfrak{h}) \subseteq U$ generated by the elements $h - (\chi - \rho)(h)$.

• **Verma properties to recall:**

(i) The unique and pairwise non-isomorphic simple quotients L_χ of the M_χ exhaust the simple modules in $\mathcal{O}$.

(ii) The natural homomorphism $Z \to \mathrm{End}_\mathfrak{g}(M_\chi) = k$ coincides with the character $\eta(\chi)$.

(iii) There is a unique indecomposable projective object $P_\chi \in \mathcal{O}$ mapping onto L_χ; these projective objects admit a filtration by Verma modules. The common value

$$d_{\chi\psi} = [M_\chi : L_\chi] = \dim \mathrm{Hom}(P_\psi, M_\chi)$$

satisfies $d_{\chi\psi} > 0$ if and only if $\chi > \psi$, and $d_{\chi\chi} = 1$.

(iv) The classes $\delta_\chi = [M_\chi]$ form an free basis of the Grothendieck group $K(\mathcal{O})$. The unique inner product $\{-,-\}$ on $K(\mathcal{O})$ for which that basis is orthonormal is also clearly W-invariant with respect to the action $w \cdot \delta_\chi = \delta_{w\chi}$.

3 Projective functors

3.1 First properties

• Some of the main actors in our story are the functors

$$F_V : \mathscr{M} \to \mathscr{M}, \quad M \mapsto V \otimes M,$$

where V is a finite-dimensional $\mathfrak{g}$-module.

• **Immediate properties:**

(i) F_V is exact and commutes with arbitrary direct sums and products.

(ii) $\mathfrak{g}$-morphisms $\varphi : V_1 \to V_2$ induce natural transformations $F_{V_1} \to F_{V_2}$.

(iii) We have $F_{V_1} \circ F_{V_2} \cong F_{V_1 \otimes V_2}$ and a biadjunction (F_{V^*}, F_V). (Here V^* is the dual of V, with respect to some anti-involution of $\mathfrak{g}$ fixing points of $\mathfrak{h}$.)

(iv) Suppose V has weights $\mu_1, \ldots, \mu_n$ (with multiplicty). Then $F_V(M_\chi)$ has a filtration with quotients $M_{\chi+\mu_i}$, $1 \le i \le n$.

• To V we also associate the (U,U)-bimodule $\Phi_V = V \otimes U$, where the left and right actions are

$$X(v \otimes u) = Xv \otimes u + v \otimes Xu, \quad (v \otimes u)X = v \otimes uX.$$

• **Lemma 3.1:**

(i) $h(\Phi_V) \cong F_V$.

(ii) $\mathrm{Hom}_{U^2}(\Phi_V, Y) \cong \mathrm{Hom}_{\mathfrak{g}}(V, Y^{\mathrm{ad}})$ for any (U, U)-bimodule Y.

(iii) Φ_V is U-generated on both sides by its subset $V = V \otimes 1$.

• **Corollary 3.2:** F_V preserves the subcategories $\mathscr{M}_f$ and $\mathcal{O}$ in $\mathscr{M}$, and also preserves projective objects in all three categories.

Proof. F_V is exact and $F_V(U) = \Phi_V$ is finitely generated by Lemma 3.1(iii), so $F_V(\mathscr{M}_f) \subseteq \mathscr{M}_f)$. Moreover, if $M \in \mathcal{O}$, then $F_V(M)$ is $\mathfrak{h}$-diagonalisable and $U(\mathfrak{n}^+)$-finite because $V \in \mathcal{O}$, and we have already seen it is finitely generated. So $F_V(\mathcal{O}) \subseteq \mathcal{O}$.

The remaining statement follows from a general fact: functors with exact right adjoints always preserve projectives. □

3.2 Another Kostant theorem

• Have a Z^2-action on the functor F_V, i.e. a ring map $Z^2 \to \mathrm{End}(F_V)$:

$$z \cdot (v \otimes m) = \sum a_i(v \otimes b_i m), \quad \text{for } z = \sum_i a_i \otimes b_i \in Z^2.$$

• This is the action obtained by transport of structure from the action of $Z^2 \subseteq U^2$ on Φ_V to F_V via the equivalence h.

• Let I_V denote the kernel of the action:

$$I_V = \{z \in Z^2 : z(V \otimes M) = 0 \text{ for all } M \in \mathscr{M}\}.$$

• Note the embedding

$$\eta^* \otimes \eta^* : Z^2 \hookrightarrow S(\mathfrak{h}) \otimes S(\mathfrak{h}) = S(\mathfrak{h} \oplus \mathfrak{h}) = P(\mathfrak{h}^* \oplus \mathfrak{h}^*);$$

since η^* identifies Z with $S(\mathfrak{h})^W$, the image of $\eta^* \otimes \eta^*$ consists of polynomials $Q(\psi, \chi)$ which are W-invariant in each variable.

• **Theorem 3.3 (Kostant):** Let Q be the image of some $z \in Z^2$. Then $z \in I_V$ if and only if $Q(\chi+\mu, \chi)$ is the zero polynomial for any weight $\mu \in P(V)$.

• **Corollary 3.4:**

(i) Z^2/I_V is finitely generated over Z.

(ii) $F_V(\mathscr{M}_{Zf}) \subseteq \mathscr{M}_{Zf}$.

Proof. Define $A = S(\mathfrak{h})$, $B = S(\mathfrak{h})^W$, and

$$J = \{Q \in A^2 : Q(\chi + \mu, \chi) = 0 \text{ for any } \mu \in P(V)\}.$$

Then J is an ideal in A^2 and $J_V = J \cap B^2$ is an ideal in B^2. Claim (i) is equivalent to saying B^2/J_V is finitely generated over B.

By the theorem, there is a B-module embedding

$$i = \oplus_\mu i_\mu : B^2/J_V \to \oplus_{\mu \in P(V)} A,$$

where $i_\mu(Q)(\chi) = Q(\chi + \mu, \chi)$. But A is finitely generated as a B-module because W is finite, so by Noetherianity of B we conclude B^2/J_V is finitely generated over B.

It remains to prove (ii). Exercise from (i): Given a $\mathfrak{g}$-module with $JM = 0$ for some finite-codimension ideal $J \subseteq Z$, cook up a finite-codimension ideal $J' \subseteq Z$ with $J'(V \otimes M) = 0$. Then since F_V commutes with direct limits, we get $F_V(\mathscr{M}_{Zf}) \subseteq \mathscr{M}_{Zf}$. □

3.3 Functor decomposition and the main results

• We have seen that F_V preserves $\mathscr{M}_{Zf}$; let $F_{V,Zf}$ denote its restriction to this subcategory.

• **Definition 3.5:** Direct summands of $F_{V,Zf}$ are known as *projective functors.*

• Every projective functor decomposes into a direct sum of indecomposable projective functors; ultimately we will describe these indecomposables.

• **Proposition 3.6:** Let F, G be projective functors.

(i) F is exact and preserves direct sums and products.

(ii) Direct summands of F are projective; the functors $F \oplus G$ and $F \circ G$ are projective.

(iii) F has projective right and left adjoints.

(iv) $F = \oplus_{\theta,\theta'} \mathrm{Pr}_{\theta'} \circ F \circ \mathrm{Pr}_\theta$ and each of these summands are projective.

• To parametrise projective functors, we require the sets

$$\Xi^0 = \{(\psi, \chi) \in (\mathfrak{h}^*)^2 : \psi - \chi \in \Lambda\}, \quad \Xi = \Xi^0/W,$$

where the quotient is by the component-wise W-action.

• Every element $\xi \in \Xi$ has a *proper representative* (ψ, χ), by which we mean that χ is dominant and $\psi \leq W_\chi(\psi)$. There is a well-defined map

$$\eta^r : \Xi \to \Theta, \quad \eta^r(\psi, \chi) = \eta(\chi).$$

• **Theorem A:**

(i) Each projective functor decomposes into a direct sum of indecomposable projective functors.

(ii) To each $\xi \in \Xi$ there corresponds an indecomposable projective functor F_ξ, unique up to isomorphism with the following properties:

- $F_\xi(M_\varphi) = 0$ if $\eta^r(\xi) \neq \eta(\varphi)$, $\varphi \in \mathfrak{h}^*$.
- If $\xi = (\psi, \chi)$ is written properly, then $F_\xi(M_\chi) = P_\psi$.

(iii) $\xi \mapsto F_\xi$ defines a bijection from Ξ to the set of isomorphism classes of indecomposable projective functors.

Among other things, the next result reveals the remarkable fact that projective functors are determined by their induced action on $K(\mathcal{O})$.

• **Theorem B:** Suppose F, G are projective functors. Then:

(i) If $[F] = [G]$, then F is naturally isomorphic to G.

(ii) If (F, G) is an adjoint pair, then $([F], [G])$ is a conjugate pair on the inner product space $K(\mathcal{O})$.

(iii) $[F]$ is W-equivariant.

• Theorems A and B allow us to compute $[F_\xi]$ explicitly. In particular, $[F_\xi](\delta_\varphi) = 0$ if $\varphi \notin W(\chi)$ and $[F_\xi](\delta_{w(\chi)}) = \sum_{\varphi > \psi} d_{\varphi,\psi}\delta_{w\varphi}$, so understanding F reduces to knowledge of the $d_{\varphi\psi}$.

• **Definition 3.7:** Let $\theta \in \Theta$ and let $F(\theta)$ denote the restriction of a projective functor to $\mathscr{M}(\theta)$. A *projective θ-functor* $F : \mathscr{M}(\theta) \to \mathscr{M}$ is any direct summand of a functor $F_V(\theta)$.

• The third and final theorem in this section underpins the proofs of the previous two.

• **Theorem C:** Let F, G be projective θ-functors, $\chi \in \eta^{-1}(\theta)$. Then

$$i_\chi : \mathrm{Hom}(F, G) \to \mathrm{Hom}(FM_\chi, GM_\chi), \quad i_\chi(\varphi) = \varphi_{M_\chi}$$

is a monomorphism, and an isomorphism if χ is dominant.

Proof sketch. By considering decompositions $F_V(\theta) = F \oplus F', G_L(\theta) = G \oplus G'$, we reduce to the case $F = F_V(\theta)$ and $G = G_L(\theta)$.

To prove injectivity of i_χ, need the following fact: If $\chi \in \eta^{-1}(\theta)$ is a weight and $u \in U_\theta$, then $uM_\chi = 0$ implies $u = 0$.

The isomorphism for χ dominant is proven by counting dimensions using Kostant's theorem 2.3.

• We need some subsidiary information before we can proceed to the proofs of the other two theorems. Namely, we will need to see that the restriction

$$F^\infty(\theta) : \mathscr{M}^\infty(\theta) \to \mathscr{M}$$

of a projective F is determined by the restrictions $F^n(\theta) : \mathscr{M}^n(\theta) \to \mathscr{M}$.

• **Proposition 3.8:** Suppose F, G are projective functors. Then any natural transformation

$$\varphi : F(\theta) \to G(\theta)$$

admits a lift $\widehat{\varphi} : F^\infty(\theta) \to G^\infty(\theta)$. If φ is an isomorphism, then so is $\widehat{\varphi}$; if $F = G$, then any idempotent φ can be lifted to an idempotent $\widehat{\varphi}$.

Proof. Let $H^n = \mathrm{Hom}(F^n(\theta), G^n(\theta))$, $1 \leq n \leq \infty$, and let $r_{nm} : H^n \to H^m$ denote the obvious restriction maps, $m \leq n$, so we have an inverse system.

Firstly, we have that $H^\infty = \varprojlim H^n$. This is because F commutes with direct limits and modules $M \in \mathscr{M}^\infty(\theta)$ can be expressed as follows:

$$M = \varinjlim M^n, \quad M^n = \{m \in M : J_\theta^n m = 0\} \in \mathscr{M}^n(\theta).$$

As in the sketch of Theorem C, we may assume $F = F_V$, $G = F_L$. Then, exercise (use Watt's theorem and Lemma 3.1):

$$H^n = (\mathrm{Hom}_{\mathfrak{g}}(L^* \otimes V, U^{\mathrm{ad}}))/J_\theta^n.$$

So H^∞ is a J_θ-adic completion. Then $H^n = H^\infty/J_\theta^n$, so in particular $\varphi \in H^1$ can always be lifted to some $\widehat{\varphi} \in H^\infty$.

Suppose φ is an isomorphism, inverse ψ. To prove $\widehat{\varphi}$ is an isomorphism, it suffices to prove $\widehat{\varphi}\widehat{\psi}$ and $\widehat{\psi}\widehat{\varphi}$ are invertible, so for that reason we can assume $F = G$ and $\varphi = 1$. But then $\widehat{\varphi} = 1 - \alpha$ for some $\alpha \in J_\theta$, which is a unit in H^∞.

We omit the proof that an idempotent φ has an idempotent lift. □

• **Theorem C + Proposition 3.8 = Corollary 3.9:** Suppose F, G are projective functors, χ a dominant weight with $\theta = \eta(\chi)$. Any isomorphism $FM_\chi \cong GM_\chi$ lifts to an isomorphism $F^\infty(\theta) \cong G^\infty(\theta)$, and any $\mathfrak{g}$-module decomposition $FM_\chi \cong \oplus_i M_i$ lifts to a decomposition $F^\infty(\theta) = \oplus F_i$ with $F_i M_\chi = M_i$.

• If F is a projective functor, then F is the direct sum of its restrictions to the subcategories $\mathscr{M}^\infty(\theta)$; that is,

$$F = \bigoplus_\theta F \circ \mathrm{Pr}(\theta).$$

• Now, by the corollary, $F \circ \mathrm{Pr}(\theta)$ splits into a direct sum of (finitely many) indecomposable projective functors, according to the direct sum decomposition of FM_χ. Thus we obtain Theorem A(i).

• **Remark 3.10:** If F is an indecomposable projective functor, then $F = F \circ \mathrm{Pr}(\theta)$ for some $\theta \in \Theta$. Thus $FM_\chi = 0$ whenever $\eta(\chi) \neq \theta$. On the other hand, if $\chi \in \eta^{-1}(\theta)$ is dominant, then $M_\chi = P_\chi$ is an indecomposable projective and hence $FM_\chi = P_\psi$ for some $\psi \in \mathfrak{h}^*$.

• *Proof of Theorem B.* For the first point, suppose $[F] = [G]$. By the previous discussion, it is equivalent to prove $FM_\chi \cong GM_\chi$ for any dominant weight χ. But FM_χ and GM_χ are projective objects in $\mathcal{O}$, whose isomorphism classes are recoverable from their images in $K(\mathcal{O})$.

For the second point, we need to prove $\{[F]x, y\} = \{x, [G]y\}$ for all $x, y \in K(\mathcal{O})$. We can assume $x = [P]$ is the class of a projective, since the classes of projective objects span $K(\mathcal{O})$. Then use the assumed adjunction and the formula

$$\{[P], [M]\} = \dim \mathrm{Hom}(P, M), \quad P \text{ projective, } M \text{ arbitrary in } \mathcal{O}.$$

We omit the rather lengthy proof of $[F]$'s W-equivariance.

• All that remains is to prove the classification results of Theorem A(ii),(iii).

Proof. Given a projective functor F, we define a quantity

$$a_F : (\mathfrak{h}^*)^2 \to \mathbb{Z}, \quad a_F(\psi, \chi) = \{d_\psi, [F]\delta_\chi\}.$$

In fact a_F lands in $\mathbb{N}$. Indeed, if χ is dominant, then FM_χ is projective and $a_F(\psi, \chi) \geq 0$ for any ψ (consider an appropriate Hom space); then use W-equivariance of $[F]$ to deduce that $a_F(\psi, \chi) \geq 0$ always.

Next consider the subsets

$$S(F) = \{(\psi, \chi) : a_F(\psi, \chi) > 0\},$$
$$S^{\max}(F) = \{(\psi, \chi) \in S(F) : |\psi - \chi| \text{ maximal}\}.$$

By non-negativity of a_F, we get that

$$F = \oplus_i F_i \quad \Rightarrow \quad S(F) = \cup_i S(F_i)$$

so that, since $S(F_V) \subseteq \Xi^0$, the same is true for $S(F)$. (Similarly $S^{\max}(F) \subseteq \cup_i S^{\max}(F_i)$.) Both $S(F)$ and $S^{\max}(F)$ are preserved by W, due to the W-equivariance of $[F]$.

Suppose F is indecomposable. Then $S^{\max}(F)/W$ consists of a single point. Indeed, if $F = F \circ \mathrm{Pr}(\theta)$ and $\chi \in \eta^{-1}(\theta)$ is dominant, then $FM_\chi = P_\psi$ and we get $S^{\max}(F) = W(\psi, \chi)$ (exercise).

To each indecomposable projective functor F we have associated a $\xi \in \Xi$, such that if ξ is written properly, then $FM_\chi = P_\psi$. And each $\xi = (\psi, \chi)$ arises thus: If V is a finite-dimensional $\mathfrak{g}$-module with extremal weight $\psi - \chi$, then $(\psi, \chi) \in S^{\max}(F_V)$ and therefore $(\psi, \chi) \in S^{\max}(F)$ for some indecomposable summand F of F_V. □

4 Applications

4.1 Equivalences between categories $\mathscr{M}(\theta)$

• **Theorem 4.1:** For $\theta, \theta' \in \Theta$, let $F_{\theta',V,\theta} = \mathrm{Pr}(\theta') \circ F \circ \mathrm{Pr}(\theta) : \mathscr{M}^\infty(\theta) \to \mathscr{M}^\infty(\theta')$. Suppose we have dominant weights $\chi \in \eta^{-1}(\theta), \psi \in \eta^{-1}(\theta')$ such that $W_\chi = W_\psi$ and $\lambda = \psi - \chi \in \Lambda$. Then

$$F_{\theta',V,\theta} : \mathscr{M}^\infty(\theta) \to \mathscr{M}^\infty(\theta'), \quad F_{\theta,V^*,\theta'} : \mathscr{M}^\infty(\theta') \to \mathscr{M}^\infty(\theta),$$

are inverse equivalences of categories, where V is a finite-dimensional $\mathfrak{g}$-module with extremal weight λ.

Proof. Let $F = F_{\theta',V,\theta}$, $G = F_{\theta,V^*,\theta'}$. Remembering λ is an extremal weight of V (so that $-\lambda$ is such for V^*), one can show that (exercise)

$$FM_\chi = M_\psi, \quad GM_\psi = M_\chi.$$

Hence $GFM_\chi = M_\chi$, so the theorem provides that $GF \cong \mathrm{Pr}(\theta)$; similarly $FG \cong \mathrm{Pr}(\theta')$. By restricting F, G to $\mathscr{M}(\theta), \mathscr{M}(\theta')$, we deduce that they are categorical equivalences. □

• The following observations of Bernstein refine earlier results of Zuckerman:

(i) Let $\mathscr{H}$ be any complete subcategory of $\mathscr{M}$ preserved by all functors F_V, e.g. $\mathscr{H} = \mathcal{O}$. The same proof method shows that the intersections of $\mathscr{H}$ with $\mathscr{M}^\infty(\theta)$ and $\mathscr{M}^\infty(\theta')$ are equivalent.

(ii) If we assume just an inequality of stabilisers $W_\psi \subseteq W_\chi$, then (in the notation of the proof) we conclude $GF \cong \mathrm{Id}^{\oplus |W_\chi : W_\psi|}$.

4.2 Lattices of two-sided ideals and submodules

• **Notation:** Suppose χ is a dominant weight with $\eta(\chi) = \theta$. Let Ω_θ be the lattice of two-sided ideals in U_θ; let Ω_χ be the submodule lattice of M_χ.

• **Theorem 4.2:** Let χ be a dominant weight, $\theta = \eta(\chi)$.

(i) The mapping

$$\nu : \Omega_\theta \to \Omega_\chi, \quad \nu(J) = JM_\chi$$

is an embedding, and a lattice isomorphism if χ is regular.

(ii) Let $\mathscr{P}$ be the class of modules isomorphic to direct sums of P_ψ for $\psi < \chi$ and $\psi \leq W_\chi(\psi)$. Then the image of ν consists of the $\mathscr{P}$-generated submodules of M_χ.

4.3 Duflo's theorem

The result in the previous section allows for an easy re-derivation of Duflo's famous theorem.

• **Theorem 4.3 (Duflo):** Let $J \in \Omega_\theta$ be a two-sided prime ideal. Then a weight $\psi \in \eta^{-1}(\theta)$ exists such that $J = \operatorname{Ann} L_\psi$.

Proof. Take $\chi \in \eta^{-1}(\theta)$ dominant. Let $L_1, \ldots, L_n$ be the composition factors of the module $M = M_\chi / J M_\chi$, with annihilators $I_i \subseteq U_\theta$. Certainly $J \subseteq I_i$ for all i, and the product $I = I_1 \cdots I_n$ annhilates M. It follows from section 4.2 that $I \subseteq J$. Invoking that J is prime gives $J = I_i$ for some i. But now from our knowledge of M_χ, we have that $L_i = L_\psi$ for some $\psi < \chi$, and the result follows. □