Uniform Continuity of Continuous Functions on Compact Metric Spaces

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A basic theorem asserts that a continuous function on a compact metric space with values in another metric space is uniformly continuous. The usual proofs based on a contradiction argument involving sequences or on the covering property of compact sets are quite sophisticated for students taking a first course on real analysis. We present a direct proof only using results that are established anyway in such an introductory course.

Let $f\colon X \to Y$ be a continuous function from the compact metric space (X, d_X) into the metric space (Y, d_Y). The function $F\colon X \times X \to \mathbb{R}$ given by

$$F(x, y) := d_Y\big(f(x), f(y)\big)$$

is continuous with respect to the product metric on $X \times X$. Fix $\varepsilon > 0$ and consider the inverse image

$$A_\varepsilon := F^{-1}\big[[\varepsilon, \infty)\big] := \big\{(x, y) \in X \times X\colon F(x, y) \geq \varepsilon\big\}.$$

As F is continuous and $[\varepsilon, \infty)$ is closed, A_ε is a closed subset of the compact metric space $X \times X$. Hence, A_ε is compact. Assume that $A_\varepsilon \neq \emptyset$. The real valued function $(x, y) \mapsto d_X(x, y)$ is continuous on $X \times X$ and hence has a minimum on the compact set A_ε. Thus, there exists $(x_0, y_0) \in A_\varepsilon$ such that

$$\delta := d(x_0, y_0) \leq d(x, y)$$

for all $(x, y) \in A_\varepsilon$. As $(x_0, y_0) \in A_\varepsilon$ we have $\delta > 0$ as otherwise $x_0 = y_0$ and hence $0 = F(x_0, y_0) \geq \varepsilon > 0$. Moreover, if $d_X(x, y) < \delta$, then (x, y) is in the complement of A_ε, and therefore

$$d_X(x, y) < \delta \quad \Rightarrow \quad d_Y\big(f(x), f(y)\big) = F(x, y) < \varepsilon. \tag{1}$$

This is exactly what is required for uniform continuity. If $A_\varepsilon = \emptyset$, then (1) holds for every $\delta > 0$. As the arguments work for every choice of $\varepsilon > 0$ this proves the uniform continuity of f.

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