Mini-workshop on Nonlinear Partial Differential Equations and Its Geometric Applications

24-25 February 2015, ANU Canberra

Decay estimates and finite time of extinction of the nonlinear Dirichlet-to-Neumann semigroup

Daniel Hauer

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WOMASY

Geometric and Harmonic Analysis meets PDE

Joint Seminar Day

Guest speaker: Xu-Jia Wang (ANU)

17th Feb. 2015, 10am-4.45pm

Lecture theatre W5A T1

Macquarie University

http://sydney.edu.au/science/maths/u/PDESeminar/womasy/

Organisers:

- Daniel Daners (USyd)
- Xuan Duong (MQ)
- Daniel Hauer (USyd)
- Ji Li (MQ)
- James McCoy (UOW)
- Adam Sikora (MQ)
- Glen Wheeler (UOW)
- Valentina-Mira Wheeler (UOW)

University of Wollongong Australia

Macquarie University Sydney ~ Australia

The University of Sydney

Outline of my talk:

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1§ The Dirichlet problem

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2§ The Dirichlet-to-Neumann map

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2 § The Dirichlet-to-Neumann map

3 § Old and new

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1§ The Dirichlet problem

2§ The Dirichlet-to-Neumann map

3§ Old and new

4§ L^2-theory & the Dirichlet-to-Neumann semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$

Outline of my talk:

5§ Decay estimates and finite time of extinction

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6§ Application to $\{e^{-t\Delta}\}_{t\geq 0}$

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(growth condition) $\quad |a(x,\xi)| \leq a(x) + b(x)|\xi|^{p-1}$ for some $a \in L^{p'}(\Omega)$ & $b \in L^\infty(\Omega)$.

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(growth condition) $\quad |a(x,\xi)| \leq a(x) + b(x)|\xi|^{p-1}$ for some $a \in L^{p'}(\Omega)$ & $b \in L^\infty(\Omega)$.

(strict monotonicity) $\quad (a(x,\xi_1) - a(x,\xi_2)) \cdot (\xi_1 - \xi_2) > 0$

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Further, let $a:\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ be a Carathéodory function satisfying

$a(x,0)=0$ and

(p-coercivity) $\qquad a(x,\xi)\xi \geq \eta|\xi|^p$ for some $\eta>0$,

for a.e. $x\in\Omega$ & $\forall\,\xi,\xi_1,\xi_2\in\mathbb{R}^d$ with $\xi_1\neq\xi_2$.

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By using the Minty–Browder method, for every $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$, there is a unique weak solution $u \in W^{1,p}(\Omega)$ of

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that is, for $\Phi \in W^{1,p}(\Omega)$ with $\mathrm{Tr}\,\Phi = \varphi$ one has $u - \Phi \in W^{1,p}_0(\Omega)$ and

$$\int_\Omega a(x,\nabla u)\nabla\xi \, dx = 0 \quad \text{for every } \xi \in W^{1,p}_0(\Omega).$$

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Then $P : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ is also called "elliptic extension" operator & has the properties:

- P is continuous
- P is injective
- For every $\varphi \in W^{1-1/p,p}(\partial\Omega)$ & $\Phi \in W^{1,p}(\Omega)$ with $Tr\,\Phi = \varphi$ there is a unique $u_\Phi \in W^{1,p}_0(\Omega)$ s.th. $P\varphi = u_\Phi + \Phi$.

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2§ The Dirichlet-to-Neumann map

2§ The Dirichlet-to-Neumann map

In this talk, I want to focus the attention on the mapping

$$\varphi_{|\partial\Omega} \longmapsto a(x, D\varphi)\cdot\nu_{|\partial\Omega} =: \Lambda\varphi,$$

where ν denotes the outward pointing unit normal vector.

2§ The Dirichlet-to-Neumann map

Under the assumption that $\nabla\varphi$ & $a(x,\nabla\varphi)$ are smooth,

multiplying $\Delta\varphi = a(x,\nabla\varphi)\cdot\nu$ by $\psi\in C^{\infty}(\overline{\Omega})$

& integrating over $\partial\Omega$ & applying Green's formula,

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Under the assumption that $P\varphi$ & $a(x,\nabla P\varphi)$ are smooth,

multiplying $\Lambda\varphi = a(x,P\varphi)\cdot\nu$ by $\psi \in C^\infty(\overline{\Omega})$

& integrating over $\partial\Omega$ & applying Green's formula,

$$\Longrightarrow \int_{\partial\Omega} \Lambda\varphi\cdot\psi \, d\sigma = \int_\Omega a(x,\nabla P\varphi)\nabla\psi \, dx$$

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Under the assumption that $P\varphi$ & $a(x,\nabla P\varphi)$ are smooth, multiplying $\Delta\varphi = a(x,P\varphi)\cdot\nu$ by $\psi \in C^\infty(\overline{\Omega})$ & integrating over $\partial\Omega$ & applying Green's formula,

$$\Longrightarrow \quad \int_{\partial\Omega} \Lambda\varphi \cdot \psi \, d\sigma = \underbrace{\int_{\Omega} a(x,\nabla P\varphi)\nabla\psi \, dx}_{<\infty \text{ if } P\varphi \in W^{1,p}(\Omega) \;\&\; \psi \in W^{1,p}(\Omega)}$$

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2§ The Dirichlet-to-Neumann map

We call $\Lambda : W^{1-1/p,p}(\partial\Omega) \to W^{-(1-1/p),p'}(\partial\Omega)$ defined by

$$\langle \Lambda\varphi, \psi\rangle := \int_\Omega a(x,\nabla P\varphi)\nabla E\psi \, dx \quad \text{for every } \varphi,\psi \in W^{1-1/p,p}(\partial\Omega)$$

the Dirichlet-to-Neumann map associated with $Au := \mathrm{div}(a(x,\nabla u))$

where $E : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ denotes the linear bounded extension operator satisfying $T_r \circ E = id$.

3 § Old and new

▷ The DtN map appears in many application:

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↳ The decomposition of the Wentzel-Robin oper.

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3 § Old and new

- The DtN map appears in many application:
  - ↳ Calderon's inverse problems
  - ↳ The decomposition of the Wentzel-Robin oper.
  - ↳ To determin the boundary regularity of Ω.
- The DtN is a nonlocal operator.
- In the past, much research has been done on the DtN map associated with linear unif. elliptic 2^{nd} order diff. operators & still continues....

3 § Old and new

▷ But not much is known so far about the DtN map ass. with nonlinear 2^{nd} order diff. operators.

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3§ Old and new

▷ In H.'14, I studied the well-posedness of the elliptic problem

$$(3.1) \qquad \Lambda \varphi = \psi \quad \text{on } \partial\Omega$$

$$(3.2) \qquad \int_{\partial\Omega} \varphi \, d\sigma = 0$$

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and established Hölder continuity of weak solutions of (3.1),

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▷ In H.'14, I studied the well-posedness of the elliptic problem

(3.1) $\quad \Delta_p \varphi = f \quad$ on $\partial\Omega$

(3.2) $\quad \int_{\partial\Omega} \varphi \, d\sigma = 0$

and established Hölder continuity of weak solutions of (3.1), more precisely, if $f \in L^q_m(\partial\Omega)$ with $q = \frac{d-1}{p-1-\varepsilon}$, $\varepsilon \in (0,1)$ if $p \le d$

$q = 1$ if $p > d$

then there are $\alpha \in (0,1)$ & $c_\alpha \ge 0$ s.th. every weak solution φ of (3.1) belongs to $C^{0,\alpha}(\partial\Omega)$ & satisfies

$$\|\varphi\|_{C^{0,\alpha}(\partial\Omega)} \le c_\alpha \left(\|f\|^{1/(p-1)}_{L^q(\partial\Omega)} + \|\nabla\varphi\|_{L^p(\Omega)} \right) + c_\alpha .$$

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▷ the "part of $\mathbb{A}$ in $\mathcal{C}(\partial\Omega)$" is m-T-accretive in $\mathcal{C}(\partial\Omega)$,

and hence $-\mathbb{A}$ generates a strongly continuous semigroup $\{e^{-t\mathbb{A}}\}_{t\ge 0}$ on $\overline{D(\mathbb{A})}^{\mathcal{C}(\partial\Omega)}$

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and hence $-\mathbb{A}$ generates a strongly continuous semigroup $\{e^{-t\mathbb{A}}\}_{t\ge 0}$ on $\overline{D(\mathbb{A})}^{\mathcal{C}(\partial\Omega)} \underset{\uparrow}{=} \mathcal{C}(\partial\Omega)$.

if $\partial\Omega \in \mathcal{C}^{2,\beta}$, $\beta\in(0,1)$

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- decay estimates as $t\to\infty$ of $\{e^{-tA}\}_{t\ge 0}$
- finite time of extinction of $\{e^{-tA}\}_{t\ge 0}$

4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$

4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\ge 0}$

In this section, we make the additional assumption that

there is a Carathéodory function $\mathcal{A} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ s.th.

$$\nabla_\xi \mathcal{A}(x,\xi) = a(x,\xi) \quad \text{for every } \xi \in \mathbb{R}^d \text{ \& a.e. } x \in \Omega .$$

4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$

Then for every $\varphi \in L^2(\partial\Omega)$,

$$\mathcal{E}(\varphi) := \begin{cases} \displaystyle\int_\Omega \mathcal{A}(x, \nabla P\varphi)\, dx & \text{if } \varphi \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega), \\ +\infty & \text{if otherwise,} \end{cases}$$

defines a convex, proper, l.s.c functional on $L^2(\partial\Omega)$, s.th. $\mathcal{E}_{|W^{1-1/p,p} \cap L^2}$ is C^1 & $\nabla_{L^2}\mathcal{E} = \Lambda$ in $L^2(\partial\Omega)$.

4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$

By the theory of nonlinear semigroups in Hilbert spaces:

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By the theory of nonlinear semigroups in Hilbert spaces:

for every $\varphi_0 \in L^2(\partial\Omega)$ there is a unique strong solution

$$\varphi \in W^{1,\infty}([\delta,\infty);L^2(\partial\Omega)) \cap C([0,\infty);L^2(\partial\Omega)) \quad \forall\, \delta>0$$

of

$$(4.1) \qquad \frac{d\varphi}{dt} + \Lambda\varphi = 0 \qquad \text{on } (0,\infty)\times\partial\Omega$$

$$(4.2) \qquad \int_{\partial\Omega} \varphi(t)\, d\sigma = \int_{\partial\Omega} \varphi_0\, d\sigma \quad \text{for all } t \geq 0,$$

$$(4.3) \qquad \varphi(0) = \varphi_0 \quad \text{in } L^2(\partial\Omega)$$

4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$

S.th.

$$\frac{d}{dt}\mathcal{E}(\varphi(t)) = -\left\|\frac{d\varphi}{dt}(t)\right\|^2_{L^2(\partial\Omega)} \qquad (4.4)$$

for a.e. $t>0$.

4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$

S.th.

$$\frac{d}{dt}\mathcal{E}(\varphi(t)) = -\left\|\frac{d\varphi}{dt}(t)\right\|^2_{L^2(\partial\Omega)} \quad \text{for a.e. } t>0. \tag{4.4}$$

$$\varphi \in \mathcal{C}([0,\infty); W^{1-1/p,p}(\partial\Omega)) \tag{4.5}$$

4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$

Thus & since Λ is accretive in $L^2(\partial\Omega)$,

setting $e^{-t\Lambda}\varphi_0 := \varphi(t)$ for every $t \geq 0$ & $\varphi_0 \in L^2(\partial\Omega)$

defines a strongly continuous semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ of contractions on $L^2(\partial\Omega)$.

4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\ge 0}$

Proposition

Let either $p \geq \frac{2d}{d+1}$ or $\mathcal{E}(\varphi) = \mathcal{E}(-\varphi)$ or $\varphi_0 \in L^q(\partial\Omega)$ for $q \geq (d-1)/(p-1-\varepsilon)$,

then

$$\{ e^{-t\Lambda}\varphi_0 \mid t \geq 1 \} \text{ is rel. compact in } W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega).$$

4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\ge 0}$

Therefore the *ω-limit set*

$$\omega(\varphi_0) := \Big\{ \psi \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega) \;\Big|\; \text{there is } t_n \uparrow +\infty \text{ s.th. } \lim_{n\to\infty} e^{-t_n\Lambda}\varphi_0 = \psi \text{ in } W^{1-1/p,p} \cap L^2 \Big\}$$

is non-empty, connected & single-valued iff $\lim_{t\to\infty} e^{-t\Lambda}\varphi_0$ ex.

4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\ge 0}$

Therefore the ω-limit set

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is non-empty, connected & single-valued iff $\lim_{t\to\infty} e^{-t\Lambda}\varphi_0$ ex.

By (4.4) $$\frac{d}{dt}\mathcal{E}(\varphi(t)) = -\left\|\frac{d\varphi}{dt}(t)\right\|^2_{L^2(\partial\Omega)}$$

for a.e. $t>0$,

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4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$

$\Rightarrow$ $\mathcal{E}$ is a "strict Liapunov function" for $\{e^{-t\Lambda}\}_{t\geq 0}$.

4§ L^2-theory & the Dirichlet-to-Neumann Semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$

$\Rightarrow$ $\mathcal{E}$ is a "strict Liapunov function" for $\{e^{-t\Lambda}\}_{t\geq 0}$.

$$\Longrightarrow \quad \omega(\varphi_0) \subseteq \nabla_{L^2}\mathcal{E}^{-1}(\{0\}) = \mathbb{R}$$

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Conservation of mass equality

$$\Longrightarrow \qquad \omega(\varphi_0) = \left\{ \fint_{\partial\Sigma} \varphi_0 \, d\sigma \right\}.$$

5§ Decay estimates and finite time of extinction

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The main tool to derive decay estimates & finite time of extinction is the following inequality:

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Definition

A continuously Fréchet-differentiable function $\mathcal{E}: V \to \mathbb{R}$ on a Banach space V satisfies a *generalised Łojasiewicz-Simon inequality* near $\varphi \in V$ if there is $\sigma > 0$ and a strictly increasing $\Theta \in C([0,\infty)) \cap C^1((0,\infty))$ with $\Theta(0) = 0$ s.th.

$$\begin{array}{l}\text{for every } v \in V\\ \|v - \varphi\|_V < \sigma\end{array} \implies \frac{1}{\Theta'(|\mathcal{E}(v) - \mathcal{E}(\varphi)|)} \le \|\mathcal{E}'(v)\|_{V'}$$

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5§ Decay estimates and finite time of extinction

Remarks:

(1) The gen. Łoj-Sim. ineq. is due to the classical ineq. of Łojasiewicz '63 & '65 in finite dimensions & and of Simon '83 in infinite dimensions.

5§ Decay estimates and finite time of extinction

Remarks:

(1) The gen. Łoj-Sim. ineq. is due to the classical ineq. of Łojasiewicz '63 & '65 in finite dimensions & and of Simon '83 in infinite dimensions.

(2) By taking $\Theta(s) = c \cdot s^{\alpha}$ for some $c > 0$ & $\alpha \in (0, \frac{1}{2}]$, the gen. Łoj.-Sim. ineq. reduces to the classical one.

5§ Decay estimates and finite time of extinction

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(2) By taking $\Theta(s) = c \cdot s^{\alpha}$ for some $c > 0$ & $\alpha \in (0, \frac{1}{2}]$, the gen. Łoj-Sim. ineq. reduces to the classical one.

(3) The gen. Łoj-Sim. ineq. allows now to take $\alpha \in (0,1)$

5§ Decay estimates and finite time of extinction

Remarks:

(4) The gen. Łoj.–Sim. ineq. has been introduced by Haraux–Jendoubi '03 & Haraux–Jendoubi–Kavian '03. First applied to nonlinear gradient systems by Chill–Fiorenza '06.

5§ Decay estimates and finite time of extinction

Suppose V is a Banach space, H is a Hilbert space such that $V \overset{d}{\hookrightarrow} H$.

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Suppose V is a Banach space, H is a Hilbert space such that $V \overset{d}{\hookrightarrow} H$.

1. Theorem (Convergence result)

If $u \in \mathcal{C}([0,\infty);H) \cap \mathcal{C}((0,\infty);V)$ is a strong solution of $\frac{du}{dt} + \nabla_H \mathcal{E}(u) = 0$ with rel. compact image in V and if $\mathcal{E}$ satisfies a gen. Łoj.-Sim. ineq. near some $\varphi \in \omega(u)$ wrt. V,

then $\lim_{t\to\infty} u(t) = \varphi$ in V.

5§ Decay estimates and finite time of extinction

Suppose V is a Banach space, H is a Hilbert space such that $V \overset{d}{\hookrightarrow} H$.

2. Theorem (Decay estimates)

Under the assumptions of Theorem 1, we have

$$|\mathcal{E}(u(t)) - \mathcal{E}(\varphi)| = O(\Psi^{-1}(t-\hat{t})) \text{ as } t \to \infty$$

& $$\|u(t) - \varphi\|_H = O(\Theta(\Psi^{-1}(t-\hat{t}))) \text{ as } t \to \infty.$$

where $\Psi \in \mathcal{E}((0,\infty))$ is a primitive of $-(\Theta')^2$ & Ψ^{-1} its inverse & $\hat{t} := t_0 - \Psi(\mathcal{E}(u(t_0)) - \mathcal{E}(\varphi))$ for $t_0 \geq 0$ large enough.

5§ Decay estimates and finite time of extinction

By taking $\Theta(s) = c \cdot s^{\alpha}$ for some $c > 0$ & $\alpha \in (0,1)$, we can deduce from Theorem 2 the following

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Corollary

- $\|u(t) - \varphi\|_H = O\left(t^{-\frac{\alpha}{1-2\alpha}}\right)$ as $t \to \infty$ if $0 < \alpha < \frac{1}{2}$
- $\|u(t) - \varphi\|_H = O\left(e^{-\frac{2t}{c^2}}\right)$ as $t \to \infty$ if $\alpha = \frac{1}{2}$
- $\|u(t) - \varphi\| \le \begin{cases} C \cdot (\hat{t} - t)^{\frac{\alpha}{2\alpha - 1}} & \text{if } 0 \le t \le \hat{t} \\ 0 & \text{if } t > \hat{t} \end{cases}$ if $\frac{1}{2} < \alpha < 1$

(finite time of extinction)

$$\hat{t} = t_0 + \frac{c^2 \alpha^2}{2\alpha - 1} \left(\mathcal{E}(u(t_0)) - \mathcal{E}(\varphi)\right)^{2\alpha - 1}.$$

6§ Application to $\{e^{-t\Delta}\}_{t\geq 0}$

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Theorem (H.'15)

Let $1<p<\infty$ and $a(x,\xi):=a(x)|\xi|^{p-2}\xi$ for some $a\in L^\infty(\Omega)$ with $a\geq a_0>0$. Then for every $\varphi\in L^2(\partial\Omega)$ we have

- $\|e^{-t\Lambda}\varphi-\overline{\varphi}\|_{L^2(\partial\Omega)}=O(t^{-\frac{1}{p-2}})$ as $t\to\infty$ if $p>2$
- $\|e^{-t\Lambda}\varphi-\overline{\varphi}\|_{L^2(\partial\Omega)}=O(e^{-t\frac{a}{2}/C^2})$ as $t\to\infty$ if $p=2$
- finite time of extinction if $p<2$.

6§ Application to $\{e^{-t\Delta}\}_{t\geq 0}$

Proof.

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Proof.

$$\|\mathcal{E}'(\varphi)\|_{W^{-(1-\frac{1}{p}),p'}(\partial\Omega)} \geq \frac{\left|\int_\Omega a(x,|\nabla\varphi|)\nabla E(\varphi-\overline{\varphi})\,dx\right|}{\|\varphi-\overline{\varphi}\|_{W^{1-1/p,p}(\partial\Omega)}}$$

6§ Application to $\{e^{-t\Delta}\}_{t\geq 0}$

Proof.

$$\|\mathcal{E}'(\varphi)\|_{W^{-(1-\frac{1}{p}),p'}(\partial\Omega)} \geq \frac{\left|\int_\Omega a(x,\nabla P\varphi)\nabla E(\varphi-\overline{\varphi})\,dx\right|}{\|\varphi-\overline{\varphi}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}}$$

Recall

For every $\varphi \in W^{1-1/p,p}(\partial\Omega)$ & $\Phi \in W^{1,p}(\Omega)$ with $\mathrm{Tr}\,\Phi = \varphi$

there is a unique $u_\Phi \in W^{1,p}_0(\Omega)$ s.th. $P\varphi = u_\Phi + \Phi$.

$$\Longrightarrow \quad \int_\Omega a(x,\nabla P\varphi)\nabla E\xi\,dx = \int_\Omega a(x,\nabla P\varphi)\nabla P\varphi\,dx$$

6§ Application to $\{e^{-t\Delta}\}_{t\geq 0}$

Proof.

$$\|\mathcal{E}'(\varphi)\|_{W^{-(1-\frac{1}{p}),p'}(\partial\Omega)} \geq \frac{\left|\int_\Omega a(x,|\nabla\varphi|)\nabla E(\varphi-\overline{\varphi})\,dx\right|}{\|\varphi-\overline{\varphi}\|_{W^{1-1/p,p}(\partial\Omega)}}$$

$$\geq C \quad \frac{\mathcal{E}(\varphi)}{\|\varphi-\overline{\varphi}\|_{W^{1-1/p,p}(\partial\Omega)}}$$

6§ Application to $\{e^{-t\Delta}\}_{t\geq 0}$

Proof.

$$\|\mathcal{E}'(\varphi)\|_{W^{-(1-\frac{1}{p}),p'}(\partial\Omega)} \geq \frac{\left|\int_{\Omega} a(x,|\nabla\varphi|)\nabla E(\varphi-\overline{\varphi})\,dx\right|}{\|\varphi-\overline{\varphi}\|_{W^{1-1/p,p}(\partial\Omega)}}$$

$$\geq c\;\frac{\mathcal{E}(\varphi)}{\|\varphi-\overline{\varphi}\|_{W^{1-1/p,p}(\partial\Omega)}}$$

$$\geq \tilde{C}\;\mathcal{E}(\varphi)^{1-1/p}$$

6§ Application to $\{e^{-t\Delta}\}_{t\geq 0}$

Proof.

$$\|\mathcal{E}'(\varphi)\|_{W^{-(1-\frac{1}{p}),p'}(\partial\Omega)} \geq \frac{\left|\int_{\Omega} a(x,|\nabla\varphi|)\nabla E(\varphi-\bar{\varphi})\,dx\right|}{\|\varphi-\bar{\varphi}\|_{W^{1-1/p,p}(\partial\Omega)}}$$

$$\geq c\,\frac{\mathcal{E}(\varphi)}{\|\varphi-\bar{\varphi}\|_{W^{1-1/p,p}(\partial\Omega)}}$$

$$\geq \tilde{c}\;\mathcal{E}(\varphi)^{1-1/p}$$

$$= \tilde{c}\;\left(\mathcal{E}(\varphi)-\mathcal{E}(\bar{\varphi})\right)^{1-1/p}$$

$\square$

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Thank You!