N- widths for the Sobolev classes on the unit sphere

This is a joint work with Gavin, conducted when I was a student at the University of Sydney. Our main interest is to find the sharp orders of the Kolmogorov and the linear n-widths of Sobolev's classes on the unit sphere $\mathbb{S}^{d-1}$. The key tool in our research is the positive cubature formulas and Marcinkiewicz-Zygmund (MZ) inequalities on the sphere. Our work also reveals a close relationship between positive cubature formulas and MZ inequalities on $\mathbb{S}^{d-1}$.

§1 N-widths on $\mathbb{S}^{d-1}$

- **Notation.** Let $\mathbb{S}^{d-1}$ denote the the unit sphere of the d-dimensional Euclidean space $\mathbb{R}^d$. Given $0 < p \le \infty$, we denote by $L^p \equiv L^p(\mathbb{S}^{d-1})$ the usual Lebesgue space on $\mathbb{S}^{d-1}$. We shall use the notation $A \sim B$ to mean that there exists an inessential constant $c > 0$, called the constant of equivalence, such that

$$c^{-1}A \le B \le cA.$$

- **Definitions.** For a given subset K of a normed linear space $(X, \|\cdot\|)$, the Kolmogorov n-width $d_n(K, X)$ is defined by

$$d_n(K,X) = \inf_{L_n} \sup_{x\in K} \inf_{y\in L_n} \|x-y\|,$$

with the left-most infimum being taken over all n-dimensional linear subspaces L_n of X, while the linear n-width $\delta_n(K, X)$ is defined by

$$\delta_n(K, X) = \inf_{T_n} \sup_{x \in K} \|x - T_n(x)\|,$$

with the infimum being taken over all linear continuous operators T_n on X with $\dim\Big(T_n(X)\Big) \le n$.

- **Sobolev's classes.** Given $r > 0$, we denote by $(-\triangle)^r$ the rth order Laplace - Beltrami operator on $\mathbb{S}^{d-1}$, defined in a distributional sense. For $1 \le p \le \infty$, the Sobolev space $W_p^r \equiv W_p^r(\mathbb{S}^{d-1})$ is defined by

$$W_p^r := \Big\{ f \in L^p : \quad (-\triangle)^{r/2}(f) \in L^p \Big\},$$

while the Sobolev class B_p^r is defined as the unit ball of W_p^r:

$$B_p^r := \{ f \in W_p^r : \quad \|(-\triangle)^{r/2}(f)\|_p \le 1\}.$$

As is well known, if $1 \le p,\ q \le \infty$ and $r > (d-1)(\frac{1}{p} - \frac{1}{q})_+$ then

$$W_p^r \subset L^q(\mathbb{S}^{d-1}).$$

Thus, both $d_n(B_p^r, L^q)$ and $\delta_n(B_p^r, L^q)$ are well defined whenever $r > (d-1)(\frac{1}{p} - \frac{1}{q})_+$. Our main interest here is the sharp asymptotic orders of $d_n(B_p^r, L^q)$ and $\delta_n(B_p^r, L^q)$ as $n \to \infty$ for all the pairs $1 \le p, q \le \infty$. We will find $\alpha(p,q,r)$ and $\beta(p,q,r)$ for which

$$d_n(B_p^r, L^q) \sim n^{\alpha(p,q,r)}, \ \ \delta_n(B_p^r, L^q) \sim n^{\beta(p,q,r)}$$

with the constants of equivalence independent of n.

In the case $d = 2$ (i.e. the periodic case) this problem was completely solved during 1940–1970s, due to the work of several famous mathematicians, including Kolmogorov, Tikhomirov, Kashin, Höllig, Maiorov etc. (We refer to [Pin] and [Te] for more information.) We shall restrict our attention to the higher dimensional case (i.e. $d \ge 3$) for the rest of the talk.

- **Previously known results for $d \geq 3$.** Define

$$A = \{(p,q): \quad 1 \leq p \leq q \leq 2\},$$
$$B = \{(p,q): \quad 1 \leq p \leq 2 \leq q \leq \infty\},$$
$$C = \{(p,q): \quad 2 \leq p \leq q \leq \infty\},$$
$$D = \{(p,q): \quad 1 \leq q \leq p \leq \infty\}.$$

Clearly, $[1,\infty]^2 = A \cup B \cup C \cup D$.

The following results were previously proved in [BKLT, Ka1, Ka2]:

$$d_n(B_p^r,\, L^q) \sim n^{-\frac{r}{d-1}+\frac{1}{p}-\frac{1}{q}}, \qquad (p,q) \in A$$

$$\delta_n(B_p^r, L^q) \sim \begin{cases} n^{-\frac{r}{d-1}+\frac{1}{p}-\frac{1}{2}}, & (p,q) \in A, \\ n^{-\frac{r}{d-1}+\frac{1}{p}-\frac{1}{q}}, & (p,q) \in C. \end{cases}$$

- **Our main result.**

  In the joint work with Gavin and Yongsheng Sun, we obtained the sharp orders of $d_n(B_p^r, L^q)$ and $\delta_n(B_p^r, L^q)$ for all the remaining pairs $(p,q) \in [1,\infty]^2$.

  Theorem. *Let* $r > (d-1)(\frac{1}{p} - \frac{1}{q})_+$. *Then*

$$d_n(B_p^r, L^q) \sim \begin{cases} n^{-\frac{r}{d-1}+\frac{1}{p}-\frac{1}{2}}, & (p,q) \in B \\ n^{-\frac{r}{d-1}}, & (p,q) \in C \cup D \end{cases}$$

$$\delta_n(B_p^r, L^q) \sim \begin{cases} n^{-\frac{r}{d-1}+\max\{\frac{1}{p}-\frac{1}{2},\frac{1}{2}-\frac{1}{q}\}}, & (p,q) \in B \\ n^{-\frac{r}{d-1}}, & (p,q) \in D. \end{cases}$$

This theorem was proved in [BD] and [BDS].

The main idea in the proof of the above theorem: Decompose the Sobolev spaces W_p^r into a countable sum of certain finite dimensional subspaces of spherical polynomials, and then estimate the n-widths of the unit balls of those subspaces. The MZ inequalities and related positive cubature formulas, discussed in the next section, will play a crucial role in this second step.

§2 MZ inequalities and cubature formulas

- **Notation.** Given an integer $n \geq 0$, the restriction to $\mathbb{S}^{d-1}$ of a polynomial in d-variables of degree n is called a spherical polynomial of degree at most n. We denote by Π_n^d the space of all spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$. Π_n^d is a linear space with dim $\Pi_n^d \sim n^{d-1}$. We refer to [Wa-Li] for harmonic analysis on $\mathbb{S}^{d-1}$.

  We denote by $d(x, y)$ the geodesic distance $\arccos x \cdot y$ between x and y on $\mathbb{S}^{d-1}$, by $B(x, r)$ the spherical cap $\{y \in \mathbb{S}^{d-1} : d(x, y) \leq r\}$.

  A finite subset Λ of $\mathbb{S}^{d-1}$ is said to be ε-separable if $\min\limits_{\substack{\xi,\xi' \in \Lambda \\ \xi \neq \xi'}} d(\xi, \xi') \geq \varepsilon$. A set Λ is maximal ε-separable if it is ε-separable and $\mathbb{S}^{d-1} = \bigcup\limits_{\xi \in \Lambda} B(\xi, \varepsilon)$.

Given $B \subset \mathbb{S}^{d-1}$ and $f \in C(\mathbb{S}^{d-1})$, we define

$$\text{osc}(f, B) := \sup_{x,y\in B} |f(x) - f(y)|.$$

For simplicity, we also write

$$\text{osc}(f; x, r) := \text{osc}(f, B(x, r)).$$

- **Our main results.**

Theorem. *If* $0 < p < \infty$, $s \in (0, \pi)$ *and* $\Lambda \subset \mathbb{S}^{d-1}$ *is* s*-separable, then for any* $f \in \Pi_n^d$ *and* $\beta \geq 1$,

$$s^{d-1} \sum_{\omega\in\Lambda} \Big(\mathit{osc}\Big(f; \omega, \beta s\Big)\Big)^p \leq c(ns)^p \|f\|_p^p,$$

where c *depends only on* d, p *and* β.

Of particular interest is the case when $s = \frac{\delta}{n}$ and δ is an absolute constant.

Corollary. *Assume* $0 < p < \infty$, $\delta \in (0, \pi)$ *and* $f \in \Pi_n^d$.

(i) For any $\beta \geq 1$ *and* δ/n*-separable subset* $\Lambda \subset \mathbb{S}^{d-1}$,

$$\left(\frac{\delta}{n}\right)^{d-1} \sum_{\omega \in \Lambda} \max_{x \in B(\omega, \frac{\beta\delta}{n})} |f(x)|^p \leq c\|f\|_p^p,$$

where c *depends only on* d, p *and* β.

(ii) There exists a constant $\delta_0 > 0$ *depending only on the dimension* d *such that for any* **maximal** δ/n*-separable subset* $\Lambda \subset \mathbb{S}^{d-1}$ *with* $\delta \in (0, \delta_0]$ *we have*

$$\|f\|_p^p \leq c\left(\frac{\delta}{n}\right)^{d-1} \sum_{\omega \in \Lambda} \min_{x \in B(\omega, \frac{\delta}{n})} |f(x)|^p,$$

where c *depends only on* d *and* p.

Combing (i) and (ii) above, we have, under the condition of (ii),

$$\|f\|_p^p \sim \left(\frac{\delta}{n}\right)^{d-1} \sum_{\omega \in \Lambda} \min_{x \in B(\omega, \frac{\delta}{n})} |f(x)|^p$$
$$\sim \left(\frac{\delta}{n}\right)^{d-1} \sum_{\omega \in \Lambda} \max_{x \in B(\omega, \frac{\delta}{n})} |f(x)|^p.$$

Our next result reveals a close relationship between MZ inequalities and cubature formulas:

Theorem. *Suppose we have a positive cubature formula of degree* $2n$ *on* $\mathbb{S}^{d-1}$*:*

$$\int_{\mathbb{S}^{d-1}} f(y)\,d\sigma(y) = \sum_{\omega\in\Lambda} \lambda_\omega f(\omega), \quad f \in \Pi_{2n}^d,$$

where $\lambda_\omega \geq 0$*. Then for any* $f \in \Pi_{[n/2]}^d$*,*

$$\|f\|_p \sim \begin{cases} \left(\sum_{\omega\in\Lambda} \lambda_\omega |f(\omega)|^p\right)^{\frac{1}{p}}, & \text{if } p \in (0,\infty), \\ \max_{\omega\in\Lambda} |f(\omega)|, & \text{if } p = \infty, \end{cases}$$

with the constants of equivalence depending only on p *and* d*.*

Conversely, suppose we have the following MZ inequalities for some $0 < p_0 < \infty$ *and large positive integer* n*:*

$$\|f\|_{p_0}^{p_0} \sim \frac{1}{n^{d-1}} \sum_{\omega\in\Lambda} |f(\omega)|^{p_0}, \quad \forall f \in \Pi_n^d,$$

where Λ *is a finite subset of* $\mathbb{S}^{d-1}$*, then there exist positive* $\lambda_\omega \sim \frac{1}{n^{d-1}}$ *for each* $\omega \in \Lambda$*, and a number* $\gamma \in (0,1)$ *independent of* n *and* Λ*, for which*

$$\int_{\mathbb{S}^{d-1}} f(y)\, d\sigma(y) = \sum_{\omega\in\Lambda} \lambda_\omega f(\omega), \quad \forall f \in \Pi_{[\gamma n]}^d.$$

For MZ inequalities and positive cubature formulas on $\mathbb{S}^{d-1}$, we refer to [BD, BDS, MNW, NPW1, NPW2]. In one-dimensional case, we refer to the remarkable paper [MT] of Mastroianni and Totik.

References.

[BKLT] B. Bordin, A. K. Kushpel, J. Levesley and S. A. Tozoni, Estimates of n-widths of Sobolev's classes on compact globally symmetric spaces of rank one, *J. Funct. Anal.* **202**(2003), no. 2, 307–326.

[BDS] G. Brown, F. Dai and Sun Yongsheng, Kolmogorov width of classes of smooth functions on the sphere $\mathbb{S}^{d-1}$, *J. Complexity* **18** (2002), no. 4, 1001–1023.

[Ka1] A.I. Kamzolov, The best approximation of the classes of functions $W_p^\alpha(\mathbb{S}^{d-1})$ by polynomials in spherical harmonics, *Math. Notes* **32**(1982), 622–626.

[Ka2] A.I. Kamzolov, On the Kolmogorov diameters of classes of smooth functions on a sphere, *Russian Math. Survey* **44**(1989), no. 5, 196–197.

[MNW] H. N. Mhaskar, F. J. Narcowich and J. D. Ward, *Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature,* Math. Comp. **70** (2001), 1113–1130 (Corrigendum: Math. Comp. **71** (2001) 453–454).

[MT] G. Mastroianni and V. Totik, *Weighted polynomial inequalities with doubling and A_∞ weights,* Constr. Approx. **16** (2000), no. 1, 37–71.

[NPW1] . Narcowich; P. Petrushev; J. Ward, Decomposition of Besov and Triebel-Lizorkin spaces on the sphere, *J. Funct. Anal.* **238** (2006), no. 2, 530–564.

[NPW2] arcowich, F. J.; Petrushev, P.; Ward, J. D. Localized tight frames on spheres, *SIAM*

J. Math. Anal. **38** (2006), no. 2, 574–594.

[Pin] A. Pinkus, n-widths in approximation theory, Springer, New York, 1985.

[Te] V. N. Temlyakov, Approximation of periodic functions, Nova Science Publishers, New York, 1993.

[WL] Wang Kunyang and Li Luoqing, Harmonic Analysis and Approximation on the unit Sphere, Science press, Beijing, 2000.