Balanced matrices and functions

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with interference from Richard Aron and Nigel Kalton

If you get bored . . .

Show that if $n \not\equiv 0 \pmod 7$, then

$$\frac{7}{n} \sum_{m=1}^{n} \frac{\cos\left(\frac{2m\pi}{n}\right)}{8\cos^3\left(\frac{2m\pi}{n}\right) + 4\cos^2\left(\frac{2m\pi}{n}\right) - 4\cos\left(\frac{2m\pi}{n}\right) - 1} \equiv n^5 \pmod 7.$$

A simple puzzle

Consider a 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

If

- the largest 3 elements in each row sum to a common value R, and
- the largest 3 elements in each column sum to a common value C,

then we say that A is **balanced**.

Example

R and C might be different!

For example, if

$$A = \begin{pmatrix} 3 & 3 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

then $R = 6$ and $C = 7$.

Question: How large (or small) can $\dfrac{C}{R}$ be?

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Origins of the problem

This question arose in the linear algebra community . . .

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General definition

Definition

An $n \times m$ matrix A with ***nonnegative*** *entries is $(n, \ell; m, k)$*-***balanced*** *(or simply balanced) if the sums of the largest k elements (out of m) in each row are all equal (to say R), and the sums of the largest ℓ elements (out of n) in each column are all equal (to say C). Let*

$$\mathrm{r}(n, \ell; m, k) = \sup \left\{ \frac{C}{R} \ : \ A \textit{ is } (n, \ell; m, k)\textit{-balanced} \right\}.$$

We shall write $\mathrm{rs}(n, \ell)$ for $\mathrm{r}(n, \ell; n, \ell)$.

Calculating $\mathrm{r}(n, \ell; m, k)$ seems in general to be quite a challenge, and only some special cases are known.

Known bounds — taking all but one entry

Theorem

For any $n, m \geq 2$*,*

$$\frac{n-1}{m-1} \leq \mathrm{r}(n, n-1; m, m-1) \leq \frac{n}{m} + \frac{n-2}{m(m-1)}.$$

If m is even then the upper bound is acheived.

Note: An alternative way of writing this result is

$$\frac{nm-m}{m(m-1)} \leq \mathrm{r}(n, n-1; m, m-1) \leq \frac{nm-2}{m(m-1)}.$$

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For even m, an extremal matrix looks like

$$A = \begin{pmatrix} 2(m-1) & \dots & 2(m-1) & 0 & \dots & 0 \\ 0 & \dots & 0 & 2(m-1) & \dots & 2(m-1) \\ m & & m & m & \dots & m \\ \vdots & & \vdots & \vdots & & \vdots \\ m & \dots & m & m & \dots & m \end{pmatrix}.$$

This has $R = m(m-1)$ and $C = (n-2)m + 2(m-1)$, which achieves the bound in the theorem.

Odd and even square matrices

Theorem

If n is even, then

$$\operatorname{rs}(n, n-1) = 1 + \frac{n-2}{n(n-1)}.$$

So $\operatorname{rs}(4,3) = \frac{7}{6}$, which answers the original question.

Theorem

For any odd $n \geq 3$,

$$1 + \frac{n-3}{(n-1)^2} \leq \operatorname{rs}(n, n-1) \leq 1 + \frac{n-2}{n(n-1)}.$$

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Ideas from the proof

The proofs are relatively elementary, just involving careful rearranging of certain sums, and the following facts.

1. Being balanced is stable under permutations of the rows and columns.
2. Being balanced is stable under taking postive multiples, and adding or subtracting the same constant to every entry — as long as the entries stay nonnegative.
(So, WLOG the smallest entry is 0.)
3. The smallest entry in each row satisfies $a_{ij} \leq \frac{R}{n-1}$.

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If you add up all the entries in the matrix you get

$$nC \leq nC + \sum \text{smallest entries in each colum}$$

$$= \sum_{i,j} a_{ij}$$

$$= nR + \sum \text{smallest entries in each row}$$

and so

$$nC \leq nR + n\frac{R}{n-1}$$

or

$$\frac{C}{R} \leq 1 + \frac{1}{n-1}.$$

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Easiest open question

It is messy, but one can prove that $\mathrm{rs}(3,2) = 1$. That is, there are no nontrivial 3×3 balanced matrices. This is the lower bound in the previous theorem.

Question

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The 5 x 5 case

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Is an extreme $(5,4)$ *balanced matrix*

$$A = \begin{pmatrix} 36 & 36 & 0 & 0 & 0 \\ 0 & 0 & 24 & 24 & 24 \\ 15 & 15 & 19 & 19 & 19 \\ 15 & 15 & 19 & 19 & 19 \\ 15 & 15 & 19 & 19 & 19 \end{pmatrix}$$

This has $\frac{C}{R} = \frac{9}{8} = \frac{45}{40}$. Looking at the theorem

$$1 + \frac{n-3}{(n-1)^2} \leq \mathrm{rs}(n, n-1) \leq 1 + \frac{n-2}{n(n-1)}$$

this is equal to the lower bound. Can you get the upper bound $\frac{C}{R} = \frac{23}{20} = \frac{46}{40}$?

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General square matrices

If you are summing the largest ℓ entries in each row and column of an $n \times n$ matrix the following is known.

Theorem

For $n \geq 2$ and $1 \leq \ell \leq n$,

$$\mathrm{rs}(n, \ell) \leq \min\left(\frac{n}{\ell}, 2\right).$$

This is far from being sharp in general!

What is the right bound?

Our original guess: For $n \geq 2$, $1 \leq \ell \leq n$, is

$$\mathrm{rs}(n, \ell) \leq 2 - \frac{\ell}{n} - \frac{(n-\ell)}{n\ell} \; ?$$

Recent examples due to Nigel Kalton indicate that this is not quite right!

Balanced functions

You can of course ask the same sorts of questions concerning functions, now taking integrals rather than sums.

Here the questions of measurability of sets starts arising, which complicates the arguments significantly.

Let S denote the unit square $[0, 1] \times [0, 1]$.

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Let S denote the unit square $[0, 1] \times [0, 1]$.

Definition

Let $\alpha \in (0,1)$. A bounded measurable function $f : S \to [0,\infty)$ is α-balanced if there exist constants R, C such that

$$\sup_{\mu(X)=\alpha} \int_X f(x,y_0)\,d\mu(x) = R, \qquad \text{for almost all } y_0 \in [0,1],$$

$$\sup_{\mu(Y)=\alpha} \int_Y f(x_0,y)\,d\mu(y) = C, \qquad \text{for almost all } x_0 \in [0,1].$$

Define

$$\mathrm{r}(\alpha) = \sup\left\{\frac{C}{R} : f \text{ is } \alpha\text{-balanced}\right\}.$$

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Define

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Bounds for functions

Theorem

For $\alpha \in (0,1)$,

$$2 - \alpha \le \mathrm{r}(\alpha) \le \min\{2, \frac{1}{\alpha}\}.$$

The lower bound comes from taking explicit functions based on the corresponding matrix problems.

The upper bound is rather harder — a major issue is measurability of various sets that are constructed.

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Ideas in the proof

To get around the measurability issues we show that any α-balanced function is nearly equal to a continuous function which is nearly α-balanced.

One then has to do some careful epsilonics, considering the cases $\alpha < \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$ separately.

Lemma (Part 1)

Suppose that $f \in L^\infty(S)$ *is* α*-balanced. Then for all* $\epsilon > 0$ *there exists* $g \in C(S)$ *and measurable sets* $X, Y \subset [0,1]$ *such that*

1. $\mu(X) > 1 - \epsilon$ *and* $\mu(Y) > 1 - \epsilon$;
2. *for all* $x \in X$*, there exists* $V_x \subset [0,1]$ *with* $\mu(V_x) = \alpha$ *such that for any set* $V \subset [0,1]$ *with* $\mu(V) = \alpha$

$$\int_V g(x,y)\,dy \le \int_{V_x} g(x,y)\,dy;$$

3. *for all* $x \in X$*,*

$$\left| \int_{V_x} g(x,y)\,dy - C \right| < \epsilon;$$

Lemma — continued

Lemma (Part 2)

4. *for all* $y \in Y$*, there exists* $H_y \subset [0,1]$ *with* $\mu(H_y) = \alpha$ *such that for any set* $H \subset [0,1]$ *with* $\mu(H) = \alpha$
$$\int_H g(x,y)\,dx \le \int_{H_y} g(x,y)\,dx;$$
5. *for all* $y \in Y$*,*
$$\left| \int_{H_y} g(x,y)\,dx - R \right| < \epsilon;$$
6. *the sets* $A = \{(x,y) : x \in X,\ y \in V_x\}$ *and* $B = \{(x,y) : y \in Y,\ x \in H_y\}$ *are measurable.*

Continuous functions

Frustrating open questions:

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