MATH 402 Homework 4

Due Friday October 6, 2017

Exercise 1. Let S be any set, and let $f, g : S \to S$ be any two functions. Recall that we say that g is the *inverse* of f (and write $f^{-1} := g$) if for every $s \in S$ we have

$$f(g(s)) = s; \qquad g(f(s)) = s.$$

A function f which has an inverse is called *invertible.*

a. A function $f : S \to S$ is called *bijective* if it is both injective ('one-to-one') and surjective ('onto'). Prove that a bijective function f must have a unique inverse.
b. Prove that if f and g are invertible with inverses f^{-1}, g^{-1}, and if $h = f \circ g$, then h is invertible with $h^{-1} = g^{-1} \circ f^{-1}$.
c. In particular, we defined a *transformation* to be a bijection of the plane, so it follows immediately from the above that a transformation has an inverse. Recall that a transformation is called an *isometry* if it preserves length. Prove that if f is an isometry, then its inverse is also an isometry.
d. Prove that if f and g are isometries, then $f \circ g$ is an isometry.
e. Combine the last two parts of the exercise, and use the fact that composition of functions is associative, to show that the set of isometries is a group. (You may need to review the definition of a group! Make sure you address each group axiom in your solution.)

Exercise 2. Prove the following theorem:

Theorem 1. *Suppose that f and g are two isometries which agree on three non-collinear points A, B, C. Prove that $f(P) = g(P)$ for* all *points P.*

Exercise 3. Prove that an isometry preserves circles: i.e. if f is an isometry, and c is a circle with radius r and centre O, then f maps c to the circle c' of radius r and centre $f(O)$.

Exercise 4. A *reflection* r is defined as an isometry which has two fixed points, and which is not the identity.

a. Prove that $r^2 = \mathrm{Id}$. Thus, a reflection is its own inverse.
b. Define what it means for a set S to be *fixed* by r. Define what it means for S to be *invariant* under r.
c. Recall that we proved that the reflection fixes the entire line ℓ determined by these two points, and we denoted this reflection by $r = r_\ell$. Now prove that the invariant lines of r_ℓ are exactly the line ℓ and the lines m which are perpendicular to ℓ

Remember that in addition to the points assigned to each question, you will receive up to five further points for neatness and organization.