MATH 402 Homework 8

Due Friday November 10, 2017

Exercise 1.

a. Show that in an omega-triangle $PQ\Omega$, the angles $\angle PQ\Omega$ and $\angle QP\Omega$ always add up to strictly less than $180°$.

b. Use this and our earlier results on omega-triangles to prove *Angle-Angle Congruence* for omega-triangles: If $PQ\Omega$ and $P'Q'\Omega'$ are two omega-triangles with $\angle PQ\Omega \cong \angle P'Q'\Omega'$ and $\angle QP\Omega \cong \angle Q'P'\Omega'$, prove that $PQ = P'Q'$. (Hint: if not, assume $PQ < P'Q'$, and choose a point Q'' on $P'Q'$ such that $PQ = P'Q''$. Consider the new omega-triangles you get with this point.)

Exercise 2. In this exercise, you will study what happens to omega points under reflections and rotations. Let ℓ be a hyperbolic line, and let r_ℓ be reflection across ℓ.

You may find it helpful to remember how reflections act, and convince yourself that it makes sense in hyperbolic geometry just as well as Euclidean geometry.

- If $P \in \ell$, then $r_\ell(P) = P$.
- If $P \notin \ell$, then r_ℓ is supposed to reflect P to the other side of the line ℓ. To find the image, draw the line m which passes through P and is perpendicular to ℓ; let Q be the point where m and ℓ intersect. Then $r_\ell(P)$ is the unique point on m on the other side of ℓ such that the distance from P to Q is the same as the distance from $r_\ell(P)$ to Q.

a. Let ℓ' be a limiting parallel to ℓ. Show that $r_\ell(\ell')$ is also a limiting parallel to ℓ. To do this, choose a point P on $r_\ell(\ell')$, and show that $r_\ell(\ell')$ is limiting parallel to ℓ at that point. That means you need to draw the perpendicular to ℓ through the point P, and consider rays interior to the angle you've just made at P. You must show that they intersect ℓ. What happens if you apply r_ℓ to the point P, the perpendicular, and the interior rays?

b. Let Ω be an omega point of ℓ. Show that r_ℓ fixes Ω. (What does it mean to "fix the omega-point Ω"? It means that the set of lines that makes up Ω is preserved by r_ℓ.)

c. Now prove that the only omega-points fixed by r_ℓ are the omega-points of ℓ. Start by assuming that ℓ' is a line with different omega-points. Suppose that r_ℓ fixes one of them, call it Ω'. Take P to be any point in ℓ, and consider the line $\overleftrightarrow{P\Omega'}$. What happens when you apply r_ℓ to this line?

d. Suppose that $R_{A,\alpha}$ is rotation about a point A by angle α. Assume that $R_{A,\alpha}$ fixes an omega-point Ω. Use what you've shown about reflections to prove that $\alpha = 0$ (i.e. $R_{A,\alpha}$ is the identity). (Hint: you can express the rotation as a composition of two reflections across lines which pass through A. Choose one of your lines so that you know what happens to Ω. What do you know about the second line?)

Remember that in addition to the points assigned to each question, you will receive up to five further points for neatness and organization.