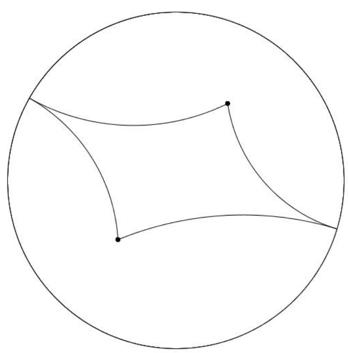
MATH 402 Final exam practice questions

Exercise 1 *Consider the following figure in hyperbolic geometry. Prove that the sum of the two angles at the real vertices is less than* $360°$. *Find the area of this figure, in terms of the constant* k^2.



Exercise 2 *(a) Draw a picture of 3 Poincaré lines* ℓ, m, *and* n *such that* ℓ *and* m *are ultraparallel, and* n *and* m *are ultraparallel, but* ℓ *and* n *intersect at a single point* P.

(b) Let $f = r_m \circ r_\ell$.

  i. *What kind of isometry is this?*

  ii. *What fixed points does it have?*

(c) Let $g = r_n \circ r_m$.

  i. *What kind of isometry is this?*

  ii. *How many omega-points (if any) are fixed by this isometry?*

  iii. g *can be written as a Möbius transformation of the form*

$$g(z) = \beta \frac{z - \alpha}{\overline{\alpha} z - 1}$$

  . *What can you say about* $|\alpha|$ *and* $|\beta|$*?*

(d) Let $h = g \circ f$.

  i. *What kind of isometry is this?*

  ii. *Is this isometry orientation-preserving?*

(e) Give the definition of a group.

(f) Recall that in Euclidean geometry, the set of translations (including the identity element, viewed as translation by zero) forms a group. Use your results from above to argue that the set of translations (including the identity) in the Poincaré disk does not *form a group.*

Exercise 3 *In hyperbolic geometry, we can have a tiling of type* $(3, 10)$. *Suppose that a tile in this tiling has area 144. Now consider a different tiling, one of type* $(5, 4)$. *Prove that a tile in this new tiling has area 72.*

Exercise 4 *Given a line ℓ and a point P not on ℓ:*

(a) *Assume we are in Euclidean geometry. What does Playfair's postulate say about P and ℓ?*

(b) *Assume we are in hyperbolic geometry.*

i. *What does the hyperbolic parallel postulate say about P and ℓ?*

ii. *The fundamental theorem of parallels in hyperbolic geometry tells us that there are two limiting parallels to ℓ through P. Why does this imply that there must be infinitely many ultraparallels to ℓ through P? (Drawing a picture might help you explain your answer.)*

iii. *In particular, through any point P not on ℓ there are at least two lines through P that are ultraparallel to ℓ. Use this to argue that if f is a hyperbolic isometry such that ℓ and all lines ultraparallel to ℓ are invariant under f, then $f = Id$.*

Exercise 5 (a) *State Pasch's axiom.*

(b) *Let ΔABC be a triangle, and let X be a point in the interior of the triangle. Prove that the ray $\overrightarrow{AX}$ passes through the opposite side, $\overline{BC}$.*

Exercise 6 *Let $ABCD$ be a Saccheri quadrilateral.*

(a) *Use $ABCD$ to construct a Lambert quadrilateral.*

(b) *Use $ABCD$ to construct a quadrilateral with all four angles congruent. What can you say about the sides of this quadrilateral?*

Exercise 7 (a) *What does it mean for an isometry to have finite order?*

(b) *What kind of Euclidean isometries can or must have finite order?*

(c) *What Euclidean isometries have order two?*

Exercise 8 *Let (z_1, z_2, z_3) be distinct complex numbers. Explain how you can use the cross-ratio to define a Möbius transformation f sending $z_1 \mapsto 1, z_2 \mapsto 0, z_3 \mapsto \infty$. Explain why f is the unique Möbius transformation with this property.*

Exercise 9 *Give careful definitions. To receive full marks, you must give the* definition *of the term, not an equivalent characterization.*
(e.g. a Euclidean translation *was defined in class as an isometry which can be written as the composition of two reflections across parallel lines; we later proved that a Euclidean translation is an isometry which can be expressed in the form $T(x, y) = (x, y) + (v_1, v_2)$. Saying the first of these things would get you full marks; saying the second would get only part marks.)*

(a) *When two points A, B are* on the same side *of a line ℓ.*

(b) *The Klein distance function.*

(c) *The inverse of a point P with respect to a circle $c = (O, r)$.*

(d) *A limiting parallel (model-independent).*

(e) *A regular tiling.*

(f) *The order of a group element $g \in G$.*

(g) *When two triangles are* similar.

(h) *A Möbius transformation.*