MATH 402 Review for August 27–31

Topics: 1.3 (Golden Ratio), 1.4 and 1.5 (Axiomatic systems), 1.6 (Euclid's axioms), Appendix A (Euclid's axioms), Appendix D (Hilbert's axioms), and a bit of 2.1 (proving things in Euclidean geometry). These were covered in lectures and in the Project (1.3). This material will also appear in Homework 1.

1. **Key things about Axiomatic Systems:**
   (a) The building blocks are *undefined terms* and *axioms/postulates* (plus basic ingredients from logic).
   (b) We use these to define new terms, and prove *theorems*.
   (c) Know these terms: *consistent*, *independent*, *complete*.
   (d) If we can find a *model* for an axiomatic system, then the system is consistent (at least, relative to whatever system we needed to define the model).

2. **Key things to know about the development of Euclidean and non-Euclidean geometry:**
   (a) Euclid wrote down five postulates. (Know what they say. You don't need to memorize the funny syntax.) The last one is called the Parallel Postulate.
   (b) The Parallel Postulate is equivalent to Playfair's Postulate. Know the statement of Playfair's Postulate.
   (c) People thought that the Parallel Postulate was weird. They tried to prove that it *wasn't independent* of the first four axioms. They tried to do this by replacing it with other axioms that contradict it, and showing that the resulting axiomatic system *wasn't consistent*.
   (d) They failed, because the Parallel Postulate *is* independent after all.
   (e) When we do geometry using only the first four axioms, this is called *neutral geometry*. The results we prove in neutral geometry are true in both Euclidean and hyperbolic geometry.
   (f) Euclid's axiomatic system wasn't perfect. He was missing some assumptions (which he used without proof). Hilbert came along and wrote down a more precise formulation which fills in these gaps, but to do this he needed 21 axioms instead of just 5.
      - For example, Hilbert needed to include axioms that imply the property of circle continuity.
      - He also needed to assume SAS congruence as an axiom (whereas Euclid had a "proof" of this fact that relied on unstated assumptions).
   (g) Know the five undefined terms in Hilbert's axiomatic system, and how we use them to build up the language of geometry. Give the following definitions, using only previously defined terms:
      - Segment, ray, angle, bisector of a segment (or midpoint), bisector of an angle, triangle.

3. **First results in Euclidean geometry:**
   (a) In Euclidean geometry we're interested in "straight-edge and compass constructions". Make sure you're comfortable with these examples:
      - Given a base $\overline{AB}$, construct an equilateral triangle ΔABC.
      - Given an angle $\angle CAB$, construct a ray $\overrightarrow{AD}$ bisecting the angle.
   (b) Remember that SAS congruence is an axiom in Hilbert's axiomatic system. List the other triangle congruence results.

Practice Questions

1. **Practice with Axiomatic Systems**
   If you want to look at more examples of working with axiomatic systems, you can read page 20–21 of the textbook, or try the sequence of exercises 1.4.3–1.4.5.

2. **Practice defining terms in Hilbert's system**
   Define the following, using only previously defined terms:

   - Equilateral triangle
   - Isosceles triangle
   - The perpendicular bisector of a segment
   - What it means for two angles to be *supplementary*

   Try to make a definition by yourself, and then look in the book to see if your definition agrees. If it's different, can you prove that it's equivalent? (For example, in the book we define an isosceles triangle to be a triangle with two edges congruent, but it turns out that's equivalent to having two angles congruent.)

3. **Practice with geometric constructions**
   Try one of the following "games", which give you puzzles involving geometric constructions: https://kasperpeulen.github.io/ or https://www.euclidea.xyz/ (or just search for "Euclidean geometry games" to find other apps—they're all pretty similar). These are great practice to get comfortable with the techniques we use to prove theorems in Euclidean geometry.