MATH 402 Worksheet 3

Wednesday 3 October, 2018

Read about the Klein model of hyperbolic geometry (section 7.2.2), and answer the following questions.

Exercise 1.

a. Show that Euclid's first two axioms hold for Klein points and Klein lines.

b. Draw a picture of some Klein points and Klein lines which shows that the hyperbolic parallel postulate holds in the Klein model.

c. Define the circle of radius r about a point O in this model, and think about why Euclid's third axiom holds. (Hint: use the continuity and the unboundedness of the logarithm function—it is similar to the discussion for the Poincaré disk.)

Angles in the Klein model

Exercise 2. Let ℓ be a Klein line which corresponds to a chord which is a diameter of the circle. Let ℓ' be a Klein line which corresponds to a chord $\overline{AB}$ which is *not* a diameter.

a. When is another Klein line m perpendicular to ℓ? (Look up the answer in the book. Draw some pictures.)

b. Define the *pole* of the chord $\overline{AB}$. (Look up the answer in the book. Draw some pictures. If you need to remind yourself of some of the words used in the definition, do so now.)

c. We say that a Klein line m is *perpendicular* to ℓ' if the Euclidean line extending the chord of m passes through both ℓ and the pole of the underlying chord $\overline{AB}$. Draw some pictures of some lines perpendicular to ℓ'.

d. Draw some pictures to illustrate the fact that in the Klein model, whenever we have a line ℓ (corresponding to either a chord or a diameter) and a point P not on ℓ, there is a unique perpendicular line to ℓ through P.

   (Note that we don't need to *prove* this fact: the constructions given earlier using ruler and compass hold in neutral geometry, and hence in hyperbolic geometry, and hence in this model. Make sure you understand this statement. The point here is simply to *illustrate* the fact which we already know to be true.)

e. Let m and m' be any two Klein lines which are parallel to each other.

   (i) Use the definition of perpendicular lines to prove that in almost all cases, there exists a unique line n which is perpendicular to both m and m'. (We say n is a *common perpendicular line.*)

   (ii) In which case does this not hold? In that case, is the problem that there is *more than one* common perpendicular line, or that there are *none*?

   (iii) Compare to Euclidean geometry: how many (if any) common perpendicular lines are there between two parallel lines in Euclidean geometry?

Limiting parallels and the angle of parallelism

Let ℓ be a Klein line and P a point not on ℓ. You already showed that there are multiple parallel lines to ℓ through the point P. In particular, there are two Klein lines which have the property that they divide the set of all lines through P into two sets: those lines which are parallel to ℓ, and those which are not. These two lines are called the *limiting parallels* to ℓ at P.

Exercise 3.

a. Draw a picture illustrating limiting parallels.
b. Is there a notion of limiting parallels in the Poincaré model?
c. Drop a perpendicular from P to ℓ (as in Exercise 3 d.), and let the intersection point with ℓ be Q. (Consider the picture on page 275.) The *angle of parallelism* for ℓ at P is defined to be the angle made by $\overleftrightarrow{PQ}$ with the limiting parallel. Draw a picture and label this angle.
d. Prove that the angle of parallelism cannot be a right angle. (One strategy: suppose it is a right angle. Consider the pole of the chord corresponding to the line $\overleftrightarrow{PQ}$. Which of the chords in your picture must extend to Euclidean lines which pass through this pole?)
e. Now prove that the angle of parallelism cannot be larger than a right angle. (Hint: if it is, show that you can construct a triangle with two right angles.)

(You might ask: we had to choose one of the limiting parallels to define our angle of parallelism. What if we had chosen the other one? We will prove later that the angle formed with each of the two limiting parallels is congruent, but we do not have the tools for that yet.)

You do not need to hand your work in, but you are expected to complete it. If you get stuck or are unsure about your answers, come to office hours. This material is examinable and will not be covered in ordinary lecture format, so you must make sure that you understand it as it is presented here.